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ELEMENTS

OF

PLANE GEOMETRY

AND

MENSURATION,

ON THE BASIS OF LEGENDRE'S ELEMENTS, WITH NUMEROUS
ALTERATIONS AND ADDITIONS:

BEING THE

FIRST PART OF A SERIES

ON

ELEMENTARY AND HIGHER

GEOMETRY, TRIGONOMETRY, AND MENSURATION;

CONTAINING MANY VALUABLE DISCOVERIES AND IMPROVEMENTS IN MATHEMATICAL
SCIENCE, ESPECIALLY IN RELATION TO THE QUADRATURE OF THE CIRCLE,
AND SOME OTHER CURVES, AS WELL AS THE CUBATURE OF CERTAIN
CURVILINEAR SOLIDS; DESIGNED AS A TEXT-BOOK FOR COLLEGIATE
AND ACADEMIC INSTRUCTION, AND AS A PRACTICAL
COMPENDIUM ON MENSURATION.

BY NATHAN SCHOLFIELD.

BIBLIOTHECA BODLEIANA

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PREFACE.

In presenting this work to the public, the author is fully impressed with the responsibility he assumes, since the press is constantly teeming with works of a similar character, from the pens of the most eminent mathematicians in this and other countries; and since we have the accumulated discoveries of ancient and modern times, the condensed wisdom of ages, the result of talent and labor, spread out on pages before us, it may by some be supposed that no new truths, no new principles or applications can be elicited, especially in elementary Geometry, which reached almost its present perfection in a remote period of antiquity. The works of Euclid, for more than twenty centuries, having been and are still, with scarcely an important improvement, the standard works on elementary Geometry; and while other sciences have been progressing and vieing with each other in the splendor of progressive acquisition, this has been handed down to us as a system of unchangeable truths, not affected by time or varying circumstances; evidently showing the science to be based on immutable principles. But although the elements of this science were thus early established, yet by the application of the principles therein contained, new truths have been constantly developing themselves in an accelerated progression, and many important mathematical truths which have been supposed unattainable by human ken have thereby been established on the same basis as the elements themselves.

It may be possible, therefore, that other mathematical principles, either elementary or general, may yet be developed by the skilful application of those already established.

It is with this view that the author has undertaken this work; having, as he deems, made some advancements in the science, not only in the elements, but also in its general application.

Although Euclid's elements deservedly possess such high regard or merit, both on account of their antiquity and the rigorous accuracy of his reasoning, yet we have numerous modern works on the subject, some of which are better adapted to the present state of science, the authors of which only aimed at the perfection of Euclid, with a more obvious connection of the parts, or greater elegance of diction; but after all, none have been able to supersede his authority in point of rigorous accuracy.

Among the many modern works on elementary Geometry may be mentioned the names of Lacroix and Legendre of the French, Hutton, Leslie and Young, of the English. Others of considerable merit might be mentioned, but these are specified on account of their popularity; Legendre particularly having given a more favorable impetus to speculative, and Hutton to practical Geometry, or Mensuration. By means of which, the science has been clothed in a more attractive form and elevated to a more favorable position. The elementary part of the work now presented to the public, the author has deemed it expedient to base on the elements of Legendre, not only on account of the elegance and general accuracy of that work, but also on account of its well deserved popularity. But while this work has been selected as the basis, others have been consulted, and whenever it was thought the subject required any change, selections have been made from other authors, under such modifications as were required to adapt them to this work, or the propositions in the original of Legendre have occasionally been slightly altered when propriety seemed to require it, and many original propositions have been added, and the work greatly extended both in plane and solid Geometry, but more particularly in solid.

It is needless here to specify all the supposed improvements introduced in the elementary parts of the work, but it is believed that some new definitions, which are here introduced, will be regarded with favor, as being more strictly in accordance with mathematical reasoning. The subject of Proportion, which is wanting in Legendre, and which is made the Fifth Book of Euclid's Elements, is here made to precede the other parts, thereby allowing those parts more purely geometrical to follow in consecutive order. The subject of Proportion is prefaced by the application of numbers to magnitudes, and the whole subject of proportional magnitudes is treated in a full, and, as the author conceives, in a perspicuous manner. The subject of incommensurable quantities or magnitudes is discussed, and it is shown that our reasoning in reference to such quantities ought not to be embarrassed on account of their incommensurability, since any magnitude may, as will be shown, be expressed by some function of any other magnitude, and hence may have some definite expression in terms of such magnitude. A new mode of treating the subject of parallels, it is thought, possesses some advantage. But a few changes have been introduced into the second and third books from those of Legendre: a few useful propositions have, however, been added. In the fourth book, by discussing the subject of Proportion and proportional figures before that of the squares of the sides of right angled triangles, we are enabled to present a new demonstration for that proposition founded on the proportionality of the sides of similar triangles, at once elegant and satisfactory, and to deduce directly from this, in the most conclusive manner, the general property that similar triangles and polygons are proportional to the squares on their homologous sides, &c.

Some curious propositions follow toward the close of this book some of which are original, and some have been selected from other authors, with such modification as was deemed necessary to correspond with the style of this work.

In the fifth book, also, will be found much new matter. It is unnecessary to particularize.

The sixth book consists of the Isoperimetry of plane figures, and is succeeded by Notes illustrating certain portions of the subject to which they refer.

The application of the principles embraced in elementary plane geometry, to Mensuration, closes this part of the series.

The second part consists of elementary solid geometry based also on Legendre's Elements, but is considerably extended beyond the ordinary limits of similar works. Some new solids are introduced, and it is believed that part relating to solids of revolution, especially of the surface and solidity of a sphere, is more rigorous and satisfactory than in any other work.

The third part consists of spherical geometry, analytical trigonometry, the application of trigonometry to the mensuration of heights and distances, and trigonometrical surveying, &c. The application of algebra to geometry; Conic Sections, are also embraced in this part.

The fourth part treats of such solids of revolution as depend on the higher geometry, and the solid sections or segments of such as are embraced in the elementary part. In this volume some new solids are introduced. By the discussions in this volume, we are enabled to arrive at most of the useful properties of such solids as are bounded by curve surfaces, including the segments of solids of revolution, by the simple elementary principles of geometry, some of which have only been heretofore obtained through the medium of fluxions, or the calculus.

The quadrature of the circle, and the rectification of the elliptical circumference and some other curves, are also discussed. Some new properties of the circle, and other curves are developed, enabling us to pursue our investigations in relation to them, by pure geometry. We are also enabled to show that, although it may not be possible for us to express the circumference of a circle numerically in terms of its diameter, yet that it may be expressed in terms of some known function of that diameter, and hence that it is possible to get an algebraic expression for its value in known quantities.

The series closes with the application of the former principles to the mensuration of such figures, whether plane or solid, as depend on the higher geometry. Some rules are obtained, much shorter and easier than any others in use, for the determination of some of the more difficult problems.

CONTENTS.

BOOK FIRST.

THE APPLICATION OF NUMBERS TO MAGNITUDES—RATIO AND PROPORTION—DEFINITION OF TERMS—EXPLANATION OF SYMBOLS—AXIOMS—POSTULATES—DEFINITIONS AND EXPLANATIONS OF PRINCIPLES USED IN PROPORTION—GENERAL DISCUSSION OF PROPORTIONAL MAGNITUDES.

BOOK SECOND.

OF LINES AND THEIR PROPERTIES—DEFINITIONS OF GEOMETRICAL TERMS—GENERAL DISCUSSIONS AND PROPOSITIONS.

BOOK THIRD.

THE CIRCLE AND MEASUREMENT OF ANGLES—DEFINITIONS—GENERAL DISCUSSISON AND PROPOSITIONS—PROBLEMS RELATING TO THE SECOND AND THIRD BOOKS.

BOOK FOURTH.

OF THE PROPERTIES AND AREAS OF FIGURES—DEFINITIONS—GENERAL PROPOSITIONS—PROBLEMS.

BOOK FIFTH.

OF REGULAR POLYGONS AND THE MEASUREMENT OF THE CIRCLE—DEFINITIONS—GENERAL PROPOSITIONS—PROBLEMS.

BOOK SIXTH.

ISOPERIMETRY—DEFINITIONS—GENERAL PROPOSITIONS.

NOTES.

OF SOME DEFINITIONS—SPECULATIONS ON SOME OF THE SUBJECTS EMBRACED IN GEOMETRICAL WORKS.

MENSURATION OF SUPERFICIES.

THE MENSURATION OF SUCH PLANES AS DEPEND ON ELEMENTARY PRINCIPLES—LINES AND SURFACES.

ELEMENTS OF GEOMETRY.

BOOK I.

GENERAL PRINCIPLES.

GEOMETRY is the science which treats of the measurement and comparison of magnitude, and the relations of locality.

Magnitude can have but three dimensions; length, breadth and thickness: all of which are necessary to constitute a body, or solid. It is important, however, to consider magnitude under three distinct denominations; of lines, surfaces and solids: and thus the science of Geometry becomes divided into three principal branches: the first part treating of lines described upon the same plane, and of the surfaces which they enclose: the second, of lines situated in different planes, and of the relations of those planes to each other; and the third part contemplating body under its several dimensions of length, breadth and thickness.

Lines are obviously the boundaries of surfaces, and surfaces are the boundaries of solids; it is equally obvious that a line being mere length, without either breadth or thickness, can exist only as the boundary of a surface, and that a surface, being absolutely without thickness, can exist only as an attribute of body.

Although, therefore, it cannot be supposed that a line, or a surface, can have separate or independent existence, the fact will not in the smallest degree interrupt or embarrass our reasonings, in considering these several attributes of body or space, each apart from the others, nothing more being requisite than the abstracting these others from our inquiry; so that in considering lines, length only is recognized, and in contemplating surfaces, length and breadth are combined and thickness excluded.

Geometry is necessarily dependent on numbers for the development and application of its principles; for lines, surfaces, or solids, can be expressed only in relation to their magnitudes or figures, and an expression of magnitude is the result of comparison with some other magnitude of the same kind as a unit.

DEFINITIONS OF TERMS USED IN GEOMETRY.

An *Axiom* is a self-evident truth.

A *Postulate* requires us to admit the possibility of an operation.

A *Theorem* is a truth which becomes evident by means of a train of reasoning called a demonstration

A *Demonstration* is *direct*, when the truth is inferred directly from the premises, as the conclusion of a regular series of inductions.

A *Demonstration* is *indirect*, when the conclusion shows that the introduction of any supposition, contrary to the truth advanced, necessarily leads to an absurdity.

A *Problem* proposes an operation to be performed.

A *Lemma* is a subsidiary truth, the evidence of which must be established, preparatory to the demonstration of a succeeding theorem.

A *Proposition* is a general term for either a theorem, a problem or a lemma.

A *Corollary* is an obvious consequence, resulting from a demonstration.

An *Hypothesis* is a supposition, and may be either true or false.

A *Scholium* is a remark on one or several preceding propositions, which tends to point out their connection, their use, their restriction, or their extension.

EXPLANATION OF SYMBOLS EMPLOYED.

1. The sign $=$ is the sign of equality between two or more magnitudes or quantities; thus, the expression $A=B$ signifies that A is equal to B.

2. The symbol $>$ placed between two magnitudes or quantities denotes that the former is greater than the latter; the symbol $<$ is used to denote that the former is less than the latter. Thus, $A>B$ indicates that the quantity represented by A is greater than that represented by B: by $A<B$ is implied that A is less than B.

3. The sign $+$ is called *plus*: it indicates addition.

4. The sign $-$ is called *minus*: it signifies subtraction.

Thus $A+B$ represents the sum of the quantities A and B: $A-B$ represents their difference, or what remains after B is taken from A; and $A-B+C$, or $A+C-B$, signifies that A and C are to be added together, and that B is to be subtracted from their sum. The double sign $\pm$ is employed to express indifferently either the sum or difference of the magnitudes or quantities between which it is placed.

5. The sign $\backsim$ is used to denote the difference between two quantities when it is not known which is the greater; thus $A \backsim B$ represents the difference between A and B, without reference to which is the greater.

6. The sign $\times$, *into or with*, indicates multiplication: thus $A\times B$ represents the product of A and B. A *point* is sometimes used instead of the sign $\times$; thus $A.B$ denotes the same as $A\times B$. The same product is also designated without any intermediate sign, by AB. The expression $A\times(B+C-D)$ represents the product of A by the quantity $B+C-D$. If $A+B$ were to be multiplied by $A-B+C$, the product would be indicated thus $(A+B)\times(A-B+C)$, those quantities enclosed within the brackets being considered as a single quantity.

7. A number placed before a letter denoting a quantity, is called a co-efficient, and serves as a multiplier to that quantity: thus 3A signifies that the quantity A is to be repeated three times: $\frac{1}{2}$A signifies half the quantity A.

8. The sign $\div$ signifies division, and is called *by*. Thus $A\div B$ signifies that the quantity A is to be divided by the quantity B; the same is also represented by placing the dividend above the divisor, thus $\frac{A}{B}$ is equivalent to the expression $A\div B$.

9. A power of a quantity is denoted by affixing to the quantity a small figure or letter indicating the degree of the power, the figure being a little elevated to distinguish it from a multiplier. Thus A^2 signifies the square of A, or that A is to be multiplied into itself; A^3 signifies the cube of A, or that the product of A into itself is to be multiplied by A; and A^n denotes that A is to be raised to the nth power, or multiplied into itself n number of times.

10. The sign $\surd$ indicates a root to be extracted; thus $\surd 2$ means the square root of 2; $\surd\overline{A\times B}$ or $\surd(A\times B)$ means the square root of the product of A and B.

11. The symbol $\therefore$ is used to denote the word *hence* or *therefore;* indicating a conclusion, depending on the foregoing premises.

AXIOMS.

1. Quantities which are equal to the same quantities, are equal to each other.
2. If equals be added to equals, the wholes will be equal.
3. If equals be taken from equals, the remainders will be equal.
4. If equals be added to unequals, the wholes will be unequal.
5. If equals be taken from unequals, the remainders will be unequal.
6. If equal quantities are multiplied by equal quantities, the products will be equal.
7. If equal quantities are divided by equal quantities, the quotients will be equal.
8. The whole of a quantity is equal to the sum of all its parts.
9. Things which coincide or fill the same space are identical or mutually equal in all their parts.

POSTULATES.

1. Let it be granted that magnitudes, as well as numbers, may be multiplied indefinitely.
2. Let it be granted that magnitudes, as well as numbers, may be divided indefinitely.
3. Let it be granted that one quantity or magnitude may be added to another quantity or magnitude. Also, that any number of quantities or magnitudes may be added to any number of other quantities or magnitudes, of the same kind.
4. Let it also be granted that any quantity or magnitude may be taken from any other greater quantity or magnitude, of the same kind.
5. And hence, that any quantity or magnitude may be increased till it is equal to a greater, or diminished till it is equal to a less.

ON THE RELATION OF MAGNITUDES TO NUMBERS, AND ON RATIOS AND PROPORTIONS.

Definitions and Explanation of Principles.

1. *Number* is the expression of an object, or an association of objects, indicating their individuality, or the magnitude of such association.

Magnitudes to be represented by numbers must be referred to some specific magnitude of the same kind as a unit; and this magnitude so represented by a unit, may be called a prime; and any magnitude may be expressed by the number of such primes contained in such magnitude.

Magnitudes so represented by numbers may be compared with each other, and hence results ratio.

2. *Ratio* is a mutual relation of two magnitudes of the same kind to each other in respect of quantity. Or it is the quotient arising from dividing one quantity by another quantity of the same kind. Thus, if A and B represent quantities of the same kind, the ratio of A to B is expressed by $\frac{A}{B}$. Or the ratio is sometimes expressed by placing a colon between the magnitudes compared, thus the ratio of A to B is $A : B$.

Of two magnitudes A and B, if A be divided into M number of units, each equal to A', then $A = M \times A'$; and if B be divided into N number of units, each equal to A', then $B = N \times A'$: M and N being integral numbers. The ratio of A to B will be the same as the ratio of $M \times A'$ to $N \times A'$; that is as the ratio of M to N, since A' is a common unit.

In the same manner the ratio of any other two magnitudes, C and D, may be expressed by $P \times C'$ to $Q \times C'$, P and Q being also integral numbers, and their ratio will be the same as that of P to Q.

3. The two magnitudes, when spoken of separately, are called the *terms* of the ratio; when spoken of together, they are called a couplet; the first term is called the antecedent and the last term the consequent.

4. A *compound ratio* is the ratio of the products of the corresponding terms of two or more simple ratios: thus the ratio of A to B is $\frac{B}{A}$, and the ratio of C to D is $\frac{D}{C}$; and the ratio compounded of these two ratios is $\frac{B}{A} \times \frac{D}{C}$ or $\frac{BD}{AC}$.

5. That class of compound ratios produced by multiplying a simple ratio into itself or into another equal ratio, that is, the square of a simple ratio, is called a *duplicate ratio*; that produced by multiplying three equal ratios together, that is, the cube of a simple ratio, is called a *triplicate ratio*, &c.; the ratio of the square roots of two magnitudes is called a *subduplicate ratio*; that of the cube roots, a *subtriplicate*, &c.

Thus the simple ratio of A to B is $A : B$.
The duplicate ratio is $A^2 : B^2$
The triplicate ratio is $A^3 : B^3$
The subduplicate ratio is $\sqrt{A} : \sqrt{B}$

The subtriplicate ratio is $\sqrt[3]{A} \,.\, \sqrt[3]{B}$

6. To multiply the antecedent by any quantity, divides the ratio by that quantity; and to divide the antecedent, multiplies the ratio.

Thus the ratio of A to B is $\frac{B}{A}$.

The ratio of $A \times 2 : B$ is $\frac{B}{2A}$.

And the ratio of $A \div 2 : B$ is $\frac{B}{\frac{1}{2}A}$.

And conversely, to multiply the consequent of a couplet by any quantity, multiplies the ratio, and to divide the consequent, divides the ratio.

7. To multiply or divide both the antecedent and consequent by the same quantity does not alter the ratio.

Thus the ratio of A to B is $\frac{B}{A}$.

The ratio of $A \times 2 : B \times 2$ is $\frac{2B}{2A} = \frac{B}{A}$.

The ratio of $A \div 2 : B \div 2$ is $\frac{\frac{1}{2}B}{\frac{1}{2}A} = \frac{B}{A}$.

8. If four magnitudes A, B, C, D, have such values that $\frac{B}{A}$ is equal to $\frac{D}{C}$, then A is said to have the same ratio to B that C has to D. When four quantities have this relation to each other, they are said to be in proportion.

Proportion, then, is an equality of ratios.

9. To indicate that the ratio of A to B is equal to the ratio of C to D, the quantities are usually written thus, A : B:: C : D, and read, A is to B as C to D; the quantities which are compared together are called the *terms of the proportion;* the first and last terms are called the *two extremes*, and the second and third terms the *two means.*

10. Of four proportional quantities the first and third are called the *antecedents*, and the second and fourth the *consequents;* and the last is said to be a *fourth proportional* to the other *three* taken in order. *Antecedents* are called homologous or like terms, and so of the *consequents.*

11. Magnitudes are said to be in *continued proportion* when every consequent is considered as the antecedent of the succeeding term; thus if A is to B as B to C and as C to D, those magnitudes are in *continued proportion;* and are written A : B::B : C::C : D, &c.

Several independent couplets having the same ratio are frequently expressed in the same way. Thus if A is to B as C to D and as E to F, they are written A : B::C : D::E : F.

12. Magnitudes are said to be in proportion by inversion, or inversely, when the consequents are taken as antecedents, and the antecedents as consequents.

13. Magnitudes are in proportion by alternation, or alternately, when antecedent is compared with antecedent, and consequent with consequent.

14. Magnitudes are in proportion by composition, when the sum of the antecedent and consequent is compared either with antecedent or consequent.

15. Magnitudes are said to be in proportion by division, when the difference of the antecedent and consequent is compared either with antecedent or consequent.

16. If the greater of two magnitudes contain the less a certain number of times without leaving a remainder, it is called a multiple of the less; and the less is, in this case, called a submultiple of the greater.

17. *Equimultiples* of two quantities are the products which arise from multiplying the quantities by the same number: thus $A\times m$, $B\times m$, are equimultiples of A and B, the common multiplyer being m.

18. *Equi-submultiples* or like submultiples are those which are contained in their respective multiples the same number of times.

19. Two quantities, A and B, are said to be reciprocally proportional, or inversely proportional, when one increases in the same ratio as the other diminishes. In such case, either of them is equal to a constant quantity divided by the other, and their product is constant.

PROPOSITION I. THEOREM.

If two magnitudes are expressed in terms of a unit of magnitude of the same kind, their sum or their difference may also be expressed in the same terms.

Let P be a unit of magnitude; and let A and B be two magnitudes, each of which may be measured by P, then will their sum or difference be also measured by P.

Suppose the quotient of $A\div P=Q$, and $B\div P=q$. Now let A be added to B, and it is evident that their sum is as much greater than either of them, as the number of units contained in their sum is greater than either; and either one will contain as many of the primes or units when associated as when single; thus A will contain as many P's while associated with B, as before; and B will contain as many P's while associated with A, as when single: hence the number of P's contained in their sum, is equal to the sum of their P's while single; and as Q is the number of P's in A, and q the number of P's in B, $Q+q$ is the number of P's in their sum, it being the value of $A+B$ expressed in terms of P.

Again, let $A-B$ be the difference of two magnitudes, and $Q-q$ the difference of their quotients when compared with P. Now $A-B$ will be as much smaller than A as the number of units in B: therefore $A-B$ will contain as many less of P's than A, as B contains units. Hence the P's contained in their difference are equal to the difference of the P's contained in the two magnitudes, viz. $Q-q$: therefore the difference of the two magnitudes is measured by P.

Cor. If P measure B and also $A-B$ or $A+B$, it must measure A; for the sum of B and $A-B$ is A, and the difference of B and $A+B$ is A.

PROPOSITION II. THEOREM.

If a magnitude is expressed in terms of the unit of a homologous magnitude, any multiple of such magnitude may also be expressed in like terms.

For if A represent the magnitude and Q the quotient arising from comparing this magnitude with the unit P, and if CA be a multiple of A; then because the magnitude A is repeated as many times as there are units in C, its measure when compared with P will be increased in the same ratio, viz. QC will be the measure of the multiple.

PROPOSITION III. THEOREM.

If a magnitude is expressed in terms of a unit of a homologous magnitude, any submultiple of such magnitude may also be expressed in terms of such unit.

If A represent the magnitude, and Q the quotient when such magnitude is compared with P; and if $\frac{A}{C}$ be a submultiple of A resulting from dividing A into as many parts, each equal to C, as there are units in C; then the number of times that P would be contained in $\frac{A}{C}$ would be to the number of times it is contained in A, as $\frac{A}{C}$ to A. Hence $A : \frac{A}{C} :: Q : \frac{Q}{C}$, and $\frac{Q}{C}$ would be the measure of the submultiple $\frac{A}{C}$ in terms of P.

PROPOSITION IV. THEOREM.

If two magnitudes be like multiples of two others, the sum or difference of the former multiples will be the same multiple of the sum or difference of the latter.

Let P, R be equimultiples of A, B; then will P±R be a like multiple of A±B.

For suppose P be divided into parts, each equal to A, and R to be divided into parts each equal to B; then by hypothesis, the number of the parts of P is the same as the number of the parts of R. Let one of the parts of P be added to or subtracted from one of the parts of R, the sum or difference will be equal to A±B; and if any other of the parts of P be added to or subtracted from any other of the like parts of R, the sum or difference will in like manner be equal to A±B: therefore as many magnitudes as there are in P equal to A, or as there are in R equal to B, so many are there in P±R equal to A±B.

Cor. Hence it may be shown that if two magnitudes are like submultiples of two others, the sum or difference of the former submultiples will be the same submultiple of the sum or difference of the latter.

Scholium. Having proved it of two magnitudes, it may be proved of three, and so on of any number.

PROPOSITION V. PROBLEM.

Two magnitudes of the same kind being given, to find their greatest common measure.

Let A and B be the proposed magnitudes, it is required to find the greatest magnitude that can measure them both.

Suppose A to be the greater of the two magnitudes, and let B be taken as often as possible from A, leaving a remainder C less than B; let C be in like manner taken as often as possible from B, leaving a remainder D less than C; now let D be in a similar way taken as often as possible from C, leaving a remainder E less than D, and so on till no remainder be left, which will be the case when the last measures the preceding one. The last remainder will be the greatest common measure sought.

For the magnitude sought, as it measures B, will measure any multiple thereof, and consequently, since it also measures A, it must measure the difference between any multiple of B and A (Prop. I.); but C is the difference between a multiple of B and A, therefore it measures C. Again, since it must measure a multiple of C, it must measure likewise the difference between a multiple of C and B (Prop. I.); it must therefore measure D; in a similar manner, by continuing this reasoning, is it to be shown that the required measure must also measure E, and so on as long as there are any remainders.

Now let us suppose that E is the last remainder; then E measures D, and therefore any multiple of it; and since the difference between C and a multiple of D is equal to E, E must measure C (Prop. I. Cor.), it must therefore measure a multiple of C, and since the difference between B and a multiple of C is equal to D, which E has already been shown to measure, E must measure B; it must therefore measure a multiple of B; and since the difference between A and a multiple of B is C, which E has been shown to measure, E must measure A; E therefore measures both A and B. Now it has been shown above that every common measure of A and B measures the last remainder, and we have just proved that the last remainder must measure A and B; consequently the last remainder is the *greatest* common measure of A and B.

Cor. Hence, if a last remainder can never be arrived at, that is, if the above process be interminable, the proposed magnitudes have not a common measure in simple finite terms; and hence they are called incommensurable.

Scholium. Since the decimals resulting from dividing one number by another, when the two numbers have no common measure, in integral numbers, are proper fractions, whose numerator is the decimal and whose denominator is unity with cyphers annexed, equal in number to the number of decimal figures, they may be changed into any equivalent fraction having any other denominator; and if any given magnitude is such as cannot be expressed definitely in integral and decimal numbers in consequence of the ratio of increase or decrease of those numbers, any other fractional number may be added to or subtracted from the numbers designed to express such magnitude; and since no limit exists to the extension and composition of fractions, and since they may be added or subtracted indefinitely, (Post. 3 and 4), any given number may by such means be increased or diminished till it will express any given magnitude. (Post. 5. See also notes).

But since a quantity requiring an expression in an indefinite number of terms, cannot be definitely expressed in visible and direct measures of known quantities, it may nevertheless be expressed in some ratio or function of some known quantity; by which may be understood some property or condition of numbers, resulting from their relations to each other according to some definite law. Thus, a multiple, a submultiple, a power, a root, and a ratio, definite or varying according to some definite law, are functions of quantities or magnitudes, and as such are definite in themselves, and determine any magnitude which they are designed to express.

PROPOSITION VI. THEOREM.

Every magnitude N, *into which a magnitude* M *may be divided, may be expressed in terms of a unit of a homologous magnitude, or some function of such unit.*

For, if it is denied, it must still be acknowledged that, however great the magnitude M may be, the unit P may be so multiplied as to represent a magnitude as great, (Post. 1.), or however small the magnitude M may be, P may be so divided as to be as small. (Post. 2.) Now to show that any magnitude N, being a part of the magnitude M, may be exactly expressed by numbers, or some function thereof, we have only to show that numbers are infinitely divisible, or that they are divisible to the same extent that magnitude is divisible; and although integral numbers and decimals express only such magnitudes as increase or decrease by unity or in tenfold ratios of unity, yet, as fractions may be indefinitely extended and compounded, between the limits of any two integral numbers or between any multiple or decimal of two integral numbers, it follows that no magnitude M, which is measured by the unit P, can be so divided, but each part of such magnitude may also be measured by P, or some function of P: for otherwise there must be a limit to the compounding and extension of fractions, and also to divisibility, which is absurd. And since every part into which a magnitude M may be divided is a definite magnitude, as well as the magnitude M, the parts of the magnitude so divided may each be expressed by numbers, or some function thereof, in terms of the unit P, which is definite also. (Prop. V. Scholium.)

Cor. Hence any magnitude whatever may be expressed either in direct terms, or some function of any other homologous magnitude. For since any magnitudes, M, N, may be expressed in some function of the unit P, and since (Def. 1 and 2), the value of M and N in terms of P is the ratio of P to M and P to N respectively; then $\frac{M}{P}$, $\frac{N}{P}$, are the values of M and N.

But suppose M cannot be definitely expressed by the unit P in simple terms, it may be expressed in terms of some function of P, which function we will represent by P′; then if N represent the magnitude N in terms of P, M will represent the magnitude M in terms of P′, and M′ will represent the same magnitude in terms of P.

Scholium. Although certain magnitudes may be integrally incommensurable (Prop. V. Cor.) with other given magnitudes, yet there is no magnitude but may be expressed by some function of any proposed magnitude; therefore, our geometrical investigations will not be at all embarrassed in considering the relations of such magnitudes; for they are, nevertheless, as definite in themselves as though they were susceptible of a finite measure in direct terms; for an incommensurable magnitude may be expressed in terms of a given magnitude by a series of ratios decreasing to infinity, according to a definite order, and hence is a definite magnitude. (See Prop. XXV. following).

PROPOSITION VII. THEOREM.

When four magnitudes or quantities are proportional, the product of the two extremes is equal to the product of the two means.

Let A, B, C, D be four magnitudes which are in proportion, and Q, q and Q′, q' their numerical representatives when compared with the unit P; then if $Q : q :: Q' : q'$, so will $Q\times q'=Q'\times q$. For since the quantities are in proportion, we have (*Def.* 8) $\frac{q}{Q}=\frac{q'}{Q'}$. Therefore $q=\frac{Q\times q'}{Q'}$ and $Q'\times q=Q\times q'$.

Scholium. As quantity represents number as well as magnitude, we shall more frequently use this term as being of more general application, since we have shown the relation of numbers to magnitude. And moreover, we shall use the former letters, A, B, C, &c., of the alphabet to denote magnitudes, and the latter letters, Q, R, S, &c., to denote the numerical representatives of magnitudes.

PROPOSITION VIII. THEOREM.

If three quantities are proportional, the product of the extremes will be equal to the square of the means.

Let A, B, C, be three magnitudes which are in proportion, and let them be represented by Q, q, and R. Then according to the conditions, $Q : q :: q : R$; and (Prop. VII.) $Q\times R=q\times q$, or $QR=q^2$.

PROPOSITION IX. THEOREM.

If the product of two quantities be equal to the product of two other quantities, two of them will be the extremes and the other two the means of a proportion.

Let $Q\times R=q\times r$; then will $Q : q :: r : R$. For if r have not to R the ratio of Q to q, let r have to $R+t$ the same ratio that Q has to q; that is, let $Q : q :: r : R+t$: then $Q\times(R+t)=q\times r$ (Prop. 7); hence $R+t=\frac{q\times r}{Q}$. But $R=\frac{q\times r}{Q}$; consequently $R+t=R$ and $t=0$. Hence the four quantities are proportional, that is $Q : q :: r : R$.

PROPOSITION X. THEOREM.

If four quantities are proportional, they will be proportional alternately or by permutation, or the antecedents will have the same ratio as the consequents.

Let P, Q, R, S, be the numerical representatives of four magnitudes in proportion such that $P : Q :: R : S$; then will $P : R :: Q : S$. For in the first proportion we have $P\times S=Q\times R$, and in the second we have $P\times S=R\times Q$, the result being the same in each case.

PROPOSITION XI. THEOREM.

If four quantities be proportional, they will be proportional when taken inversely.

Let $Q' : Q :: q' : q$, then will $Q : Q' :: q : q'$. For from the first proportion we have $Q'\times q=Q\times q'$, and in the second we have $Q\times q'=Q'\times q$, which two results are manifestly equal.

PROPOSITION XII. THEOREM.

If four quantities are proportional, they will be proportional by composition or division.

Let $P : Q :: R : S$; then will $P\pm Q : P :: R\pm S : R$. For from the first proportion we have $P\times S=Q\times R$ or $Q\times R=P\times S$. Add each of the members of the last equation to or subtract it from $P.R$ and we shall have $P.R\pm Q.R=P.R\pm P.S$; or $(P\pm Q)\times R=(R\pm S)\times P$. But $(P\pm Q)$ and R may be considered as the two extremes, and $R\pm S$ and P the two means of a proportion; hence $P\pm Q : P :: R\pm S : R$.

PROPOSITION XIII.

Equimultiples of any two quantities have the same ratio as the quantities themselves.

Let Q and q represent any two quantities; and let mQ and mq be any equimultiples of them; then $mQ : mq :: Q : q$. For $\frac{q}{Q}$ is the ratio of Q to q, and $\frac{mq}{mQ}=\frac{q}{Q}$ is the ratio of mQ to mq. The ratios are therefore equal.

Cor. 1. Hence, any equi-submultiple of two quantities have the same ratio as the quantities themselves; since Q and q may be considered as submultiples of mQ and mq.

Cor. 2. Hence also, if two quantities are increased or diminished by like parts of each, they will still have the same ratio.

PROPOSITION XIV. THEOREM.

If of four proportional quantities there be taken any equimultiples of the two antecedents, and any equimultiples of the two consequents, the four resulting quantities will be proportional.

Let P, Q, R, S, be the numerical representatives of four quantities in proportion; and let m and n be any assumed numbers; then will $m.P : n.Q :: m.R : n.S$.

For since in $P : Q :: R : S$ we have $P \times S = Q \times R$, and in $mP : nQ :: mR : nS$ we have $mP \times nS = nQ \times mR$, or $mnP.S = mnQ.R$, or $P \times S = Q \times R$; hence the results of each proportion are manifestly equal.

PROPOSITION XV. THEOREM.

If of four proportional quantities the two consequents be either augmented or diminished by the antecedents, or by quantities having the same ratio as the antecedents, the resulting quantities and the antecedents will be proportional.

If $P : Q :: R : S$, then will $P : Q \pm P :: R : S \pm R$. For the ratio of the first couplet of the last proportion is $\frac{Q \pm P}{P}$ or $\frac{Q}{P} \pm 1$. The ratio of the second couplet is $\frac{S \pm R}{R}$ or $\frac{S}{R} \pm 1$; and if we put the first ratio equal to the second, we have $\frac{Q}{P} \pm 1 = \frac{S}{R} \pm 1$, or $\frac{Q}{P} = \frac{S}{R}$; and resolving this equation into a proportion, we have $P : Q :: R : S$, which is the same as the former proportion.

Again, let P' and R' be two quantities having the same ratio as the two antecedents P and R; then will $P : Q\pm P' :: R : S\pm R'$. For the ratio of the first couplet is $\frac{Q\pm P'}{P}$ and the ratio of the second couplet is $\frac{S\pm R'}{R}$, which ratios are manifestly equal, since R' by hypothesis has the same ratio to R as P' has to P.

Cor. 1. Hence, if the consequents are either augmented or diminished by any equimultiple or any like submultiple of the antecedents, the quantities will still be proportional.

Cor. 2. And conversely, if the antecedents are augmented or diminished in like manner by any like multiple or like submultiple of the consequents, the quantities will be proportional.

Cor. 3. The difference between an antecedent and its consequent, is to either term, as the difference between the other antecedent and consequent, to the like term.

PROPOSITION XVI.

If any number of quantities are proportional, then any one of the antecedents will be to its consequent as the sum of all the antecedents to the sum of all the consequents.

Let $M : N :: P : Q :: R : S$, &c.; then will $M : N :: (M+P+R) : (N+Q+S)$. For since $M : N :: P : Q$, we have $M\times Q = N\times P$. And since $M : N :: R : S$, we have $M\times S = N\times R$. Add $M\times N = M\times N$, and we have $M.N+M.Q+M.S = M.N+N.P+N.R$, or $M\times(N+Q+S) = N\times(M+P+R)$. Therefore $M : N :: (M+P+R) : (N+Q+S)$.

PROPOSITION XVII. THEOREM.

If in a proportion the antecedent is greater in one couplet than that in the other, the consequent in the former will exceed the consequent of the latter.

Let $P : Q :: R : S$ be the proportion, and let $P > R$; then will $Q > S$. For by inversion (Prop. XI.) $Q : P :: S : R$; and whatever be the difference between P and R, it is possible for a multiple thereof to exceed S, and consequently such a multiple of P must contain R oftener than an equimultiple of R. Let mP and mR be those equimultiples of P and R.

$$\begin{array}{c} Q : P :: S : R \\ mP \quad\quad mR \end{array}$$

By Prop. XIII, the ratio of Q to mP is the same as that of S to mR; consequently mP does not contain Q oftener than mR contains S; but mP does contain S oftener than mR

contains it: consequently mP contains S oftener than it contains Q. Therefore $Q > S$.

Cor. 1. It follows, by inversion, that, if one consequent be greater than another, the antecedent of the former will be greater than that of the latter.

Cor. 2. Consequently, if the antecedents be equal, the consequents will be equal, and conversely.

Cor. 3. Hence, if in two proportions there be three corresponding terms in each respectively equal, the fourth terms will be equal.

PROPOSITION XVIII. THEOREM.

Of four proportional quantities, the sum of the greatest and least terms exceeds the sum of the other two.

Let the quantities $P : Q :: R : S$, and let P be the greatest term; then S will be the least (Prop. XVII.), and $(P+S) > (Q+R)$.

For (Prop. XII.) $P : P-Q :: R : R-S$, and alternately $P : R :: P-Q : R-S$; and P being greater than R, $P-Q$ is greater than $R-S$, (Prop. XVII.): therefore if $Q+S$ be added to each, there will result $P+S > Q+R$.

But if Q be the largest term, then by inversion $Q : P :: S : R$; so R is the least term: and by reasoning as above, there will result $(Q+R) > (P+S)$.

Cor. In the proportion $P : Q :: Q : R$, $(P+R) < 2Q$; so that the mean term of three proportional numbers is less than half the sum of the extremes.

PROPOSITION XIX. THEOREM.

If there be four proportional quantities and four other proportional quantities, having the antecedents the same in both, the consequents will be proportional.

Let $P : Q :: R : S$, and $P : M :: R : N$;

Then will $Q : S :: M : N$.

For by alternation we have from the first proportion

$$P : R :: Q : S, \text{ or } \frac{R}{P} = \frac{S}{Q},$$

and from the second we have

$$P : R :: M : N, \text{ or } \frac{R}{P} = \frac{N}{M}.$$

Whence (Ax. 1.), $\frac{S}{Q} = \frac{N}{M}$;

by resolving the equation into a proportion, we have

$$Q : S :: M : N.$$

Cor. If there be two sets of proportionals, having an antecedent and consequent of the first equal to an antecedent and consequent of the second, the remaining terms will be proportional.

PROPOSITION XX. THEOREM.

If there be two series of magnitudes, such that the first term is to the second in the first series as the first term to the second in the other series, and the second term to the third in the former as the second to the third in the latter, and so on; then as the first term is to the last in the one series, so is the first to the last in the other series.

First, let there be three magnitudes in each series, viz: M, N, O, in one, and P, Q, R, in the other, furnishing the two proportions $M : N :: P : Q$ and $N : O :: Q : R$; then also $M : O :: P : R$.

For take mM and mP, any equimultiples of M and P; also take nO and nR, any equimultiples of O and R:

$$M,\ O,\ P,\ R,$$
$$mM,\ nO,\ mP,\ nR.$$

then it is to be proved that if $mM>nO$, we must have $mP>nR$; and if $mP>nR$, then also $mM>nO$. (Prop. XIII.)

Let us suppose $mM>nO$; since mM and mP are equimultiples of M and P, and nO and nR equimultiples of O R, it follows (Prop. XIII.) $mM : N :: mP : Q$, and $nO : N :: nR : Q$.

Let N' be a submultiple of N less than $mM-nO$, and take Q' an equi-submultiple of Q; then N' must be contained oftener in mM than in nO. Now (Prop. XIII.) and by inversion $N' : mM :: Q' : mP$; and $N' : nO :: Q' : nR$, therefore Q' is contained as often in mP as N' is contained in mM; and consequently Q' is contained in mP oftener than N' is contained in nO: but N' is contained in nO as often as Q' is contained in nR (Def. 8.); therefore Q' is contained in mP oftener than Q' is contained in nR; consequently $mP>nR$. In a similar manner, if we had supposed $mP>nR$, it would have resulted that $mM>nO$; consequently (Prop. XIII.) $M : O :: P : R$.

Now let there be four magnitudes in each series represented by M, N, O, T, and P, Q, R, S, furnishing the additional proportion $O : T :: R : S$. Then as shown above $M : O :: P$ R, and since $O : T :: R : S$, it follows by the preceding case that $M : T :: P : S$, and so on for any number of magnitudes.

Cor. 1. If the consequents of one proportion be the ante-

cedents in another, a third proportion may be found with the same antecedents as the first proportion and the same consequents as the second.

Cor. 2. Also if the antecedents in two proportions be the same, the consequents of one are as the consequents of the other, each to each; or if the consequents be the same, then the antecedents of the one are as those of the other, each to each. This immediately follows from the last corollary by inverting the terms of the proportions.

PROPOSITION XXI. THEOREM.

If the antecedents in one proportion be the same as those in another, then the first antecedent is to the sum or difference of the first consequents, as the second antecedent is to the sum or difference of the second consequents.

Let the proportions be $P : Q :: R : S$, and $P : M :: R : N$, then will $P : Q \pm M :: R : S \pm N$.

For by (Prop. XIX.) $Q : M :: S : N$, therefore (Prop. XV.) $Q : Q \pm M :: S : S \pm N$; but from the first proportion $Q : P :: S : R$; hence (Prop. XV.) $P : Q \pm M :: R : S \pm N$.

Cor. 1. It is likewise obvious that this result, combined with the first of the proposed proportions, gives (Cor. 2. last Prop.) $Q : Q \pm M :: S : S \pm N$.

Cor. 2. If the terms of both proportions are all homologous, then, by alternation (Prop. X.), $P : R :: Q \pm M : S \pm N$, and $Q : S :: Q \pm M : S \pm N$.

Scholium. It is obvious that the reasoning might be extended to three proportions, then to four, and so on to any number.

PROPOSITION XXII. THEOREM.

If there be two sets of proportional quantities, the products of the corresponding terms will be proportional.

Let $M : N :: O : P$ be one set of proportional quantities, and $Q : R :: S : T$ another set of proportional quantities, then will $M.Q : N.R :: O.S : P.T$.

For since $M \times P = N \times O$, and $Q \times T = R \times S$, we shall have $M.P.Q.T = N.O.R.S$, or $M.Q.P.T = N.R.O.S$. Therefore, (Prop IX.) $M.Q : N.R :: O.S : PT$.

PROPOSITION XXIII. THEOREM.

If any quantities are proportional, their squares, or cubes, or any like powers or roots, will also be proportional.

Let $P : Q :: R : S$, then will $P^2 : Q^2 :: R^2 : S^2$; whence from the first proportion we have $P\times S=Q\times R$, or $PS=QR$; and from the second $P^2\times S^2=Q^2\times R^2$, or $P^2S^2=Q^2R^2$.

Now, if we take the first equation, viz: $PS=QR$, and square both sides, we have $PS\times PS=QR\times QR$, or $P^2S^2=Q^2R^2$, which result is the same as that obtained from the second proportion; and this equation may be resolved into the proportion $P^2 : Q^2 :: R^2 : S^2$. And the same result may be obtained, if we take the ^{n}th power of the quantities in the proportion. Thus, $P^n : Q^n :: R^n : S^n$.

Again, if we extract the roots of the quantities, those roots will also be proportional. Thus $\sqrt{P} : \sqrt{Q} :: \sqrt{R} : \sqrt{S}$, and $\sqrt[n]{P} : \sqrt[n]{Q} :: \sqrt[n]{R} : \sqrt[n]{S}$. For since $P : Q :: R : S$, therefore $P\times S=Q\times R$; hence, taking like roots of each, $\sqrt{P\times S}=\sqrt{Q\times R}$, and $\sqrt[n]{P\times S}=\sqrt[n]{Q\times R}$; consequently, (Prop. IX.) $\sqrt{P} : \sqrt{Q} :: \sqrt{R} : \sqrt{S}$, and $\sqrt[n]{P} : \sqrt[n]{Q} :: R\sqrt[n]{\ } : \sqrt[n]{S}$.

Scholium. Although the like powers, or like roots, of proportional quantities are themselves proportional, it should be observed that they are not proportional in the same ratio as their prime numbers.

For in the numerical proportion $2 : 4 :: 3 : 6$, $2\times 6=4\times 2$, and $2^2 : 4^2 :: 3^2 : 6^2$; hence, $2^2\times 6^2=4^2\times 3^2$, or $4\times 36=16\times 9$; and $4 : 16 :: 9 : 36$, which ratio is evidently different from that of the prime numbers, viz: $2 : 4 :: 3 : 6$.

PROPOSITION XXIV.

If any number of quantities be continued proportionals, the ratio of the first to the third will be duplicate or the square of the ratio of the first to the second; and the ratio of the first to the fourth will be triplicate or the cube of that of the first and second; and so on.

Let q, mq, m^2q, m^3q, &c., be continued proportionals, then $\frac{mq}{q}=m=$the ratio of the first to the second, and $\frac{m^2q}{q}=m^2=$ the ratio of the first to the third, and $\frac{m^3q}{q}=m^3=$the ratio of the first to the fourth.

Scholium. These ratios, though inverse, give results similar to direct ratios, but in inverse order; since, by reversing the order of the proportion, they would be direct.

PROPOSITION XXV.

If one magnitude contain another, and leave a remainder such that the greater of the two magnitudes is to the small-

er, as the smaller is to this remainder, then the two magnitudes will be incommensurable.

Let A, the greater of two magnitudes, contain the smaller, B, any number of times, leaving for a remainder a magnitude C, such that A : B::B : C; then A and B have not a common measure in direct terms.

For let C, D, E, &c., be the successive remainders in the process for finding the common measure, (Prop. V.); then C cannot measure B, otherwise B would measure A (Prop. X.), and there could be no remainder; but C is contained as often in B as B is contained in A, (Def. 8.) Let then P be the greatest multiple of B which is contained in A, and let Q be an equimultiple of C, which must be the greatest contained in B; then (Prop. XIV.) A : P::B : Q, and (Props. X. and XV.) A : B::A—P : B—Q; but A—P=C, and B—Q=D, by hypothesis: therefore A : B::C : D, but B : C::C : D; hence D cannot measure C, for if it could, C would measure B. Let now P′ be the greatest multiple of C in B, and Q′ the like multiple of D; then pursuing the same course as before, there results the proportion C : D::D : E, so that E cannot measure D; and so on for each succeeding remainder. It appears, therefore, that no remainder can ever measure the preceding one, consequently the process for finding the common measure of A and B will be interminable, and therefore (Prop. V. Cor.) these two magnitudes are incommensurable.

Cor. Since each of the successive remainders have the same ratio to the magnitudes from which they are immediately derived, and since this ratio is constantly decreasing in relation to the former magnitudes, and in a continued proportion, it follows that the two magnitudes can only be expressed by a series of ratios in continued proportion, decreasing to infinity, according to a definite order, which series, being regarded as the function of one of the given magnitudes, expresses the value of the other.

BOOK II.

OF LINES AND THEIR PROPERTIES.

Definitions.

1. *Extension* is the principal subject for geometrical investigation: its existence may be exhibited under one, two, or three dimensions; length, breadth, and height or thickness; and hence results magnitude.

2. *Magnitude*, being the result of extension, must necessarily occupy space; therefore all magnitudes possess locality.

3. *Locality*, or *position*, is a certain condition of an object in relation to space and in reference to other objects; and is a quality by which a given object may be distinguished from others.

4. *Locality* may exist without extension, but extension cannot exist without locality.

5. A *point* is locality without extension.

6. If locality is extended in one dimension only, the product is a line. A *line*, therefore, has length without breadth or thickness.

Hence, the motion of a point generates a line; thus if the point A be made to move through the distance from A to B, it will generate the line AB. A.————————B.

Hence, the *extremities* of a line are points, and the intersections of lines with each other are also points.

7. Lines are either *right*, or *curved*, or *mixed* of these two.

8. A *right line*, or *straight line*, lies all in the same direction between its extreme points, and if traced by a point, the point moves always in the same direction. Hence, also, a right line is the shortest distance between two points.

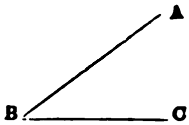
Thus the line AB is the shortest distance between the two points forming the extremities of the line.

A————————B

9. A *curve* line is such that if traced by a point, the point continually changes its direction, as the line CD.

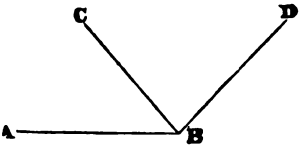
10. The word line, without qualification, is used to designate a right line; and the word *curve*, a *curve* line.

11. If two lines meet each other in a point, the inclination or opening between them is called an angle. Thus if the line AB meets the line CB in the point B, the inclination of these lines to each other determines the angle at B.



An angle is designated by a letter at the vertex or point where the lines meet, as B; but if two or more angles have the same vertex, then each angle is designated by naming the sides about the angle; naming the letter at the vertex always in the middle.

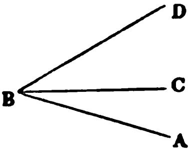
Thus the angle CBA, or ABC, designates the angle formed by the intersection of the lines CB and AB at the vertex B; and by the angle DBC or CBD, the angle formed by the meeting of the lines CB and DB is in like manner designated.



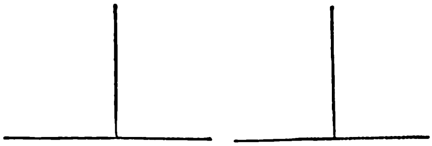
Angles are greater or less, as the lines forming the angles are more or less inclined or opened, and like other magnitudes or quantities are susceptible of addition, subtraction, multiplication and division.

Thus the angle ABD is the sum of the two angles ABC and CBD, and the angle CBD is the difference of the two angles ABD and ABC.

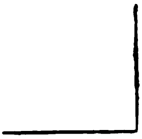
Hence the quantity of an angle depends not upon the length of the lines forming that angle, but altogether upon their relative position or inclination.



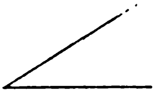
13. One right line is said to be *perpendicular* to another, when it makes with it equal adjacent angles A perpendicular at the extremity of a line is that which makes an angle with it equal to the adjacent angle which would be formed by producing the line beyond that extremity.

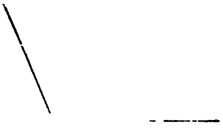


13. A *right angle* is the angle formed by a right line and a perpendicular to it.



14. An *acute angle* is less than a right angle.



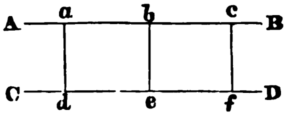


15. An *obtuse angle* is greater than a right angle.

16. By the distance of a point from a right line is meant the perpendicular from that point to the line; and one line is said to be *equidistant* from another, when every point in the one is equidistant from the other.

17. Right lines are *parallel*, when they are at the same perpendicular distance from each other in all their parts.

Thus, if the two right lines AB and CD are so posited, that perpendiculars, *ad*, *be*, *cf*, &c., from the several points *a*, *b*, *c*, &c., in one line, drawn to the other, are equal each to each, wherever posited, the lines AB and CD are parallel.

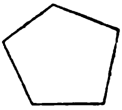


18. A *surface* is that which has two dimensions of extension, viz: length and breadth without thickness.

19. A *plane* is a surface in which if two points be assumed at pleasure and connected by a right line, that line will lie wholly in the surface.

20. Every surface which is not a plane surface, or composed of plane surfaces, is a *curved surface*.

21. A *plane figure* is an enclosed plane surface.



22. If it be bounded by right lines only it is called a *rectilineal figure*.

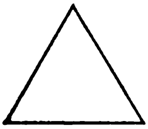
23. A *polygon* is a name used to comprehend every species of rectilineal figures, without regard to the number of sides; but figures are more particularly distinguished as follows, viz:

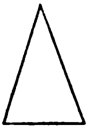
A figure of three sides is called a *triangle*.
The *quadrilateral* has four sides.
The *pentagon* has five sides.
The *hexagon* has six sides.
The *heptagon* has seven sides.
The *octagon* has eight sides; and so on.

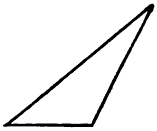
24. All the sides of the polygon taken together form the *contour* or *perimeter* of the polygon.

25. Among triangles we may distinguish

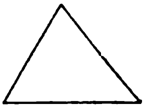
The *equilateral triangle*, which has its three sides equal;
The *isosceles triangle* which has two of its sides equal;
The *scalene triangle*, which has its three sides unequal.



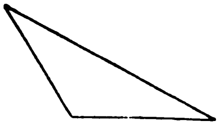




The *acute angled triangle*, which has three acute angles.

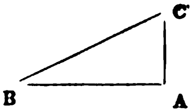


The *obtuse angled triangle* which has one obtuse angle.



26. A *right angled triangle* is one which has a right angle.

27. In a right angled triangle, the side opposite the right angle is called the *hypothenuse*. If, for example, the angle A is right, the side BC is the hypothenuse.

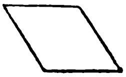


Any side of a triangle may be considered as its *base*, but it is usual in the case of the isoceles triangle to confine this term to that side that is not equal to either of the others.

28. A *rhomboid* or *parallelogram* is a quadrilateral whose opposite sides are parallel.

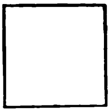
29. If only two of the opposite sides are parallel, the figure is a *trapezium*.

30. A *rhombus* is a rhomboid, the adjacent sides of which are equal.



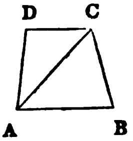
31. A *rectangle* is a right angled rhomboid.

32. A *square* is a right angled rhombus.



33. The right line which joins the vertices of two opposite angles of a quadrilateral is called a *diagonal*.

Thus the line AC joining the vertices of the opposite angles DAB, DCB, of the quadrilateral ABCD, is a diagonal.



34. Plane figures are *equal*, when, by supposing them to be applied to each other, they would coincide throughout; and they are said to be *equivalent*, when they enclose equal portions of space, and are at the same time incapable of such coincidence.

35. An *equilateral polygon* is one which has all its sides equal. An *equiangular polygon* is one which has all its angles equal.

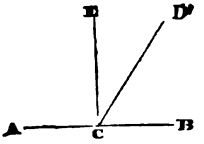
36. Two polygons are *mutually equilateral*, when they have their sides equal each to each and placed in the same order, that is, when following their perimeters in the same direction, the first side of one is equal to the first side of the other, the second of one to the second of the other, the third to the third, and so on. The phrase, *mutually equiangular*, has a corresponding signification with respect to the angles. In both cases, the equal sides, or the equal angles, are called *homologous sides* or *angles*.

POSTULATES.

1. Grant that a right line may be drawn from one point to another.
2. And that it may be either increased till it is equal to a greater right line, or diminished till it be equal to a less.
3. Grant also, that an angle may be increased till it is equal to a greater angle, or diminished till it is equal to a less.
4. And lastly, that from a point either within or without a right line a perpendicular thereto may be drawn.

PROPOSITION I. THEOREM.

From the same point in a given right line, more than one perpendicular thereto cannot be drawn.



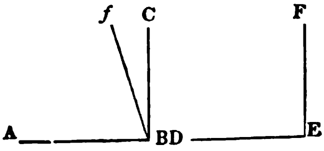
Let CE be perpendicular to the right line AC or AB, CB being the production of AC; and if the proposition be denied, let some other line, as CD, drawn from the same point C, be also perpendicular to AC.

Then because the angles ACE and BCE are equal (Def. 12.), the angle ACE must be greater than the angle DCB, since the sum of the two angles DCB and DCE is only equal to the angle BCE or ACE. But CD is perpendicular to AB by hypothesis, therefore the angle DCB must be equal to the angle BCE or ACE (Def. 12.), which is a manifest absurdity; therefore CD cannot be perpendicular to AC or AB.

PROPOSITION II. THEOREM.

All right angles are equal to each other.

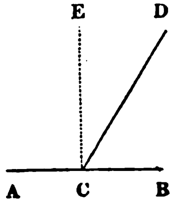
Let ABC be a right angle, and DEF any other right angle, then, if it be denied that these two angles are equal to each other, one of them, as ABC, must be supposed greater than the other, so that DEF must be equal to some portion of ABC.



Let AB*f* represent that portion; then, because AB*f* is a right angle, B*f* is perpendicular to AB (Def. 12.); but ABC is also a right angle, therefore BC is likewise perpendicular to AB, that is, from the same point B in the right line AB, two perpendiculars thereto are drawn, which is impossible. (Prop. I.) Therefore ABC cannot be greater than DEF, and in a similar manner it may be proved that DEF cannot be greater than ABC; the two angles are therefore equal.

PROPOSITION III. THEOREM.

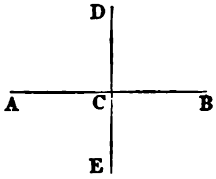
The adjacent angles which one right line makes with another which it meets, are together equal to two right angles.



Let AB and CD be the right lines, meeting each other at C, then will the angle ACD+the angle DCB, be equal to two right angles.

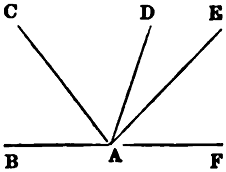
At the point C, erect CE perpendicular to AB. The angle ACD is the sum of the angles ACE, ECD; therefore ACD+BCD is the sum of the three angles ACE, ECD, BCD: but the first of those three angles is a right angle; and the other two together make up the right angle BCE; hence the sum of the two angles ACD and BCD is equal to two right angles.

Cor. 1. If one of the angles ACD, BCD, is right, the other is right also.



Cor. 2. If the line DE is perpendicular to AB, reciprocally AB will be perpendicular to DE.

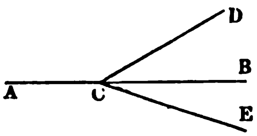
For, since DE is perpendicular to AB, the angle ACD must be equal to its adjacent one DCB, and both of them must be right, (Def. 13.) But since ACD is a right angle, its adjacent one ACE must also be right, (Cor. 1.); hence the angle ACE=ACD, (Prop. II.); therefore AB is perpendicular to DE.



Cor. 3. The sum of all the successive angles, BAC, CAD, DAE, EAF, formed on the same side of a straight line BF, is equal to two right angles; because their sum is equal to that of the two adjacent angles, BAC, CAF.

PROPOSITION IV. THEOREM. (*Converse of Prop. III.*)

If a right line meet two other right lines at a common point, making the sum of the two adjacent angles equal to two right angles, the two right lines which are met will form one and the same right line.

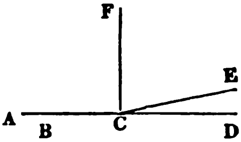


Let the right line CD meet the two lines AC, CB, at their common point C, making the sum of the two adjacent angles DCA, DCB, equal to two right angles; then will CB be the prolongation of AC, or AC and CB will form one and the same right line.

For, if CB is not the prolongation of AC, let CE be that prolongation: then the line ACE being straight, the sum of the angles ACD, DCE, will be equal to two right angles (Prop. III.) But by hypothesis, the sum of the angles ACD, DCB, is also equal to two right angles: therefore, ACD+DCB must be equal to ACD+DCE; and taking away the angle ACD from each, there remains the angle DCE equal to the angle DCB, which can only be the case when the lines CE and CB coincide: hence, AC, CB, form one and the same right line.

PROPOSITION V. THEOREM.

Two right lines, having two points common to both, form but one continued right line.

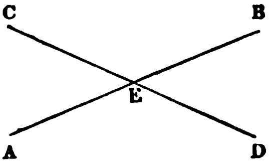


Let A, B, be the points through which two right lines pass, then they must necessarily coincide between A and B (Def. 8.); but if they do not coincide throughout, let ACD be the direction of one, and ACE that of the other; and at the point C, where they separate, let there be CF perpendicular to ACD.

Then, because CF is perpendicular to the right line ACD, FCD is a right angle (Def. 12.); and since by hypothesis ACE is a right line, and FCA a right angle, the angle FCE is also a right angle (Prop. III. Cor. 1.), but all right angles are equal to each other (Prop. II.); therefore the whole angle FCD is equal to the part FCE, which is absurd.

PROPOSITION VI. THEOREM.

If two right lines intersect each other, the opposite angles formed at their intersection will be equal.

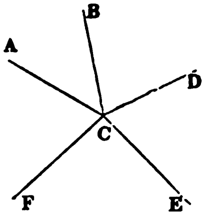


If the two right lines AB, CD, intersect at E, the opposite angles, CEB, AED, will be equal.

For the sum of the angles CEA, CEB, is equal to two right angles (Prop. III.) Also the sum of the angles CEA, AED, is equal to two right angles; that is, the sum of the angles CEA, CEB, is equal to the sum of the angles CEA, AED; and taking away from each of these equal sums the common angle CEA, the remaining angles CEB, AED must be equal.

In a similar manner it might have been shown that the opposite angles CEA, DEB, are equal.

Scholium. The four angles formed about a point by two straight lines, which intersect each other, are together equal to four right angles; for the sum of the two angles AEC, CEB, is equal to two right angles; and the sum of the other two, AED, DEB, is also equal to two right angles: therefore the sum of the four is equal to four right angles.



In general, if any number of straight lines CA, CB, CD, &c. meet in a point C, the sum of all the successive angles ACB, BCD, DCE, ECF, FCA, will be equal to four right angles; for, if four right angles were formed about the point C, by two lines perpendicular to each other, the same space would be occupied by the four right angles, as the successive angles ACB, BCD, DCE, ECF, FCA.

PROPOSITION VII. THEOREM. (*Converse of Prop. VI.*)

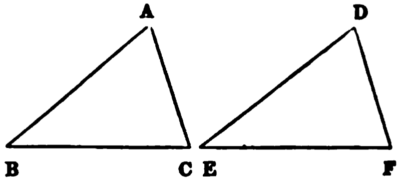
If the opposite angles formed by four right lines meeting in a point are equal, those lines shall form but two right lines.

Let the four right lines AE, BE, CE, DE, meet in the point E (see the diagram to last proposition), so that the opposite angles CEB, AED, may be equal; and also the other opposite angles CEA, DEB; then AEB and CED shall be right lines.

For, since the sum of the angles CEA, CEB, is by hypothesis equal to the sum of the angles DEB, DEA; and the sum of all four is, by the corollary to last proposition, equal to four right angles; it follows that each of the above sums must be equal to two right angles, so that the right line CE makes with the two AE, BE, adjacent angles, which are together equal to two right angles; therefore AEB is a straight line (Prop. IV.) In a similar manner it may evidently be proved that CED is a right line; hence the four lines form but two distinct right lines.

PROPOSITION VIII. THEOREM.

If two sides and the included angle in one triangle be equal to two sides and the included angle in another triangle, those triangles shall be equal.



Let the triangles ABC, DEF, have the sides AB, AC, and the included angle A in the one, equal to the two sides DE, DF, and the included angle D in the other; then shall the angle B be equal to the angle E; the angle C equal to the angle F, and the side BC equal to the side EF.

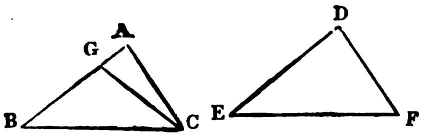
For, since AB is equal to DE, and the angle A equal to the angle D, a triangle equal to DEF may be conceived to be formed, of which AB shall be one side, and A an angle; now it is obvious that the side of this triangle which corresponds to DF, must fall upon AC, otherwise the angles A and D would be unequal; nor can this side extend beyond or fall short of the point C, for DF is equal to AC. The two extremities, therefore, of the base would coincide with the points B, C; the two bases, therefore, would coincide throughout (Prop. V.), so that the triangle so formed would entirely coincide with the triangle ABC, and it is at the same time equal to the triangle DEF; hence the triangles ABC, DEF, are equal.

Cor. 1. If a perpendicular from one of the angles of a triangle to the opposite side bisect that side, it shall also bisect the angle, and the sides containing that angle shall be equal; that is, the triangle will be either isosceles or equilateral.

Cor. 2. When two triangles have these three things equal, namely, the side ED=BA, the side DF=AC, and the angle D=A, the remaining three are also respectively equal, viz: the side EF=BC, the angle E=B, and the angle F=C.

PROPOSITION IX. THEOREM.

Two triangles are equal, if two angles and the interjacent side of one is equal to two angles and the interjacent side in the other.



Let the triangles ABC, DEF, have the two angles ABC, ACB, and the interjacent side BC, in the one, equal respectively to the two angles, E, F, and interjacent side EF in the other, the two triangles will be equal.

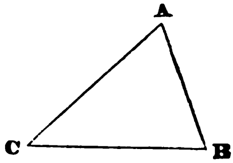
In proof of this it will be necessary only to show that the side AB must be equal to the side DE (Prop. VIII.) If this equality be denied, let one of these sides, as AB, be supposed longer than the other, and let BG be equal to ED. Join GC, then, since GB, BC, and the included angle B are respectively equal to DE, EF, and the included angle E, the angle BCG must be equal to the angle F (Prop. VIII.); but by hypothesis, the angle F is equal to the angle ACB; hence, then the angle GCB is equal to the angle ACB, a part to the whole, which is absurd; therefore, AB cannot be longer than DE, and in like manner it may be shown that DE cannot be longer than AB; AB is therefore equal to DE, and consequently the triangle ABC is equal to the triangle DEF.

Cor. 1. From this proposition immediately follows the converse of the corollary to Prop. VIII., viz: if a perpendicular from one of the angles of a triangle to the opposite side bisect this angle, it shall also bisect the side on which it falls, so that the sides including the proposed angle must be equal. (Prop. VIII. Cor. 1.)

Cor. 2. Whenever, in two triangles, these three things are equal, namely, the angle E=B, the angle F=C, and the included side EF equal to the included side BC, it may be inferred that the remaining three are also respectively equal, namely, the angle D=A, the side ED=BA, and the side DF=AC.

PROPOSITION X. THEOREM.

The sum of any two sides of a triangle is greater than the third side.

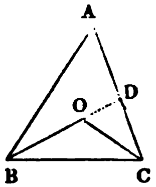


Let ABC be a triangle: then will the sum of two of its sides, as AC, CB, be greater than the third side AB.

For the straight line AB is the shortest distance between the points A and B (Def. 8.); hence AC+CB is greater than AB.

PROPOSITION XI. THEOREM.

If from any point within a triangle, two right lines be drawn to the extremities of either side, their sum will be less than the sum of the other two sides of the triangle.



Let any point, as O, be taken within the triangle BAC, and let the lines OB, OC, be drawn to the extremities of either side, as BC; then will OB+OC<BA+AC.

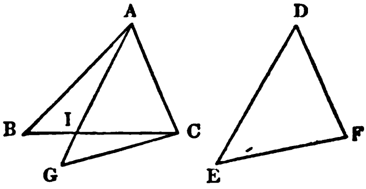
Let BO be produced till it meets the side AC in D: then the line OC is shorter than OD+DC (Prop. X.): add BO to each, and we have BO+OC<BO+OD+DC (Ax. 4.), or BO+OC<BD+DC.

Again, BD<BA+AD: add DC to each, and we have BD

$+DC<BA+AC$. But it has just been found that $BO+OC<BD+DC$; therefore, still more is $BO+OC<BA+AC$.

PROPOSITION XII. THEOREM.

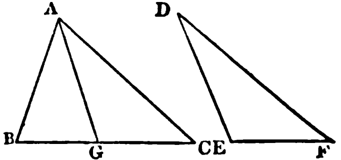
If two sides of one triangle be respectively equal to two sides of another, but include a greater angle; the third side of the former shall exceed the third side of the latter.



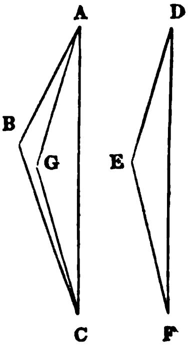
Let BAC and EDF be two triangles, having the side $AB=DE$, $AC=DF$, and the angle $A>D$; then will $BC>EF$.

Make the angle $CAG=D$; take $AG=DE$, and draw CG. The triangle GAC is equal to DEF, since, by construction, they have an equal angle in each, contained by equal sides, (Prop. VIII.); therefore CG is equal to EF. Now, there may be three cases in the proposition, according as the point G falls without the triangle ABC, or upon its base BC, or within it.

First Case. The straight line $GC<GI+IC$, and the straight line $AB<AI+IB$; therefore, $GC+AB<GI+AI+IC+IB$, or, which is the same thing, $GC+AB<AG+BC$. Take away AB from the one side, and its equal AG from the other; and there remains $GC<BC$ (Ax. 5.); but we have found $GC=EF$, therefore $BC>EF$.



Second Case. If the point G fall on the side BC, it is evident that GC, or its equal EF, will be shorter than BC, since $BC=BG+GC$.



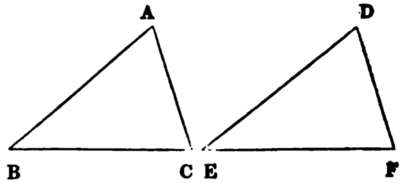
Third Case. Lastly, if the point G fall within the triangle BAC, we shall have, by the preceding theorem, $AG+GC<AB+BC$; and taking AG from the one, and its equal AB from the other, there will remain $GC<BC$, or $BC<EF$.

Scholium. Conversely, if two sides BA, AC, of the triangle BAC, are equal to the two ED, DF, of the triangle EDF, each to each, while the third side BC of the first triangle is greater than the third side EF of the second; then will the angle BAC of the first triangle be greater than the angle EDF of the second.

For, if not, the angle BAC must be equal to EDF, or less than it. In the first case, the side BC would be equal to EF, (Prop. V.): in the second, CB would be less than EF; but either of these results contradicts the hypothesis; therefore, BAC is greater than EDF.

PROPOSITION XIII. THEOREM.

Two triangles are equal, which have the three sides of the one, respectively equal to the three sides of the other.



Let the side ED=BA, the side EF=BC, and the side DF=AC; then will the angle D=A, the angle E=B, and the angle F=C.

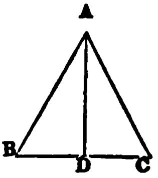
For, if the angle D were greater than A, while the sides ED, DF, were equal to BA, AC, each to each, it would follow by the last proposition, that the side EF must be greater than BC; and if the angle D were less than A, it would follow, that the side EF must be less than BC: but EF is equal to BC, by hypothesis: therefore, the angle D can neither be greater nor less than A: hence it must be equal to it. In the same manner it may be shown that the angle E is equal to B, and the angle F to C; hence the two triangles are equal (Def. 34.)

Scholium. It may be observed that the equal angles lie opposite the equal sides: thus, the equal angles D and A, lie opposite the equal sides EF and BC.

PROPOSITION XIV. THEOREM.

The angles opposite the equal sides of an isosceles triangle are equal.

Let the sides AB, AC, of the triangle ABC be equal, then will the angle C be equal to the angle B.



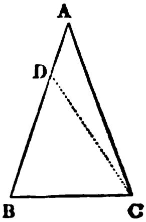
For, let AD be the line bisecting the angle A: then, in the two triangles ABD, ACD, two sides, AB, AD, and the included angle in the one, are equal to the two sides AC, AD, and the included angle in the other; hence the angle B is equal to the angle C (Prop. VIII. Cor. 2.)

Cor. 1. It also follows (Prop. VIII. Cor. 2.) that BD is equal to CD, and that the angle ADB is equal to the angle ADC; therefore, the line bisecting the vertical angle of an isosceles triangle bisects the base at right angles; and conversely, the line bisecting the base of an isosceles triangle at right angles, bisects also the vertical angle.

Cor. 2. Every equilateral triangle is also equiangular.

PROPOSITION XV. THEOREM.

Conversely, if two angles of a triangle are equal, the opposite sides are equal, and the triangle is isiosceles.

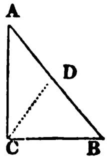


In the triangle ABC, let the angles ABC, ACB, be equal; then, if it be supposed that one of the opposite sides, as AB, is longer than the other AC, let BD be equal to AC; then the triangle DCB is obviously less than the triangle ABC. But, since CB, BD, and the included angle, are equal to BC, CA, and the included angle, by hypothesis, it follows that the same triangles are equal (Prop. VIII.), which is impossible; therefore AB cannot be longer than AC, and in a similar manner it may be shown that AC cannot be longer than AB; therefore these two sides are equal.

Cor. Therefore every equiangular triangle is equilateral.

PROPOSITION XVI. THEOREM.

In any triangle the greater angle is opposite the longer side; and conversely, the longer side is opposite the greater angle.



First, Let the angle C be greater than the angle B; then will the side AB, opposite C, be greater than AC, opposite B.

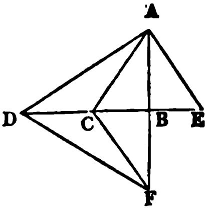
For, make the angle BCD=B. Then, in the triangle CDB, we shall have CD=BD (Prop. XV.) Now, the side AC<AD+CD; but AD+CD= AD+DB=AB; therefore AC<AB.

Secondly, Suppose the side AB>AC; then will the angle C, opposite to AB, be greater than the angle B, opposite to AC.

For, if the angle C<B, it follows, from what has just been proved, that AB<AC; which is contrary to the hypothesis. If the angle C=B, then the side AB=AC (Prop. XV.), which is also contrary to the supposition. Therefore, when AB> AC, the angle C must be greater than B.

PROPOSITION XVII. THEOREM.

The perpendicular drawn from a point to a right line is shorter than any other line drawn thereto from the same point; and those lines which meet the proposed line at equal distances from the perpendicular are themselves equal, and the more remote from the perpendicular the point of meeting is, the longer is the line drawn.



Let A be the given point, DE the given line, AB the perpendicular, and AD, AC, AE, the oblique lines.

Produce the perpendicular AB till BF is equal to AB, and draw FC, FD.

First. The triangle BCF is equal to the triangle BCA, for they have the right angle CBF=CBA, the side CB common, and the side BF=BA: hence the third sides, CF and CA, are equal (Prop. VIII. Cor. 2.) But ABF, being a right line, is shorter (Def. 8.) than ACF, which is a broken line; therefore, AB, the half of ABF, is shorter than AC, the half of ACF; hence, the perpendicular is shorter than any oblique line.

Secondly. Let us suppose BC=BE; then will the triangle CAB be equal to the triangle BAE; for BC=BE, the side AB is common, and the angle CBA=ABE: hence the sides AC and AE are equal (Prop. VIII. Cor. 2.): therefore, two oblique lines, equally distant from the perpendicular, are equal.

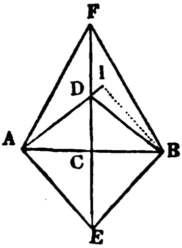
Thirdly. In the triangle DFA, the sum of the lines AC, CF, is less than the sum of the sides AD, DF (Prop. XI.); therefore AC, the half of the line ACF, is shorter than AD, the half of the line ADF: hence, the oblique line, which is farther from the perpendicular, is longer than the one which is nearer.

Cor. 1. The perpendicular measures the shortest distance of a point from a line. Hence only one perpendicular can be drawn from a point to a given line.

Cor. 2. From the same point to the same straight line, only two equal straight lines can be drawn; for, if there could be more, we should have at least two equal oblique lines on the same side of the perpendicular, which is impossible.

PROPOSITION XVIII. THEOREM.

If a perpendicular be drawn from the middle point of a right *line, every point in the perpendicular will be equidistant from the extremities of the line; and every point without the perpendicular, or the perpendicular produced, will be unequally distant from those extremities.*



Let AB be the given right line, C the middle point, and ECF the perpendicular.

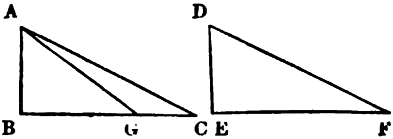
First. Since AC=CB, the two oblique lines AD, DB, are equally distant from the perpendicular, and therefore equal (Prop. XVII.) So, likewise, are the two oblique lines AE, EB, the two AF, FB, and so on. Therefore every point in the perpendicular is equally distant from the extremities A and B.

Secondly. Let I be a point out of the perpendicular. If IA and IB be drawn, one of these lines will cut the perpendicular in D; from which, drawing DB, we shall have BD =DA. But the straight line IB is less than ID+DB, and ID+DB=ID+DA=IA; therefore, IB<IA; hence, every point out of the perpendicular, is unequally distant from the extremities A and B.

Cor. If a straight line have two points D and F, equally distant from the extremities A and B, it will be perpendicular to AB at the middle point C.

PROPOSITION XIX. THEOREM.

Two right angled triangles are equal, when the hypothenuse and one side in one triangle are respectively equal to the hypothenuse and a side in the other.



In the two right angled triangles BAC, EDF, let the hypothenuse AC=DF, and the side BA=ED; then will the side BC=EF, the angle BAC=D, and the angle C=F.

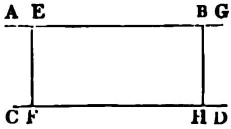
If the side BC is equal to EF, the like angles of the two triangles are equal (Prop. XIII. Sch.) Now, if it be possible, suppose these two sides to be unequal, and that BC is the greater.

On BC take BG=EF, and draw AG. Then, in the two triangles BAG, DEF, the angles B and E are equal, being right angles, the side BA=ED by hypothesis, and the side BG=EF by construction: consequently, AG=DF (Prop.

VIII. Cor. 2.) But by hypothesis AC=DF; and therefore AC=AG (Ax. 1.) But the oblique line AC cannot be equal to AG, which lies nearer the perpendicular AB (Prop. XVII.); therefore, BC and EF cannot be unequal, and hence the angle A=C, and the angle BAC=F; and therefore, the triangles are equal.

PROPOSITION XX. THEOREM.

If two right lines which lie in the same plane are equidistant in two points, they are equidistant in all their points, and hence are parallel.



Let AB and CD be two right lines lying in the same plane, and let them be so related, that the lines EF and GH drawn from the points E and G in the line AB, perpendicular to CD, are equal each to each; and the lines AB and CD will be equidistant in all their parts.

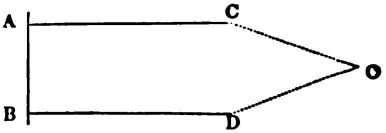
For let the two lines AB and CD be brought into such position that the points E and G in the line AB may coincide with the points F and H in the line CD, and (Prop. V.) the two lines will coincide in all their parts, and form but one line.

Now, as the line CD is a right line, and as all the points in the line AB coincide with corresponding points in the line CD when brought in the same position, if posited by the side of each other they will touch in all their parts, and the points E and G in the line AB will have the same relation to each of the other points in the line AB, as the points F and H in the line CD, to each of the other points in the line CD, and each point in the line AB will be similarly related to the corresponding points in the line CD, viz: they all touch each other. Now, if the line AB is moved, so that the points E and G may be equidistant from the points F and H, viz: so that the line EF, joining the points E and F, may be equal to the line GH joining the points G and H, the points E and G will still have the same relation to every other point in the line AB as before; and because the points E and G are similarly related to the points F and H, every other point in the AB is similarly related to its corresponding point in the line CD; that is, every point in the line AB is equidistant from the line CD, and hence the lines AB and CD are parallel.

Scholium. It is evident that parallel lines would never meet each other, however far they may be produced; for every part by which the lines are produced, when added, becomes a part of the parallel lines, which, by definition, are equidistant in all their parts.

PROPOSITION XXI. THEOREM.

If two right lines are perpendicular to a third line, they are parallel to each other.

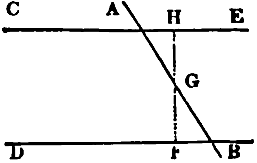


Let the two lines AC, BD, be perpendicular to AB, then will they be parallel.

For, if it be possible, let the productions of the lines AC and BD, meet in some point O on either side of AB, and if OCA and ODB are right lines and perpendicular to AB, there would be two perpendiculars, OA, OB, drawn from the same point on the same right line, which is impossible, (Prop. XVII. Cor. 1.)

PPOPOSITION XXII. THEOREM.

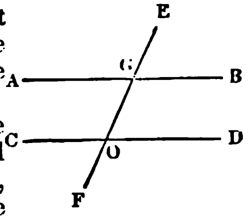
If a right line meet two other right lines, making the sum of the interior angles on the same side of the first mentioned line equal to two right angles, the two lines will be parallel.



Let the line AB meet the two lines EC and BD, making the angles BAC, ABD, together equal to two right angles; then the lines EC and BD will be parallel.

From G, the middle point of BA, draw the right line HGF, perpendicular to EC. It will also be perpendicular to BD. For, the sum BAC+ABD is equal to two right angles, by hypothesis; the sum BAC+BAE is likewise equal to two right angles (Prop. III.); and taking away BAC from both, there will remain the angle ABD=BAE.

Again, the angles HGA, BGF, are equal (Prop. VI.); therefore, the triangles HGA and BGF, have each a side and two adjacent angles equal; therefore they are themselves equal, and the angle GHA is equal to the angle GFB: but GHA is a right angle by construction; therefore, GFB is a right angle; hence the two lines EC, BD, are perpendicular to the same straight line, and are therefore parallel (Prop. XXI.)



Scholium. When two parallel straight lines AB, CD, are met by a third line FE, the angles which are formed take particular names.

Interior angles on the same side, are those which lie within the parallels, and on the same side of the secant line: thus, OGB, BGE, are interior angles on the same side; and so also are the angles OGA, GOC.

Alternate angles lie within the parallels, and on different sides of the secant line: AGO, DOG, are alternate angles; and so also are the angles COG, BGO.

Alternate exterior angles lie without the parallels, and on different sides of the secant line: EGB, COF, are alternate exterior angles; so also, are the angles AGE, FOD.

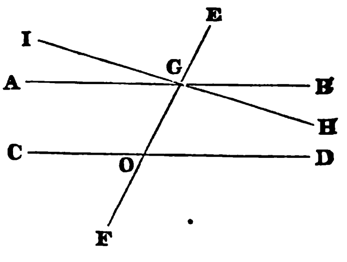
Opposite exterior and interior angles lie on the same side of the secant line, the one without and the other within the parallels, but not adjacent: thus, EGB, GOD, are opposite exterior and interior angles; and so also, are the angles AGE, GOC.

Cor. 1. If a straight line EF, meet two straight lines CD, AB, making the alternate angles AGO, GOD, equal to each other, the two lines will be parallel. For, to each add the angle OGB; we shall then have, $AGO+OGB=GOD+OGB$: but $AGO+OGB$ is equal to two right angles (Prop. III.); hence $GOD+OGB$ is equal to two right angles: therefore, CD, AB, are parallel.

Cor. 2. If a straight line EF meet two straight lines CD, AB, making the exterior angle FGB equal to the interior and opposite angle GOD, the two lines will be parallel. For, to each add the angle OGB; we shall then have $EGB+OGB=GOD+OGB$; but $EGB+OGB$ is equal to two right angles; hence, $GOD+OGB$ is equal to two right angles; therefore, CD, AB, are parallel.

PROPOSITION XXIII. THEOREM.

If a right line meet two parallel right lines, the sum of the interior angles on the same side will be equal to two right angles.



Let the parallels AB, CD, be met by the secant line FE: then will OGB+GOD, or OGA+GOC, be equal to two right angles.

For, if OGB+GOD be not equal to two right angles, let IGH be drawn, making the sum OGH+GOD equal to two right angles; then IH and CD will be parallel (Prop. XXII.), and hence we shall have two lines GB, GH, drawn through the same point G and parallel to CD, which is impossible (Ax. 12.); hence, GB and GH should coincide, and OGB+GOD is equal to two right angles. In the same manner it may be proved that OGA+COC is equal to two right angles.

Cor. 1. If OGB is a right angle, GOD will be a right angle also; therefore, every straight line perpendicular to one of two parallels, is perpendicular to the other.

Cor. 2. If a right line meet two parallel lines, the alternate angles will be equal.

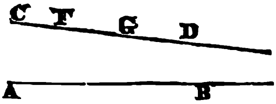
Let AB, CD, (see diagram to the scholium of the preceding proposition), be the parallels, and FE the secant line. The sum OGB+GOD is equal to two right angles. But the sum OGB+OGA is also equal to two right angles (Prop. III.) Taking from each the angle OGB, and there remains OGA=GOD. In the same manner we may prove that GOC =OGB.

Cor. 3. If a straight line meet two parallel lines, the opposite exterior and interior angles will be equal. For the sum OGB+GOD is equal to two right angles. But the sum OGB +DGB is also equal to two right angles. Taking from each the angle OGB, and there remains GOD=FGB. In the same manner we may prove that AGE=GOC.

Cor. 4. We see that of the eight angles formed by a line cutting two parallel lines obliquely, the four acute angles are equal to each other, and so also are the four obtuse angles.

PROPOSITION XXIV. THEOREM.

If two points in a right line be unequally distant from another right line, the former, by being produced on the side of the least distance, will continue to approach nearer and nearer to the latter, or its production, till at length it will meet it.



Let the two points F, G, in the right line CD, be unequally distant from the right line AB, then the line CD. by being produced, shall approach the line AB, or its prolongation, till it meets it.

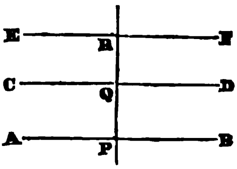
For the right lines AB and CD may be conceived to be generated by the motion of a point, and may be produced at pleasure by the continued motion of the point; and, (Def. 8.). if a right line is traced by the motion of a point, that point moves always in the same direction. Now, if in the course of the production of two lines lying in the same plane, they are found to approach each other in any degree: or if the point G is found to be nearer to the line AB, than the point F; then, because the lines always preserve the same direction, the point D will be nearer the line AB than the point G, and so on for any production of the two lines: and hence, there must be a point more or less distant to which both of the lines tend, and in which they would meet if produced.

Cor. Hence, from the same point more than one parallel to a right line cannot be drawn.

PROPOSITION XXV. THEOREM.

Two right lines which are parallel to a third line are parallel to each other.

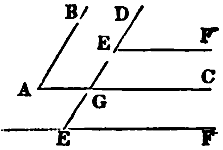
Let CD and AB be parallel to the third line EF; then are they parallel to each other.



Draw PQR perpendicular to EF, and cutting AB, CD. Since AB is parallel to EF, PR will be perpendicular to AB (Prop. XXIII. Cor. 1.); and since CD is parallel to EF, PR will for a like reason be perpendicular to CD. Hence AB and CD are perpendicular to the same straight line; hence they are parallel (Prop XXI.)

PROPOSITION XXVI. THEOREM.

If two angles have their sides parallel and lying in the same direction, they will be equal.



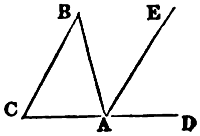
Let ABC and DEF be the two angles, having AB parallel to ED, and AC to EF; then will the angles be equal.

For, produce DE, if necessary, till it meets AC in G. Then, since EF is parallel to GC, the angle DEF is equal to DGC (Prop. XXIII. Cor. 3.); and since DG is parallel to AB, the angle DGC is equal to BAC; hence, the angle DEF is equal to BAC (Ax. 1.)

Scholium. The restriction of this proposition to the case where the side EF lies in the same direction with AC, and ED in the same direction with AB, is necessary, because if FE were produced toward H, the angle DEH would have its sides parallel to those of the angle BAC, but would not be equal to it. In that case, DEH and BAC would be together equal to two right angles. For, DEH+DEF is equal to two right angles (Prop. III.); but DEF is equal to BAC: hence, DEH+BAC is equal to two right angles.

PROPOSITION XXVII. THEOREM.

In every triangle, the sum of the three angles is equal to two right angles.



Let ABC be any triangle: then will the angle C+A+B be equal to two right angles.

For, produce the side CA towards D, and at the point A, draw AE parallel to BC. Then, since AE, CB, are parallel, and CAD cuts them, the exterior angle DAE will be equal to its interior opposite one ACB (Prop. XXIII. Cor. 3.); in like manner, since AE, CB, are parallel, and AB cuts them, the alternate angles ABC, BAE, will be equal: hence the three angles of the triangle ABC make up the same sum as the three angles CAB, BAE, EAD; hence, the sum of the three angles is equal to two right angles (Prop. III.)

Cor. 1. Two angles of a triangle being given, or merely their sum, the third will be found by subtracting that sum from two right angles.

Cor. 2. If two angles of one triangle are respectively equal to two angles of another, the third angles will also be equal, and the two triangles will be mutually equiangular.

Cor. 3. In any triangle there can be but one right angle: for if there were two, the third angle must be nothing. Still less, can a triangle have more than one obtuse angle.

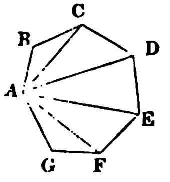
Cor. 4. In every right angled triangle, the sum of the two acute angles is equal to one right angle.

Cor. 5. Since every equilateral triangle is also equiangular (Prop. XIV. Cor. 2.), each of its angles will be equal to the third part of two right angles; so that, if the right angle is expressed by unity, the angle of an equilateral triangle will be expressed by $\frac{2}{3}$.

Cor. 6. Of every triangle ABC, the exterior angle BAD is equal to the sum of the two interior opposite angles B and C. For, AE being parallel to BC, the part BAE is equal to the angle B, and the part DAE is equal to the angle C.

PROPOSITION XXVIII. THEOREM.

The sum of all the interior angles of a polygon is equal to twice as many right angles as the figure has sides, less four right angles.

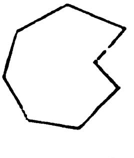


Let ABCDEFG be the proposed polygon. If from the vertex of any one angle A, diagonals AC, AD, AE, AF, be drawn to the vertices of all the opposite angles, it is plain that the polygon will be divided into five triangles, if it has seven sides; into six triangles, if it has eight; and, in general, into as many triangles, less two, as the polygon has sides; for, these triangles may be considered as having the point A for a common vertex, and for bases, the several sides of the polygon, excepting the two sides which form the angle A. It is evident also, that the sum of all the angles in these triangles, does not differ from the sum of all the angles in the polygon: hence the sum of all the angles of the polygon is equal to two right angles, taken as many times as there are triangles in the figure: in other words, as there are units in the number of sides diminished by two.

Cor. 1. The sum of the angles in a quadrilateral is equal to two right angles multiplied by 4—2, which amounts to four right angles; hence, if all the angles of a quadrilateral are equal, each of them will be a right angle: and if the sum of any two of the angles is equal to two right angles, the sum of the other two will also be equal to two right angles.

Cor. 2. The sum of the angles of a pentagon is equal to two right angles multiplied by 5—2, which amounts to six right angles: hence, when a pentagon is equiangular, each angle is equal to the fifth part of six right angles, or to $\frac{6}{5}$ of one right angle.

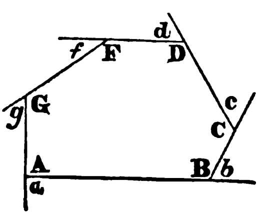
Cor. 3. The sum of the angles of a hexagon is equal to $2\times(6-2)$, or eight right angles; hence in the equiangular hexagon, each angle is the sixth part of eight right angles, or $\frac{4}{3}$ of one.



Scholium. When this proposition is applied to polygons which have *re-entrant* angles, each re-entrant angle must be regarded as greater than two right angles. But to avoid all ambiguity, we shall henceforth limit our reasoning to polygons with *salient* angles, which might otherwise be named *convex polygons.* Every convex polygon is such that a straight line, drawn at pleasure, cannot meet the contour of the polygon in more than two points.

PROPOSITION XXIX. THEOREM.

In any polygon the exterior angles formed by producing each side, amount to four right angles.



Let the sides of the polygon ABCDFG, be produced, in the same direction; then will the sum of the exterior angles $a+b+c+d+f+g$, be equal to four right angles.

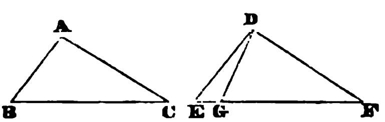
For, each interior angle, plus its exterior angle, as $A+a$, is equal to two right angles (Prop. III.) But there are as many exterior as interior angles, and as many of each as there are sides of the polygon; hence, the sum of all the interior and exterior angles is equal to twice as many right angles as the polygon has sides. Again, the sum of all the interior angles is equal to two right angles, taken as many times, less two, as the polygon has sides (Prop. XXVIII.); that is, equal to twice as many right angles as the figure has sides, wanting four right angles. Hence, the interior angles plus four right angles, is equal to twice as many right angles as the polygon has sides, and consequently, equal to the sum of the interior angles plus the exterior angles. Taking from each the sum of the interior angles, and there remain the exterior angles, equal to four right angles.

PROPOSITION XXX. THEOREM.

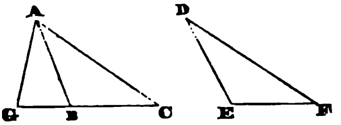
Two triangles are equal, if two sides and an opposite angle in one, are respectively equal to two sides and a correspond-

ing opposite angle in the other; provided the other opposite angles in each triangle are either both acute or both obtuse.

In the triangles ABC, DEF, let the sides AB, AC, be respectively equal to the sides DE, DF, and let the angle C, opposite to the side AB, be equal to the angle F, opposite to the side DE: then will BC be equal to EF, provided the angles B and E are either both acute, or both obtuse.



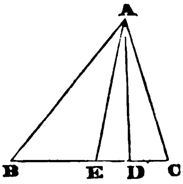
First, let B and E be acute: then, if the equality of BC and EF be denied, one of them as EF must be longer than the other. Let then FG be taken equal to BC, and draw DG, which will be equal to AB (Prop. VIII.Cor. 2.), and, therefore, equal to DE; consequently the angle E is equal to the angle DGE (Prop. XIV.): DGE is, therefore, an acute angle, but this angle, together with DGF, make up two right angles, (Prop. III.); DGF is therefore an obtuse angle. But since the triangles ABC, DGF, are equal, the angle DGF must be equal to the angle B, and, therefore, acute, which is impossible; so that FG cannot be equal to BC; and the demonstration would have been the same had BC been supposed longer than EF; these two sides are, therefore, equal.

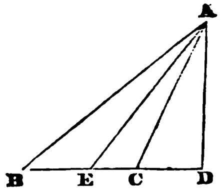


Next, let the angles B and E be obtuse: then, if EF be supposed longer than BC, produce the latter beyond the vertex B, till CG be equal to EF; join AG. Then, as before shown, AG is equal to DE, or to AB, and the angle AGC, which is equal to the angle ABG (Prop. XIV.), is acute, since ABD is obtuse; but the same angle must be obtuse, because the triangles AGC, DEF, are equal (Prop. VIII.), which is impossible; whence EF cannot be longer than BC, and had BC been supposed longer than EF, a similar absurdity would obviously have followed; hence in this case also the sides BC, EF, are equal, and therefore, (Prop. XIII.) the triangle ABC is equal to the triangle DEF.

PROPOSITION XXXI. THEOREM.

In any triangle, the difference of the angles at the base is double the angle contained by a line drawn from the vertex perpendicular to the base, and another line bisecting the angle at the vertex.





Let ABC denote any triangle ; AD perpendicular to the base (produced if necessary), AE a line bisecting the angle BAC : the difference of the angles at B and C, is double the angle EAD.

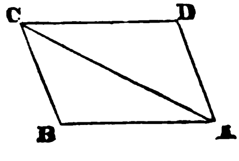
The angle ABC is equal to the difference of the angles ADC and BAD (Prop. XXVII. Cor. 4.) ; that is, of a right angle and BAD. Also, the angle ACB is equal to the difference of a right angle and DAC. Wherefore the difference of the angles ACB and ABC is equal to the difference of the angles BAD and DAC, that is, to the difference of the angles EAC, EAD, and DAC, that is to twice the angle DAE.

If the perpendicular fall on the base produced, as in the second figure, then as before, the angle ABC is equal to the difference of a right angle, and BAD (Prop. XXVII. Cor. 4.) ; and the angle ACB is equal to the *sum* of the angles CDA and DAC, that is, to a right angle and DAC ; wherefore the difference of the angles ACB and ABC is equal to the sum of the angles BAD, CAD; that is, to twice EAC and twice CAD, that is, to twice EAD.

PROPOSITION XXXII. THEOREM.

The opposite sides and angles of a rhomboid are equal.

Let ABCD be a rhomboid, the opposite sides and angles are equal.



Draw the diagonal AC ; then, since AB, DC, are parallel, the alternate angles BAC, DCA, are equal (Pr. XXIII. Cor. 3.), and because AD, BC, are also parallel, the alternate angles DAC, BCA, are likewise equal : hence the two angles BAC, DAC, are together equal to the two angles CDA, BCA ; that is, the opposite angles BAD, DCB, are equal. Again, since the angles BAC, BCA, and the interjacent side of the triangle ABC are respectively equal to the angles DCA, DAC, and the interjacent side of the triangle CDA, the triangles are equal (Prop. IX.) ; therefore the side AB is equal to the side CD, the side BC to DA, and the angle B to the angle D ; hence, in a rhomboid, the opposite sides and angles are equal.

Cor. 1. From this proposition, and proposition XXVIII. Cor. 1., it follows, that if one angle of a rhomboid be right, all the angles will be right.

Cor. 2. Therefore in the rectangle and square (see Definitions) all the angles are right, and in the latter all the sides are equal.

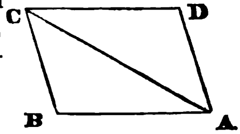
Cor. 3. The diagonal divides a rhomboid into two equivalent triangles.

Cor. 4. Parallels included between two other parallels are equal.

PROPOSITION XXXIII. THEOREM.

Conversely, if the opposite sides of a quadrilateral be equal, or if the opposite angles be equal, the figure will be a rhomboid.

In the quadrilateral ABCD let the opposite sides be equal, the figure will be a rhomboid.



Let the diagonal AC be drawn, then the triangles ABC, ADC, are equal, since the three sides of the one are respectively equal to those of the other, (Prop. XIII.); therefore the angles BAC, DCA, opposite the equal sides BC, DA, are equal; therefore DC is parallel to AB; the angles ACB, CAD, opposite the equal sides AB, DC, are also equal; BC is therefore parallel to AD (Prop. XXII. Cor. 1.): hence ABCD is a rhomboid.

Next, let the opposite angles be equal.

Then the sum of the angles BAD, ADC, must be equal to the sum of the angles DCB, CBA; therefore each sum is equal to two right angles (Prop. XXVIII. Cor. 1.); therefore AB, DC, are parallel (Prop. XXII. Cor. 1.) For similar reasons AD, BC, are parallel; therefore the figure is a rhomboid.

PROPOSITION XXXIV. THEOREM.

If two of the opposite sides of a quadrilateral are both equal and parallel, the figure is a rhomboid.

In the quadrilateral ABCD (preceding diagram), let AB be equal and parallel to DC, then will ABCD be a rhomboid.

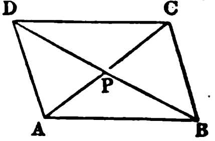
For the diagonal AC makes the alternate angles BAC, DCA, equal (Prop. XXIII. Cor. 2.); so that in the triangles

ABC, CDA, two sides, and the included angle in each, are respectively equal; these triangles are, therefore, equal, (Prop. VIII.): the angle ACB is, therefore, equal to the angle CAD; hence AD is parallel to BC (Prop. XXII. Cor. 1.), and the other two sides are parallel by hypothesis; therefore ABCD is a rhomboid.

Cor. If, in addition, the parallel sides be each equal to a third side, the rhomboid will be either a rhombus or a square, according as it has, or has not, a right angle.

Scholium. It has been proved (Prop. VIII.) that two triangles are equal, when two sides and the included angle in the one, are respectively equal to two sides and the included angle in the other; we may now infer further, that two triangles are equivalent or equal in surface, when two sides of the one are respectively equal to two sides of the other, and the sum of the included angles equal to two right angles.

For, let the triangles ADC, BCD, having two sides, AD, DC, in the one equal to the two BC, CD, in the other, be placed as in the margin, a side of the one coinciding with the equal side in the other; let also the included angles ADC, BCD, be together equal to two right angles, and let AB, BD be drawn.



Then, since the angles ADC, BCD, are together equal to two right angles, the lines AD, BC, are parallel (Prop. XXII. Cor. 1.), but they are also equal by hypothesis; hence, by the above proposition, the figure ABCD is a rhomboid; now, the triangle ADC is half the rhomboid (Prop. XXXII. Cor. 3.), so also is the triangle BCD; these triangles are, therefore, equivalent.

PROPOSITION XXXV. THEOREM.

The diagonals of a rhomboid bisect each other.

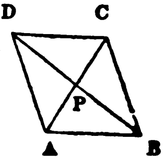
The diagonals AC, BD, of the rhomboid ABCD (see preceding diagram) are mutually bisected in the point P.

For, since AB, CD, are parallel, the angles PAB, PBA, are respectively equal to the angles PCD, PDC (Prop. XXIII. Cor. 2.), and AB being also equal to CD, the triangles PAB, PCD, are equal (Prop. IX.) therefore the sides AP, CP, opposite the equal angles ABP, CDP, are equal, as also the sides BP, DP, opposite the other equal angles. The diagonals of a rhomboid, therefore, bisect each other.

PROPOSITION XXXI. THEOREM.

Conversely, if the diagonals of a quadrilateral bisect each other, the figure is a rhomboid.

If the diagonals AC, BD, bisect each other, ABCD is a rhomboid.



For, the two sides AP, BP, and included angle, being equal to the two sides CP, PD, and included angle, the side AB is equal to the side CD (Prop. VIII. Cor. 2.) For similar reasons AD is equal to CB; hence (Prop. XXXIII.) the quadrilateral is a rhomboid.

It may be remarked that in this and the four succeeding books of these elements, the lines concerned in each proposition are supposed to be all situated in the same plane.

BOOK III.

THE CIRCLE AND THE MEASUREMENT OF ANGLES.

DEFINITIONS.

1. Every line which is not a right line, or composed of right lines, is a *curve* line.

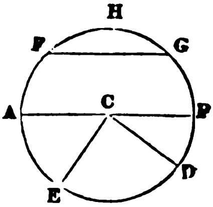
2. A *circle* is the surface terminated by a curve line, which is in every part equally distant from a point within, called the *centre*.

3. The boundary of a circle is called its *circumference*.

4. The *radius* is a right line drawn from the centre of a circle to the circumference, and it follows from Def. 2. that all radii are equal.

5. The *diameter* of a circle is a right line drawn through the centre. Hence, according to Def. 4, it is twice the radius.

In the annexed diagram the surface enclosed by the boundary curve line, is the circle; the curve line AFHGBDE, the circumference; the lines CE, CD, drawn from the centre to the circumference, are radii; and the right line AB drawn through the centre C, terminating in the circumference, the diameter.



6. An *arc* is any portion of a circumference.

7. A *chord* of an arc is a right line joining the extremities of that arc. Hence, it is said to *subtend* the arc.

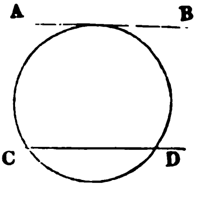
8. A *segment* of a circle is the surface included between an arc and its chord.

The surface FHG, included between the arc FHG and the chord FG is a segment; so likewise is the surface included by the same chord and the arc FAEDBG.

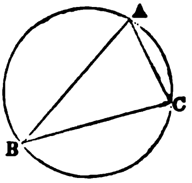
9. A *sector* of a circle is the surface included between two radii, and the intercepted arc. The surface ECD is a sector of a circ e.

10. A *tangent* is a line which touches a circumference, so that if it is produced, it will not cut it. The line AB is a tangent.

11. A *secant* is a line that cuts a circle lying partly within and partly without. The line CD is a secant.



12. One circle is said to *touch* another, when they have one point common, and but one.



13. A line is *inscribed in a circle*, when its extremities are in the circumference, as AB or BC.

14. An angle is *inscribed in a circle*, when its sides are inscribed.

The angle ABC is an inscribed angle.

15. An inscribed triangle is one which, like BAC, has its three angular points in the circumference.

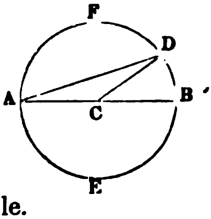
16. A polygon is inscribed in a circle, when its sides are inscribed; and under the same circumstances, a circle is said *to circumscribe* a polygon.

17. A circle is inscribed in a polygon, when all its sides touch the circumference, and the polygon is said to *circumscribe* the circle.

18. By an angle in the segment of a circle, is meant one which has its vertex in the arc, and whose sides intercept the chord And by an angle at the centre, is meant one which has its vertex in the centre. In both cases, the angles are said to be subtended by the chords, or the arcs, which their sides include.

PROPOSITION I. THEOREM.

A diameter divides a circle and its circumference into two equal parts.



Let AEDF be a circle, and AB a diameter. Now, if the figure AEB be applied to AFB, their common base AB retaining its position, the curve line AEB must fall exactly on the curve line AFB, otherwise there would, in the one or the other, be points unequally distant from the centre, which is contrary to the definition of a circle.

PROPOSITION II. THEOREM.

Every chord is less than the diameter.

Let AD (see preceding diagram) be any chord. Draw the radii CA, CD, to its extremities. We shall then have $AD < AC + CD$ (Prop. X. B. II.); or $AD < AB$.

Cor. Hence, the greatest line which can be inscribed in a circle is its diameter.

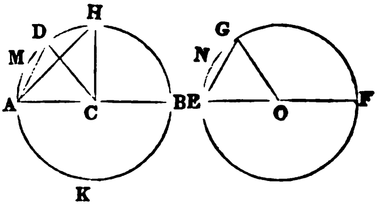
PROPOSITION III. THEOREM.

A right line cannot meet the circumference of a circle in more than two points.

For, if it could meet it in three, those three points would be equally distant from the centre; and hence, there would be three equal right lines drawn from the same point to the same right line, which is impossible, (Prop. XVII. Cor. 2. B. II.)

PROPOSITION IV. THEOREM.

In the same circle or in equal circles, equal arcs are subtended by equal chords; and conversely, equal chords subtend equal arcs.



If the radii AC, EO, are equal, and the arcs AMD, ENG; then the chord AD will be equal to the chord EG.

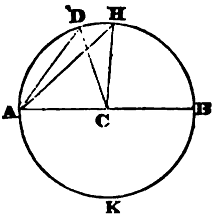
For, since the diameters AB, EF are equal, the semicircle AMDB may be applied exactly to the semicircle ENGF, and the curve line AMDB will coincide entirely with the curve line ENGF. But the part AMD is equal to the part ENG by hypothesis; hence the point D will fall on G; therefore the chord AD is equal to the chord EG.

Conversely. Suppose again the radii AC, EO, to be equal, if the chord AD is equal to the chord EG, the arcs AMD, ENG, will also be equal.

For, if the radii CD, OG, be drawn, the triangles ACD, EOG, having all their sides respectively equal, namely, AC =EO, CD=OG, and AD=EG, are themselves equal; and, consequently, the angle ACD is equal to EOG (Prop. VIII. Cor 2. B. II.) Now, placing the semicircle ADB on its equal EGF, since the angles ACD, EOG are equal, it is plain that the radius CD will fall on the radius OG, and the point D on the point G; therefore the arc AMB is equal to the arc ENG.

PROPOSITION V. THEOREM.

In the same circle, or in equal circles, a greater arc is subtended by a greater chord; and conversely, the greater chord subtends the greater arc.



Let the arc AH be greater than the arc AD: then will the chord AH be greater than the chord AD.

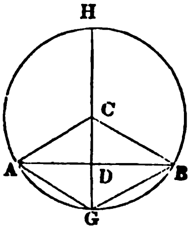
For, draw the radii CD, CH. The two sides AC, CH, of the triangle ACH, are equal to the two AC, CD, of the triangle ACD, and the angle ACH is greater than ACD: hence, the third side AH is greater than the third side AD (Prop. XII. B. II.): therefore the chord, which subtends the greater arc, is the greater.

Conversely, if the chord AH is greater than AD, it will follow, on comparing the same triangles, that the angle ACH is greater than ACD (Prop. XII. Sch. Book II.); and hence that the arc AH is greater than AD; since the whole is greater than its part.

Scholium. The arcs here treated of are each less than the semicircumference. If they were greater, the reverse property would have place; for, as the arcs increase, the chords would diminish, and conversely. Thus, the arc AKBD is greater than AKBH, and the chord AD, of the first, is less than the chord AH of the second.

PROPOSITION VI. THEOREM.

A radius, perpendicular to a chord, bisects the chord, and bisects also the subtended arc of the chord.



Let AB be a chord, and CG the radius perpendicular to it: then will AD=DB, and the arc AG=GB.

For, draw the radii CA, CB. Then the two right angled triangles ADC, CDB, will have AC=CB, and CD common: hence AD is equal to DB (Prop. XIX. B. II.)

Again, since AD, DB, are equal, CG is a perpendicular erected from the middle of AB: hence, every point of this perpendicular must be equally distant from its two extremities A and B (Prop. VIII. B. II.) Now, G is one of these points; therefore AG, BG, are equal. But if the chord AG is equal to the chord GB, the arc AG will be equal to the arc GB (Prop. IV.); hence, the radius CG, at right angles to the chord AB, divides the arc subtended by that chord into two equal parts at the point G.

Scholium. The centre C, the middle point D, of the chord

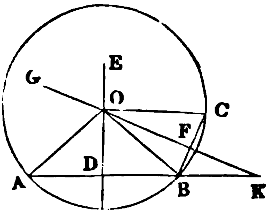
AB, and the middle point G, of the arc subtended by this chord, are three points of the same line perpendicular to the chord. But two points are sufficient to determine the position of a straight line; hence every straight line which passes through two of the points just mentioned, will necessarily pass through the third, and be perpendicular to the chord.

It follows, likewise, that the perpendicular raised from the middle of a chord passes through the centre of the circle, and through the middle of the arc subtended by that chord.

For, this perpendicular is the same as the one let fall from the centre on the same chord, since both of them pass through the centre and middle of the chord.

PROPOSITION VII. THEOREM.

Through three given points not in the same right line, one circumference may be made to pass, and but one.



Let A, B, and C, be the given points.

Draw AB, BC, and bisect these right lines by the perpendiculars DE, FG: we say first, that DE and FG, will meet in some point O.

For, they must necessarily cut each other, if they are not parallel. Now, if they were parallel, the line AB, which is perpendicular to DE, would also be perpendicular to FG, and the angle K would be a right angle (Prop. XXIII. Cor. 1. B. II.) But BK, the prolongation of BD, is a different line from BF, because the three points A, B, C, are not in the same right line; hence there would be two perpendiculars, BF, BK, let fall from the same point B, on the same straight line, which is impossible (Prop. I. B. II.); hence, DE, FG, will always meet in some point O.

And moreover, this point O, since it lies in the perpendicular DE, is equally distant from the two points, A and B, (Prop. XVIII. B. II.); and since the same point O lies in the perpendicular FG, it is also equally distant from the two points B and C, hence the three distances OA, OB, OC, are equal; therefore the circumference described from the centre O, with the radius OB, will pass through the given points A, B, C.

We have now shown that one circumference can always be made to pass through three given points, not in the same right line: we say farther, that but one can be described through them.

ich can pass through three given points.

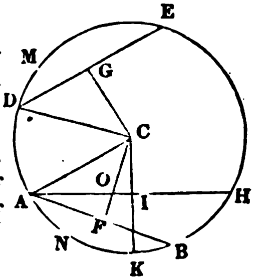
Cor. 1. Two circumferences cannot meet in more than two oints; for, if they have three common points, there would e two circumferences passing through the same three points; hich has been shown by the proposition to be impossible.

Cor. 2. Therefore, from any point not the centre, more an two equal lines cannot be drawn to the circumference.

Cor. 3. Consequently, that point from which three equal es may be drawn is the centre.

PROPOSITION VIII. THEOREM.

ual chords are equally distant from the centre; and of unqual chords, the less is at the greatest distance from the entre.



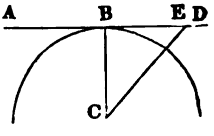
First. Suppose the chord AB=DE. sect these chords by the perpendilars CF, CG, and draw the radii , CD.

In the right angled triangles CAF, G, the hypothenuses CA, CD, are al; and the side AF, the half of , is equal to the side DG, the half DE: hence the triangles are equal, 1 CF is equal to CG, (Prop. XIX. II.); hence, the two equal chords AB, DE, are equally tant from the centre.

Secondly. Let the chord AH be greater than DE. The AKH will be greater than DME (Prop. V.): cut off from

PROPOSITION IX. THEOREM.

A perpendicular at the extremity of a radius is a tangent of the circumference; and conversely, a tangent to the circle is perpendicular to the radius drawn to the point of contact.



Let ABD be perpendicular to the radius CB, it shall touch the circle in the point B.

For to show that this point B is the only one which the circle has in common with AD, let CE be drawn to any other point E in that line, then CE being longer than CB, (Prop. XVII. B. II.) the point E must necessarily lie without the circumference; B, therefore, is the only point in AD which is also in the circumference; AD is therefore a tangent to the circle.

Conversely. Let now AD touch the circle in B, and let BC be drawn from the point of contact to the centre; AD shall be perpendicular to BC.

For every point in AD, except B, lies without the circumference: to suppose that any point lay within the circumference, the line AD must be supposed to pass through the circle, and thus cut the circumference in some other point besides B; but this is contrary to the hypothesis; therefore a line drawn from C to any other point, E in AD, must be longer than CB; this therefore being the shortest line, is perperdicular to AD (Prop. XVII. B. II.)

Cor. 1. From the same point in a circumference only one tangent can be drawn, for two lines could not be both perpendicular to the diameter at the same point.

Cor. 2. Tangents at each extremity of a diameter are parallel, (Prop. XXII. Cor. 1. B. II.)

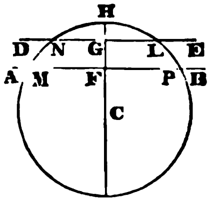
Scholium. At a given point B, only one tangent AD can be drawn to the circumference; for, if another could be drawn, it would not be perpendicular to the radius CB; hence in reference to this new tangent, the radius CB would be an oblique line, and the perpendicular let fall from the centre upon this tangent would be shorter than CB; hence this supposed tangent would enter the circle, and be a secant.

PROPOSITION X. THEOREM.

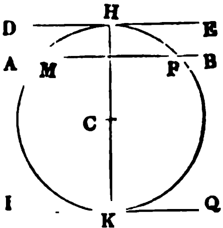
The arcs of a circle intercepted by two parallels, are equal.

There may be three cases.

First. If the two parallels are secants, draw the radius CH perpendicular to the chord MP. It will, at the same time be perpendicular to NL (Prop. XXIII. Cor. I. B. II): therefore, the point H will be at once the middle of the arc MHP, and of the arc NHL (Prop. VI.); therefore, we shall have the arc MH=HP, and the arc NH=HL; and therefore MH—NH=HP—HL; in other words, MN=PL.



Second. When, of the two parallels, AB, DE, one is a secant, the other a tangent, draw the radius CH to the point of contact H; it will be perpendicular to the tangent DE (Prop. IX.), and also to its parallel MP. But, since CH is perpendicular to the chord MP, the point H must be the middle of the arc MHP (Prop. VI.); therefore the arcs MH, HP, included between the parallels AB, DE, are equal.



Third. If the two parallels DE, IQ, are tangents, the one at H, the other at K, draw the parallel secant AB; and, from what has just been shown, we shall have MH=HP, MK=KP; and hence the whole arc HMK=HPK. It is farther evident that each of these arcs is a semicircumference.

PROPOSITION XI. THEOREM.

Conversely, if two lines intercept equal arcs of a circle and do not cut each other within the circle, the lines will be parallel.

If the two lines be DE and IQ (see preceding diagram) which touch the circumference, and if at the same time HMK and HPK are equal, KH must be the diameter (Prop. I.), and therefore DE and IQ are parallel (Prop. IX. Cor. 2.)

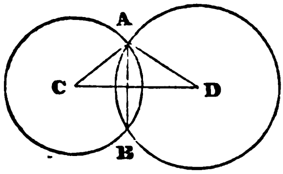
But if only one of the lines, as DE, touch, while the other AB cuts the circumference, making the arcs HM and HP equal, then the diameter HK which bisects the arc MHP bisects the chord MP (Prop. VI.) and is also perpendicular to the tangent DE (Prop. IX.); therefore DE and AB are parallel.

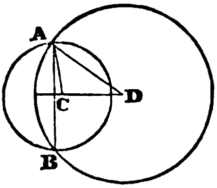
If both lines cut the circle, as DE and AB (see diagram above), and intercept equal arcs MN and LP, let the radius CH bisect one of the chords, as DE; it will also bisect the arc NHL (Prop. VI.): so that HN is equal to HL: and since NM is by hypothesis equal to LP, the whole arc HNM is equal to the whole HLP; therefore the chord AB is bisected by the radius CH; hence, as both chords are bisected by this radius, they are perpendicular to it (Prop. VI.); that is, they are parallel.

PROPOSITION XII. THEOREM.

If two circles cut each other in two points, the line which joins their centres will be perpendicular to the chord which joins the points of intersection, and will divide it into two equal parts.

For, let the line AB join the points of intersection. It will be a common chord to the two circles. Now if a perpendicular be erected from the middle of this chord, it will pass through each of the two centres C and D (Prop. VI. Sch.) But no more than one straight line can be drawn through two points; hence the straight line, which passes through the centres, will bisect the chord at right angles.

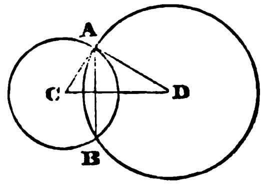




PROPOSITION XIII. THEOREM.

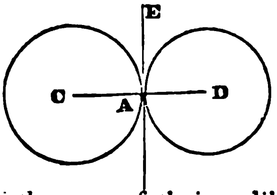
If the distance between the centres of two circles is less than the sum of their radii, the greater radius being at the same time less than the sum of the smaller and the distance between the centres, the two circumferences will cut each other.

For, to make an intersection possible, the triangle CAD must be possible. Hence, not only must we have $CD < AC + AD$, but also the greater radius $AD < AC + CD$ (Prop. X. B. II.) And, whenever the triangle CAD can be constructed, it is plain that the circles described from the centres C and D, will cut each other in A and B.



PROPOSITION XIV. THEOREM.

If the distance between the centres of two circles is equal to the sum of their radii, the two circles will touch each other externally.

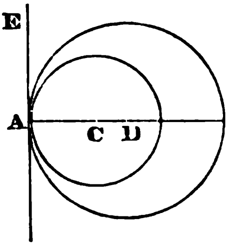


Let C and D be the centres at a distance from each other equal to CA+AD.

The circles will evidently have the point A common, and they will have no other: because if they had two points common, the distance between their centres must be less than the sum of their radii.

PROPOSITION XV. THEOREM.

If the distance between the centres of two circles is equal to the difference of their radii, the two circles will touch each other internally.



Let C and D be the centres at a distance from each other equal to AD—CA.

It is evident, as before, that they will have the point A common: they can have no other: because, if they had, the greater radius AD must be less than the sum of the radius AC and the distance CD between the centres (Prop. XIII.); which is contrary to the supposition.

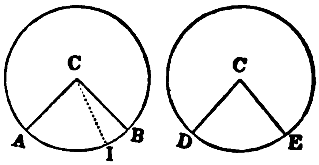
Cor. Hence, if two circles touch each other, either externally or internally, their centres and the point of contact will be in the same right line.

Scholium. All circles which have their centres on the right line AD, and which pass through the point A, are tangent to each other. For they have only the point A common, and if through the point A, AE be drawn perpendicular to AD, the straight line AE will be a common tangent to all the circles.

PROPOSITION XVI. THEOREM.

In the same circles, or in equal circles, equal angles, having their vertices at the centre, intercept equal arcs on the circumstance; and conversely, if the arcs intercepted are equal, the angles contained by the radii will be equal.

Let C and C be the centres of equal circles, and the angle ACB=DCE.



First. Since the angles ACB, DCE, are equal, they may be placed upon each other; and since their sides are equal, the point A will evidently fall on D, and the B on E. But, in that case, the arc AB must also fall on the arc DE; for if the arcs did not exactly coincide, there would, in the one or the other, be points unequally distant from the centre, which is impossible; hence the arc AB is equal to DE.

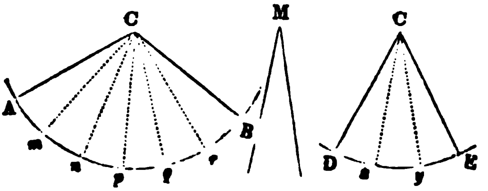
Secondly. If we suppose AB=DE, the angle ACB will be equal to DCE. For, if these angles are not equal, suppose ACB to be the greater, and let ACI be taken equal to DCE. From what has just been shown, we shall have AI=DE: but, by hypothesis, AB is equal to DE; hence AI must be equal to AB, or a part to the whole, which is absurd (Ax. 8.); hence, the angle ACB is equal to DCE.

Cor. Since the sum of the angles formed by right lines meeting in a common point, are equal to four right angles (Prop. VI. Sch. B II.), if these lines be produced to meet the circumference of a circle, it follows from this proposition that the whole circumference is intercepted by four right angles.

PROPOSITION XVII. THEOREM.

In the same circle, or in equal circles, if two angles at the centre are in the proportion to each other as two assumed numbers, the intercepted arcs will be to each other in the proportion of the same numbers, and we shall have the angle to the angle, as the arc to the corresponding arc.

Suppose, for example, that the angles ACB, DCE, are to each other as 6 is to 3; or, which is the same thing, suppose that the angle M, which may serve as a common measure, is contained 6 times in the angle ACB, and 3 times in

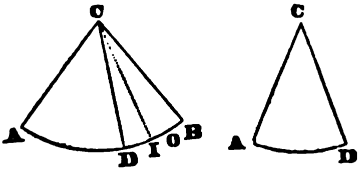


DCE. The seven partial angles AC*m*, *m*C*n*, *n*C*p*, &c., into which ACB is divided, being each equal to any of the three partial angles into which DCE is divided; each of the partial arcs A*m*, *mn*, *np*, &c., will be equal to each of the partial arcs D*x*, *xy*, &c. (Prop. XVI.) Therefore the whole arc AB will be to the whole arc DE, as 6 is to 3. But the same reasoning would evidently apply, if in place of 6 and 3 any numbers whatever were employed; hence, if the ratio of the angles ACB, DCE, can be expressed in whole numbers, the arcs AB, DE, will be to each other as the angles ACB, DCE.

Scholium. Conversely, if the arcs, AB, DE, are to each other as two whole numbers, the angles ACB, DCE, will be to each other as the same whole numbers, and we shall have ACB : DCE::AB : DE. For the partial arcs, A*m*, *mn*, &c., and D*x*, *xy*, &c., being equal, the partial angles AC*m*, *m*C*n*, &c., and DC*x*, *x*C*y*, &c., will also be equal.

PROPOSITION XVIII. THEOREM.

Whatever be the ratio of two angles, they will always be to each other as the arcs intercepted between their sides; the arcs being described from the vertices of the angles as centres with equal radii.



Let ACB be the greater and ACD the less angle.

Let the less angle be placed on the greater. If the proposition is not true, the angle ACB will be to the angle ACD as the arc AB is to an arc greater or less than AD. Suppose this arc to be greater, and let it be represented by AO; we shall thus have the angle ACB : angle ACD::arc AB : AO. Next, conceive the arc AB to be divided into equal parts, each of which is less than DO; there will be at least one point of division between D and O; let I be that point; and draw CI. The arcs AB, AI, will be to each other as two whole numbers, and, by the preceding theorem, we shall have the angle ACB : angle ACI::arc AB : arc AI. Comparing these two proportions with each other, we see that the antecedents are the same: hence, the consequents are proportional (Prop. XIX. B. I.); and thus we find the angle ACD : angle ACI::arc AO : arc AI. But the arc AO is greater than the arc AI; hence, if this proportion is true, the angle ACD must be greater than the angle ACI; on the contrary, however, it is less; hence the angle ACB cannot be to the angle ACD as the arc AB is to an arc greater than AD.

By a process of reasoning entirely similar, it may be shown that the fourth term of the proportion cannot be less than AD; hence it is AD itself; therefore we have

Angle ACB : angle ACD::arc AB : arc AD.

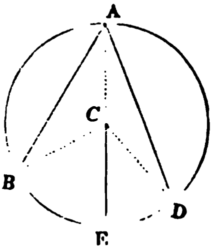
Cor. Since the angle at the centre of a circle, and the arc intercepted by its sides, have such a connection, that if the one be augmented or diminished in any ratio, the other will be augmented or diminished in the same ratio, we are authorized to establish the one of those magnitudes as the measure of the other; and we shall henceforth assume the arc AB as the measure of the angle ACB. It is only necessary that, in the comparison of angles with each other, the arcs which serve to measure them, be described with equal radii, as is implied in all the foregoing propositions.

Scholium. 1. Since the whole circumference is intercepted by four right angles (Prop. XVI. Cor.), a right angle is measured by one fourth part of the circumference, and this would be an obvious unit of measure for all other angles, and with this measure an acute angle would be expressed by some number between 0 and 1; an obtuse angle by some number between 1 and 2. This mode of expressing angles would not, however, be the most convenient in practice. It has been found more simple to meaſure them by arcs of a circle, on account of the facility with which arcs can be made equal to given arcs, and for various other reasons. At all events, if the measurement of angles by arcs of a circle is in any degree indirect, it is still equally easy to obtain the direct and absolute measure by this method; since, on comparing the arc which serves as a measure to any angle, with the fourth part of the circumference, we find the ratio of the given angle to a right angle, which is the absolute measure.

Scholium 2. All that has been demonstrated in the last three propositions, concerning the comparison of angles with arcs, holds true equally, if applied to the comparison of sectors with arcs; for sectors are not only equal when their angles are so, but are in all respects proportional to their angles; hence, two sectors ACB, ACD, taken in the same circle, or in equal circles, are to each other as the arcs AB, AD, the bases of those sectors. It is hence evident that the arcs of the circle, which serve as a measure of the different angles, are proportional to the different sectors, in the same circle, or in equal circles.

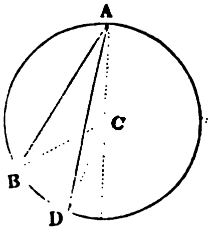
PROPOSITION XIX. THEOREM.

An inscribed angle is measured by half the arc included between its sides.



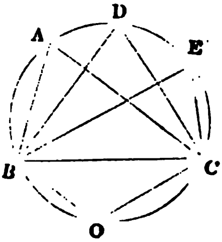
Let BAD be an inscribed angle, and let us first suppose that the centre of the circle lies within the angle BAD. Draw the diameter AE, and the radii CB, CD.

The angle BCE, being exterior to the triangle ABC, is equal to the sum of the two interior angles CAB, ABC, (Prop. XXVII. Cor. 6. B. II.): but the triangle BAC being isosceles, the angle CAB is equal to ABC; hence, the angle BCE is double of BAC. Since BCE lies at the centre, it is measured by the arc BE: hence BAC will be measured by the half of BE. For a like reason, the angle CAD will be measured by the half of ED; hence, BAC+CAD, or BAD, will be measured by half of BE+ED, or of BED.

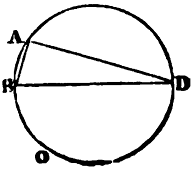


Suppose, in the second place, that the centre C lies without the angle BAD. Then drawing the diameter AE, the angle BAE will be measured by the half of BE, the angle DAE by the half of DE: hence their difference BAD will be measured by the half of BE *minus* the half of ED, or by the half of BD.

Hence every inscribed angle is measured by half of the arc included between its sides.

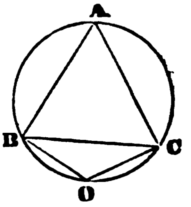


Cor. 1. All the angles BAC, BDC, and BEC, inscribed in the same segment, are equal; because they are all measured by the half of the same arc BOC.

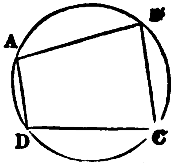


Cor. 2. Every angle BAD, inscribed in a semicircle, is a right angle; because it is measured by half the semi-circumference BOD, that is, by the fourth part of the whole circumference.

Cor. 3. Every angle BAC, inscribed in a segment greater than a semicircle, is an acute angle; for it is measured by half of the arc BOC, less than a semi-circumference.



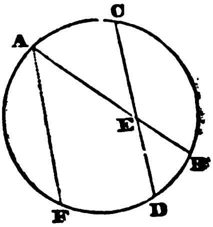
And every angle BOC, inscribed in a segment less than a semicircle, is an obtuse angle; for it is measured by half of the arc BAC, greater than a semi-circumference.



Cor. 4. The opposite angles A and C, of an inscribed quadrilateral ABCD, are together equal to two right angles: for the angle BAD is measured by half the arc BCD, the angle BCD is measured by half the arc BAD; hence the two angles BAD, BCD, taken together, are measured by the half of the circumference; hence their sum is equal to two right angles.

PROPOSITION XX. THEOREM.

The angle formed by two chords, which intersect each other, is measured by half the sum of the arcs included between its sides.



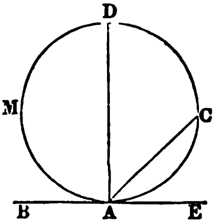
Let AB, CD, be two chords intersecting each other at E: then will the angle AEC, or DEB, be measured by half of AC+DB.

Draw AF parallel to DC: then will the arc DF be equal to AC (Prop. X.), and the angle FAB equal to the angle DEB (Prop. XXIII. Cor. 3. B. II.) But the angle FAB is measured by half the arc FDB (Prop. XIX.); therefore, DEB is measured by half of FDB; that is, by half of DB+DF, or half of DB+AC.

Again, as the angle AEC=BED, and that BED is measured by half the sum of the arcs AC and BD, the two angles are measured by the sum of those arcs; and also as the sum of the four angles made by two lines intersecting each other, are equal to four right angles, and that the opposite vertical angles are equal, and since the circumference measures four right angles (Prop. XVIII. Sch. 1.), it follows that the two are measured by the sum of those arcs, and also that either of them is measured by half of that sum.

PROPOSITION XXI. THEOREM.

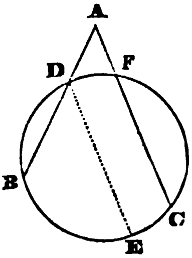
The angle formed by a tangent and a chord is measured by half the arc included between its sides.



Let BE be the tangent, and AC the
:ord.
From A, the point of contact, draw the
imeter AD. The angle BAD is a right
gle (Prop. IX.), and is measured by
f the semi-circumference AMD; the
gle DAC is measured by the half of
: hence, BAD+DAC, or BAC, is mea-
ed by the half of AMD plus the half
DC, or by half the whole arc AMDC.
: might be shown, by taking the difference between the
les DAE, DAC, that the angle CAE is measured by half
arc AC, included between its sides.
or. Hence the angle formed by a tangent and chord is
al to an angle described in the opposite segment, and
ich subtends the chord. For (Prop. XIX.) an angle de-
bed in a circle is measured by half the arc included be-
en its sides.

PROPOSITION XXII. THEOREM.

angle formed by two secants is measured by half the difference of the arcs included between its sides.

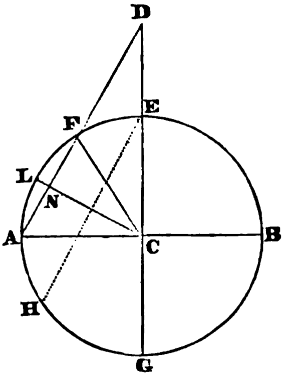


et AB, AC, be two secants: then will
angle BAC be measured by half the
rence of the arcs BEC and DF.
raw DE parallel to AC: then will the
EC be equal to DF, and the angle
: equal to the angle BAC. But BDE
easured by half the arc BE; hence,
is also measured by half the arc BE;
is, by half the difference of BEC and
or half the difference of BEC and

PROPOSITION XXIII. THEOREM.

from the centre of a circle, a line be drawn at right angles a radius of the circle and extend to any point without the ircle, and if a secant be drawn from such point, to the ex-

tremity of the radius, cutting off an arc from the circle, then the angle formed by those lines without the circle will be equal to an angle measured by half the arc cut off by the secant.



Let CD be a line drawn from the centre C of the circle AB perpendicular to the radius CA, and from the point D draw the secant DA to the extremity of the radius, cutting off AF an arc of the circle; then will the angle formed by the meeting of the lines at D be equal to an angle measured by half the arc AF cut off by the secant DA.

From the point E, where the line CD cuts the circle, draw the line EH parallel to DA and produce the line DC to the circumference at G. Now because two lines which are parallel intercept equal arcs, on the circumference of a circle (Prop. X.), the arc EF=the arc AH, being arcs intercepted by the parallel lines AD and HE. And because the quadrant AE=the quadrant AG, and because the arc FE=the arc AH, the quadrant AE—arc FE=the quadrant AG—arc AH (Ax. 3.) Hence the arc AF=arc HG.

And as the lines GE and ED are one continued straight line, meeting the two parallel lines HE and AD, the angles E and D formed by its meeting with those lines will be equal (Prop. XXII. B. I. Cor. 3.), viz: the angle CDA=the angle CEH.

Again, as an inscribed angle is measured by half the arc included between its sides (Prop. XIX.), the angle GEH or GDA is measured by half the arc HG or its equal FA; or the angle ADC=the angle ACL=half the angle ACF.

Cor. 1. The line CL, bisecting the angle ACF, divides the triangle ADC into two triangles ACN and CND, which are equiangular and the angles of which are similar to those of the triangle ADC.

For (Prop. VI.) the line CL bisecting the arc AF is perpendicular to the chord AF, and since the chord AF is a part of the right line AD, it is perpendicular to that also; and since we have shown the angle C in the triangle ACN to be equal to the angle D in the triangle CDN, and since the two triangles have each a right angle at N, their remaining angles are equal (Prop. XXVII. Cor. 2. B. II.); hence (Def.

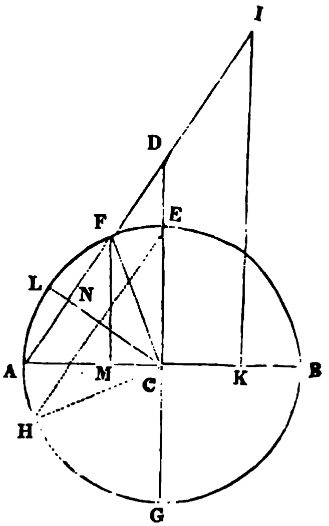
35. B. II.) the two triangles ACN and CDN are equiangular. Again, since in the two right angled triangles ACD and CDN, the angle D is common to both, their remaining angles, viz: the angles DAC and DCN are equal (Prop. XXVII. Cor. 2. B. II.); hence also, the two triangles ADC and CDN are equiangular.

Cor. 2. Hence, generally, if a perpendicular be let fall rom the right angle of a right angled triangle, on the oppote side, this perpendicular will divide the triangle into two iangles whose angles are similar, and similar to those of former triangle.

PROPOSITION XXIV. THEOREM.

rom any point of the base of a right angled triangle, as centre, or from any point of the base produced opposite right angle as a centre, a circle be described passing rough the angle formed by the base with the hypothenuse, arc of the circle cut off by the hypothenuse, or the hypoenuse produced, will be equal to twice the measure of the pposite acute angle.

'or since it was shown op. XXIII.) that the arc suring the angle ADC is half the arc AF cut off he line AD, let AD be inued to I, and draw IK llel to DC ; and because riangles ACD and AKI each a right angle, viz : ngles at C and K being angles, and because have also the angle at A non, it follows (Prop. II. Cor. 2. B. II.) that remaining angles are equal. Hence the angle is measured by half the AF.



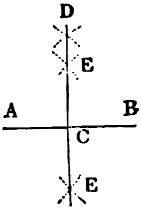
gain, from any point F e line AD draw FM parl to DC, and from the e reasoning the angle AFM is equal to the angle ADE, l therefore the angle AFM is measured by half the arc ; and as this is true in every position of FM, DC, or IK, perpendicular to the base AC or AC produced, it follows that the affirmation expressed in the proposition is established.

Scholium. From the last two propositions some important trigonometrical principles may be deduced.

PROBLEMS RELATING TO THE SECOND AND THIRD BOOKS.

PROBLEM I.

To divide a given right line into two equal parts.



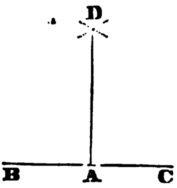
Let AB be the given right line.

From the points A and B as centres, with a radius greater than the half of AB, describe two arcs cutting each other in D; the point D will be equally distant from A and B. Find in like manner, above or beneath the line AB, a second point E, equally distant from the points A and B; through the two points D and E, draw the line DE: it will bisect the line AB in C.

For, the two points D and E, being each equally distant from the extremities A and B, must both lie in the perpendicular raised from the middle of AB (Prop. XVIII. Cor. B. II.) But only one right line can pass through two given points; hence the line DE must itself be that perpendicular, which divides AB into two equal parts at the point C.

PROBLEM II.

At a given point, in a right line, to erect a perpendicular to this line.



Let A be the given point, and BC the given line.

Take the points B and C at equal distances from A; then from the points B and C as centres, with a radius greater than BA, describe two arcs intersecting each other in D; draw AD: it will be the perpendicular required.

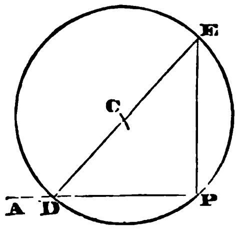
For, the point D, being equally distant from B and C, must be in the perpendicular raised from the middle of BC (Prop. XVIII. B. II.); and since two points determine a line, AD is that perpendicular.

Scholium. The same construction serves for making a right angle BAD, at a given point A, on a given right line BC.

PROBLEM III.

To erect a perpendicular at or near the end of a given line.

From any point C taken without the line with a radius equal to the distance CP describe a circumference, and from D, where it cuts AP or its prolongation, draw the diameter DE; then EP will be the perpendicular required, as is manifest from (Prop. XIX. Cor. 2.)

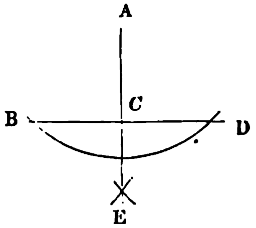


PROBLEM IV.

From a given point without a right line to let fall a perpendicular on this line.

First. Let A be the point, and BD the straight line.

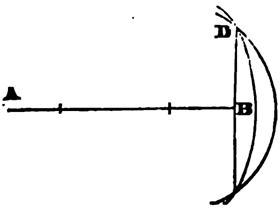
From the point A as a centre, and with a radius sufficiently great, describe an arc cutting the line BD in the two points B and D; then mark a point E, equally distant from the points B and D, and draw AE: it will be the perpendicular required.



For, the two points A and E are each equally distant from the points B and D; hence the line AE is a perpendicular passing through the middle of BD (Prop. XVIII. Cor. B. II.)

Second. When the given point is opposite the extremity of the given line, or nearly so, proceed as follows:

Let AB be the given line, and let D be the given point. Then from the other extremity of the line AB, or from any other point in AB, with a radius equal to its distance from

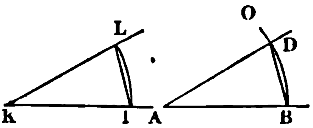


D, describe an arc; then from any second point in AB, with its distance from D as a radius, describe another arc, and through their points of intersection draw a line which will be the perpendicular required, as is obvious from (Prop. XII.)

PROBLEM V.

At a point in a given line to make an angle equal to a given angle.

Let A be the given point, AB the given line, and IKL the given angle.

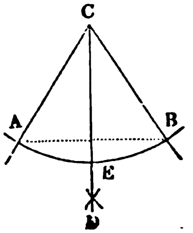


From the vertex K, as a centre, with any radius, describe the arc IL, terminating in the two sides of the angle. From the point A as a centre, with a distance AB, equal to KI, describe the indefinite arc BO; then take a radius equal to the chord LI, with which, from the point B as a centre, describe an arc cutting the indefinite arc BO, in D; draw AD; and the angle DAB will be equal to the given angle K.

For, the two arcs BD, LI, have equal radii, and equal chords; hence they are equal (Prop. IV.); therefore the angles BAD, IKL, measured by them, are equal.

PROBLEM VI.

To divide a given arc or a given angle into equal parts.



First. Let it be required to divide the arc AEB into two equal parts. From the points A and B, as centres, with the same radius, describe two arcs cutting each other in D; through the point D and the centre C, draw CD: it will bisect the arc AB in the point E.

For, the points C and D are each equally distant from the extremities A and B of the chord AB; hence the line CD bisects the chord at right angles (Prop. XVIII. Cor. B. II.); hence, it bisects the arc AB in the point E (Prop. VI.)

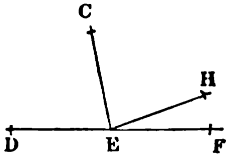
Secondly. Let it be required to divide the angle ACB into two equal parts. We begin by describing, from the vertex C as a centre, the arc AEB; which is then bisected as above. It is plain that the line CD will divide the angle ACB into two equal parts.

Scholium. By the same construction, each of the halves AE, EB, may be divided into two equal parts; and thus, by successive divisions, a given angle, or a given arc may be divided into four equal parts, into eight, into sixteen, and so on.

PROBLEM VII.

Two angles of a triangle being given, to find the third.

Draw the indefinite line DEF; at any point as E, make the angle DEC equal to one of the given angles, and the angle CEH equal to the other: the remaining angle HEF will be the third angle required; because those three angles are together equal to two right angles (Prop. III and XXVII. B. II.)

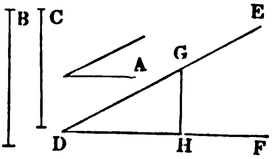


PROBLEM VIII.

Two sides of a triangle and the angle which they contain being given, to construct the triangle.

Let the lines B and C be equal to the given sides, and A the given angle.

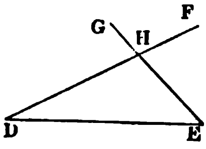
Having drawn the indefinite line DE, at the point D, make the angle EDF equal to the given angle A; then take DG=B, DH=C, and draw GH; DGH will be the triangle required (Prop. VIII B. II.)



PROBLEM IX.

A side and two angles of a triangle being given, to describe the triangle.

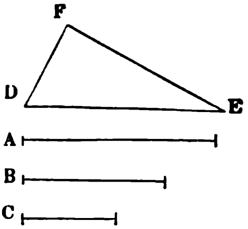
The two angles will either be both adjacent to the given side, or the one adjacent, and the other opposite: in the latter case, find the third angle (Prob. VII.); and the two adjacent angles will thus be known: draw the straight line DE equal to the given side: at the point D, make an angle EDF equal



to one of the adjacent angles, and at E, an angle DEG equal to the other: the two lines DF, EG, will cut each other in H; and DEH will be the triangle required (Prop. IX. B. II.)

PROBLEM X.

The three sides of a triangle being given, to construct the triangle.



Let A, B, and C, be the sides.

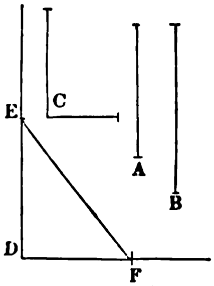
Draw DE equal to the side A; from the point E as a centre, with a radius equal to the second side B, describe an arc; from D as a centre, with a radius equal to the third side C, describe another arc intersecting the former in F; draw DF, EF; and DEF will be the triangle required (Prop. XIII. B. II.)

Scholium. If one of the sides were greater than the sum of the other two, the arcs would not intersect each other: but the solution will always be possible, when the sum of two sides, any how taken, is greater than the third.

PROBLEM XI.

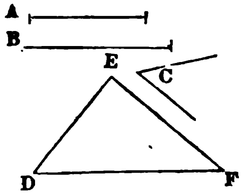
Two sides of a triangle and the angle opposite one of them being given, to describe that triangle.

Let A and B be the given sides, and C the given angle. There are two cases.

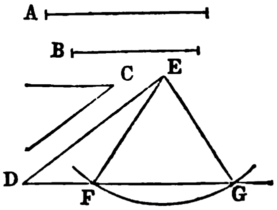


First. When the angle C is a right angle, or when it is obtuse, make the angle EDF=C; take DE=A; from the point E as a centre, with a radius equal to the given side B, describe an arc cutting DF in F; draw EF: then DEF will be the triangle required.

In this first case, the side B must be greater than A; for the angle C, being a right angle, or an obtuse angle, is the greatest angle of the triangle, and the side opposite to it must, therefore, also be the greatest (Prop. XVI. B. II.)



Secondly. If the angle C is acute, and B greater than A, the same construction will again apply, and DEF will be the triangle required.

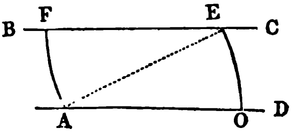


But if the angle C is acute, and the side B less than A, then the arc described from the centre E, with the radius EF=B, will cut the side DF in two points F and G, lying on the same side of D: hence there will be two triangles, DEF, DEG, either of which will satisfy the conditions of the proposition.

Scholium. If the arc described with E as a centre, should be tangent to the line DG, the triangle would be right angled, and there would be but one solution. The problem would be impossible in all cases, if the side B were less than the perpendicular let fall from E on the line DF.

PROBLEM XII.

Through a given point, to draw a parallel to a given right line.



Let A be the given point, and BC the given line.

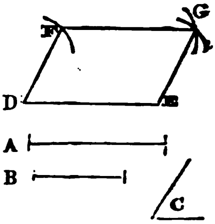
From the point A as a centre, with a radius greater than the shortest distance from A to BC, describe the indefinite arc EO; from the point E as a centre, with the same radius, describe the arc AF; make ED=AF, and draw AD: this will be the parallel required.

For, drawing AE, the alternate angles AEF, EAD, are evidently equal; therefore, the lines AD, EF, are parallel, (Prop. XXII. Cor. 1. B. II.)

PROBLEM XIII.

The adjacent sides of a rhomboid, with the angle which they contain, being given, to describe the rhomboid.

Let A and B be the given sides, and C the given angle.



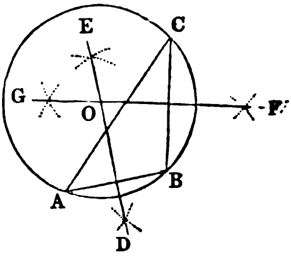
Draw the line DE=A; at the point D make the angle EDF=C; take DF=B; describe two arcs, the one from F as a centre, with a radius FG=DE, the other from E as a centre, with a radius EG=DF; to the point G, where these arcs intersect each other, draw FG, EG; DEGF will be the parallelogram required.

For, the opposite sides are equal, by construction; hence the figure is a parallelogram (Prop. XXXIII. B. II.); and it is formed with the given sides and the given angle.

Cor. If the given angle is a right angle, the figure will be a rectangle; if, in addition to this, the sides are equal, it will be a square.

PROBLEM XIV.

To find the centre of a given circle, or arc.

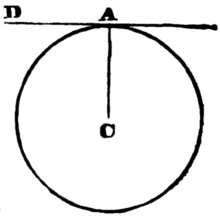


Take three points, A, B, C, any where in the circumference, or the arc; draw AB, BC, or suppose them to be drawn; bisect those two lines by the perpendiculars DE, FG: the point O, where these perpendiculars meet, will be the centre sought (Prop. VI. Sch.)

Scholium. The same construction serves for making a circumference pass through three given points A, B, C; and also for describing a circumference, in which a given triangle ABC shall be inscribed.

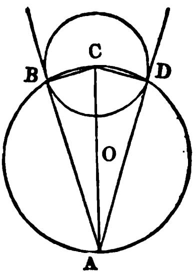
PROBLEM XV.

Through a given point to draw a tangent to a given circle.



If the given point A lies in the circumference, draw the radius CA, and erect AD perpendicular to it; AD will be the tangent required (Prop. IX.)

If the point A lies without the circle,



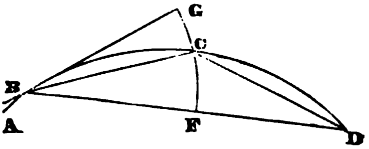
join A and the centre, by the right line CA: bisect CA in O; from O as a centre, with the radius OC, describe a circumference intersecting the given circumference in B; draw AB; this will be the tangent required.

For, drawing CB, the angle CBA being inscribed in a semicircle is a right angle (Prop. XIX. Cor. 2.); therefore AB is a perpendicular at the extremity of the radius CB; hence, it is a tangent.

Scholium. When the point A lies without the circle, there will evidently be always two equal tangents AB, AD, passing through the point A: they are equal, because the right angled triangles CBA, CDA, have the hypothenuse CA common, and the side CB=CD; hence they are equal (Prop. XIX. B. II.); hence AD is equal to AB, and also the angle CAD to CAB. And as there can be but one line bisecting the angle BAC, it follows that the line which bisects the angle formed by two tangents, must pass through the centre of the circle.

PROBLEM XVI.

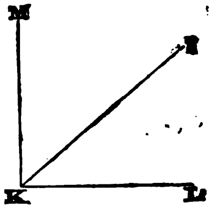
From a given point B *in the arc* BCD *of a circle, to draw a tangent thereto, without having the centre.*

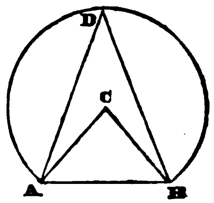


Take two equal distances BC, CD, on the arc; join BD, and from B as a common centre, with a radius equal to the distance BC, describe an arc FG; make the distance CG equal to the distance CF, and through G draw the straight line BG, which will be the tangent required; for if the chords BC, CD be drawn, the angle CBD will be equal to the angle CDB (Prop. XIX. Cor. 1.); and therefore the angle GBC will be equal to the angle CBF, that is, to CDB, an angle in the alternate segment; hence BG is a tangent at B.

PROBLEM XVII.

On a given right line AB, *to describe a segment capable of containing a given angle* IKL.



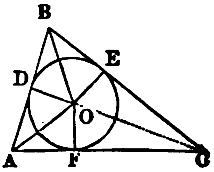


Draw KM perpendicular to KL, one of the sides of the given angle; then, at each extremity of AB, make an angle equal to IKM, the sides AC, BC, of which will meet in a point C, from which as a centre, with a radius equal to CA, or CB, describe the arc ADB, and the required segment will be completed; for the three angles of the triangle ABC are together equal to twice the angle MKL, and, by construction, the two angles CAB, CBA, are together equal to twice IKM; therefore the angle C is equal to twice IKL, but it is also equal to twice the inscribed angle D; consequently the segment ADB contains an angle equal to the given angle, IKL.

Scholium. If the given angle were obtuse, the centre must lie without the segment (Prop. XIX. Cor. 3.); if it were right, the segment would be a semicircle; and consequently the centre would be the middle of AB.

PROBLEM XVIII.

To inscribe a circle in a given triangle.



Let ABC be the given triangle.

Bisect the angles A and B, by the lines AO and BO, meeting in the point O; from the point O, let fall the perpendiculars OD, OE, OF, on the three sides of the triangle: these perpendiculars will all be equal.

For, by construction, we have the angle DAO=OAF, the right angle ADO=AFO; hence the third angle AOD is equal to the third AOF (Prop. XXVII. Cor. 2. B. II.) Moreover, the side AO is common to the two triangles AOD, AOF; and the angles adjacent to the equal side are equal: hence the triangles themselves are equal (Prop. IX. B. II.); and DO is equal to OF. In the same manner it may be shown that the two triangles BOD, BOE, are equal; therefore OD is equal to OE; hence the three perpendiculars OD, OE, OF, are all equal.

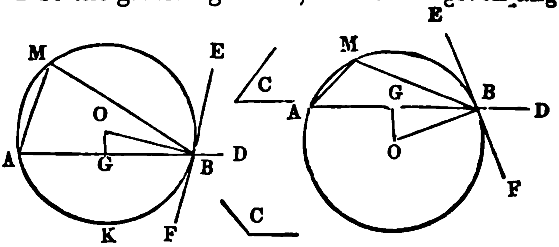
For, if from the point O as a centre, with the radius OD, a circle be described, this circle will evidently be inscribed in the triangle ABC; for the side AB, being perpendicular to the radius at its extremity, is a tangent; and the same thing is true of the sides BC, AC.

Scholium. The three lines which bisect the angles of a triangle meet in the same point.

PROBLEM XIX.

On a given right line, to describe a segment that shall contain a given angle; that is to say, a segment such, that all the angles inscribed in it shall be equal to the given angle.

Let AB be the given right line, and C the given angle.



Produce AB towards D; at the point B, make the angle DBE=C; draw BO perpendicular to BE, and GO perpendicular to AB, through the middle point G; and from the point O, where these perpendiculars meet, as a centre, with a distance OB, describe a circle: the required segment will be AMB.

For, since BF is a perpendicular at the extremity of the radius OB, it is a tangent, and the angle ABF is measured by half the arc AKB (Prop. XXI.) Also, the angle AMB, being an inscribed angle, is measured by half the arc AKB: hence all the angles inscribed in the segment AMB are equal to the given angle C.

Scholium. If the given angle were a right angle, the required segment would be a semicircle, described on AB as a diameter.

BOOK IV.

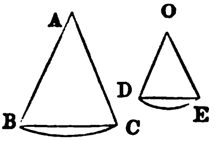
OF THE PROPORTIONS AND AREAS OF FIGURES.

Definitions.

1. Similar figures are those which have their angles respectively equal each to each, and their homologous sides proportional.

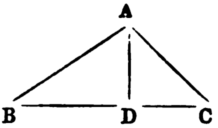
2. In two different circles, similar arcs, sectors or segments, are those which correspond to equal angles at the centre.

Thus, if the angles A and O be equal, the arc BC will be similar to the arc DE, the sector ABC to the sector ODE.

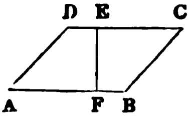


3. The *base* of any rectilineal figure is the side on which the figure is supposed to stand.

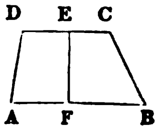
4. The *altitude* of a triangle is the perpendicular distance from the vertex of an angle to the opposite side, as a base. Thus AD is the altitude of the triangle BAC.



5. The *altitude* of a *rhomboid* is the perpendicular distance of one of its sides considered as a base, from the opposite side. Thus EF is the altitude of the rhomboid DB.

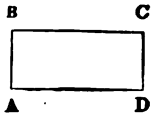


6. The *altitude* of a *trapezium* is the perpendicular distance between its parallel sides. Thus EF is the altitude of the trapezium DB.



7. The *area* and *surface* of a figure are terms very nearly synonymous. The area is the superficial contents contained within the line or lines by which the figure is bounded, and is numerically expressed by the number of times which the figure contains some other area, that is assumed for its measuring unit.

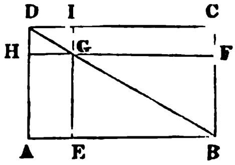
8. Figures have equal areas, when they contain the same measuring unit an equal number of times.



9. A rectangle is said to be contained by any two of the right lines which are adjacent to one of its angles. Thus the rectangle ABCD is contained by the lines AB and BC, or by BC and CD.

A rectangle, or polygon, may, for the sake of brevity, be designated by two letters at its opposite angles, as AC or BD; and the rectangle contained by the lines AB, BC, may be expressed AB.BC; and the square of a line, as AB, may be written AB^2.

10. If, within a rhomboid, or parallelogram, two right lines be drawn parallel to the adjacent sides, so as to intersect the diagonal in the same point; then, of the four rhomboids into which the figure is divided, those two through which the diagonal passes are said to be about the diagonal, and the other two are called their complements.



Thus of the four rhomboids into which the lines HF, IE, divide the rhomboid AC, the rhomboids HI and EF are about the diagonals; and AG and GC are the complements.

PROPOSITION I. THEOREM.

The complements of the rhomboids about the diagonals of a rhomboid are equivalent.

Thus, in the above diagram, the rhomboids AG, GC, are equivalent.

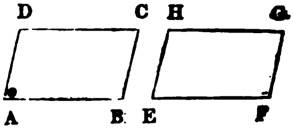
The triangle ABD is equal to the triangle CDB; the triangle HGD to the triangle IDG, and the triangle EBG to the triangle FGB (Prop. XXXII. Cor. 3. B. II.); take the triangles HGD, EBG, from the triangle ABD, and there will remain the rhomboid AG; take, in like manner, from the other half of the rhomboid AC the triangles IDG, FGB, equal to the former two, and there will remain the rhomboid GC: these rhomboids, therefore, are equivalent.

PROPOSITION II. THEOREM.

Rhomboids are equal, which have two equal sides and the included angle in each equal.

Let the sides AB, AD, and the angle A in the rhomboid

AC be respectively equal to the sides EF, EH, and the angle E in the rhomboid EG; these rhomboids are equal.

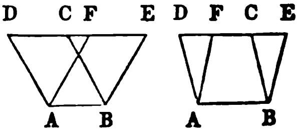


For the opposite sides of rhomboids being equal, it follows that the four sides of the rhomboid AC are respectively equal to those of the rhomboid EG; and since the angle A in the rhomboid AC is equal to the angle E in the rhomboid EG, it follows that the remaining angles in the rhomboids are respectively equal. Hence, if the rhomboid AC be applied to the rhomboid EG, so that the side AB shall coincide with the side EF, then since the angles A and B are respectively equal to the angles E and F, and since the sides containing those angles are also equal, the two rhomboids will coincide in all their parts, and hence are equal.

PROPOSITION III. THEOREM.

Rhomboids, and also triangles, which have equal bases and equal altitudes are equivalent.

Let the rhomboids AC, AE, standing upon the same base AB, have equal altitudes; or which amounts to the same thing, let the opposite sides DC, FE, lie in the same line DE, parallel to the base; these rhomboids are equal.



For DC is equal to FE, each being equal to AB (Prop. XXXII. B. II.); consequently DF is equal to CE: and since DA, AF, are respectively equal to CB, BE, the triangle ADF is equal to the triangle BCE. Take the former triangle from the quadrilateral ABED, and there will remain the rhomboid AE; take the latter triangle from the same space, and there will remain the rhomboid AC; these rhomboids are, therefore equivalent.

Cor. 1. Rhomboids whose bases and altitudes are respectively equal are equivalent, for the equal bases being placed the one upon the other must coincide.

Cor. 2. Triangles whose bases and altitudes are respectively equal are equivalent, as they are the halves of equivalent rhomboids (Prop. XXXII. Cor. 3. B. II.)

Cor. 3. Every rhomboid is equivalent to a rectangle of equal base and altitude.

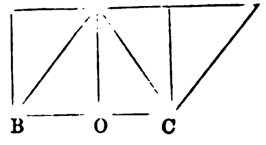
Cor. 4. A line bisecting the opposite sides of a rhomboid divides the rhomboid into two equal parts.

Scholium. It is very evident that the converse of the above proposition is not true, that is to say, it cannot be inferred that two equivalent rhomboids shall have the bases and altitudes equal; for it has been shown (Prop. I.) that the rhomboids AG, GC, are equivalent (see the diagram) where the base GF must be longer than the base GE, provided AB is longer than AD, for then the angle ADB being greater than ABD (Prop. XIX. B. II.), the angle EGB, which is equal to ADB, is greater than EBG; consequently, EB is longer than EG, but EB is equal to GF, therefore GF is longer than GE.

2. It is however, true, that equivalent rhomboids upon the same base have equal altitudes, for if the altitude of one be supposed less than that of the other, and the side opposite its base be prolonged, a portion of the other rhomboid must be cut off thereby, and the remaining portion still be equal to the former rhomboid, by the proposition, which is absurd; the altitudes therefore are equal. Having shown this, we may further prove that *equivalent rhomboids of equal altitudes have also equal bases*, for they are equivalent to rectangles of the same bases and altitudes: now, any side of a rectangle may be considered as the base; taking then those sides as bases which are equal to the altitude of the rhomboids, the other sides or altitudes are, as shown above, equal, and these altitudes are the bases of the rhomboids: the bases are therefore equal.

PROPOSITION IV. THEOREM.

Every triangle is half the parallelogram or rhomboid which has the same base and the same altitude.



Let the parallelogram BCDA, and the triangle BCA have the same base BC, and the same altitude, then will the triangle be half the parallelogram.

For (Prop. XXXII. Cor. 3. B. II.), the triangles ABC, ACD, are equal.

Cor. 1. Hence a triangle ABC is half of the rectangle BCEF, which has the same base CB, and the same altitude AO: for the rectangle BCEF is equivalent to the parallelogram ABCD.

Cor. 2. All triangles, which have equal bases and altitudes, are equivalent.

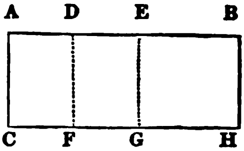
Cor. 3. Hence triangles having equal altitudes are to each other as their bases; conversely, triangles having equal bases are to each other as their altitudes.

Cor. 4. Therefore a triangle is equivalent to a rhomboid of equal base and of half its altitude, or to one of equal altitude and of half its base.

PROPOSITION V. THEOREM.

If there be two right lines, of which one is divided into parts, the sum of the rectangles contained by the undivided line and the several parts of the other, will be equal to the rectangle contained by the two whole lines.

Let the lines be AB, AC, of which the former is divided into the parts AD, DE, EB, then the rectangles contained by AC, and each of these parts, are together equal to the whole rectangle AH, contained by AB, AC.



Let DF, EG, be parallel to AC, then the angles FDE, GEB, being each equal to the angle A, the rhomboids AF, DG, EH, are rectangles, and DF, EG, being each equal to AC (Prop. XXXII. Cor. 4. B. II.), these rectangles are contained by AC, and the several parts of AB, and as they make up the whole rectangle AH, they are together equal to it.

Cor. 1. The square of a line is equivalent to twice the rectangle of the whole line and the half thereof.

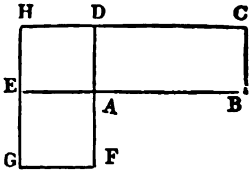
Cor. 2. Hence, rectangles having the same altitude are to each other as their bases.

PROPOSITION VI. THEOREM.

Any two rectangles are to each other as the products of their bases multiplied by their altitudes.

Let ABCD, AEGF, be two rectangles; then will the rectangle,

ABCD : AEGF::AB.AD : AF.AE.



Having placed the two rectangles, so that the angles at A are vertical, produce the sides GE, CD, till they meet in H. The two rectangles ABCD, AEHD, having the same altitude AD, are to each other as their bases AB, AE : in like manner the two rectangles AEHD, AEGF, having the same altitude AE, are to each other as their bases AD, AF : thus we have the two proportions,

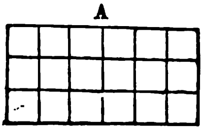
$$ABCD : AEHD : : AB : : AE,$$
$$AEHD \cdot AEGF : : AD : AF.$$

Multiplying the corresponding terms of these proportions together, and observing that the term AEHD may be omitted, since it is a multiplier of both the antecedent and the consequent, we shall have

$$ABCD \cdot AEGF : : AB \times AD : AE \times AF.$$

Scholium. The product of the base by the altitude, may therefore be assumed as the measure of a rectangle.

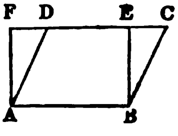
But the more common and simple method is to assume the square as the measuring unit of surfaces; and that square has been adopted, whose side is the unit of length, or a linear unit. For example, if the length of the rectangle A is six units and its altitude three, then 6×3, or 18, represents the number of superficial units or squares contained in the rectangle, and each side of these squares is equal to a linear unit of the same name as the square. If the side of the square is a linear inch, the rectangle contains 18 square inches; if the side of the square is a foot, the rectangle contains 18 square feet.



PROPOSITION VII. THEOREM.

The area of any rhomboid is equal to the product of its base by its altitude.

For, the parallelogram ABCD is equivalent to the rectangle ABEF, which has the same base AB, and the same altitude BE (Prop. III. Cor.): but this rectangle is measured by AB×BE (Prop. VI. Sch.); therefore, AB×BE is equal to the area of the parallelogram ABCD.



Cor. Parallelograms of the same base are to each other as their altitudes; and parallelograms of the same altitude are to each other as their bases: for, let B be the common base, and C and D the altitudes of two parallelograms:

then, $B \times C : B \times D : : C : D$, (Prop. XIII. B. I.)

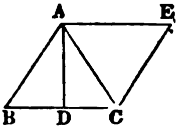
And if A and B be the bases, and C the common altitude, we shall have

$$A\times C : B\times C :: A : B.$$

And parallelograms, generally, are to each other as the products of their bases and altitudes.

PROPOSITION VIII. THEOREM.

The area of a triangle is equal to the product of its base by half its altitude.

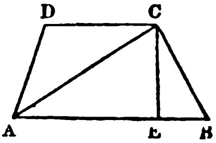


For, the triangle ABC is half of the parallelogram ABCE, which has the same base BC, and the same altitude AD (Prop. IV.); but the area of the parallelogram is equal to $BC\times AD$ (Prop. VII.); hence that of the triangle must be $\frac{1}{2}BC\times AD$, or $BC\times\frac{1}{2}AD$.

Cor. Two triangles of the same altitude are to each other as their bases, and two triangles of the same base are to each other as their altitudes. And triangles generally, are to each other, as the products of their bases and altitudes.

PROPOSITION IX. THEOREM.

The area of a trapezium is equal to its altitude multiplied by half the sum of its parallel bases.



Let ABCD be a trapezium, CE its altitude, AB and CD its parallel bases; then will its area be equal to $CE\times\frac{1}{2}(AB+CD.)$

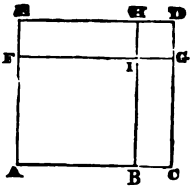
Join AC; then since $CE\times\frac{1}{2}AB$ is the measure of the triangle ABC (Prop. VIII.), and since $CE\times\frac{1}{2}DC$ is the measure of the triangle ADC, the measure of both the triangles is $CE\times\frac{1}{2}(AB+CD.)$ But the sum of the two triangles $ABC+ACD$ is equal to the trapezium ABCD (Ax. 8.), therefore the area of $ABCD=CE\times\frac{1}{2}(AB+CD)$, (Ax. 1.)

PROPOSITION X. THEOREM.

If a line be divided into two parts, the square described on the whole line is equivalent to the sum of the squares described on the parts, together with twice the rectangle contained by the parts.

Let AC be the line, and B the point of division; then, is

$$AC^2 \text{ or } (AB+BC)^2=AB^2+BC^2+2AB\times BC.$$



Construct the square ACDE; take AF=AB; draw FG parallel to AC, and BH parallel to AE.

The square ACDE is made up of four parts; the first ABIF is the square described on AB, since we made AF=AB: the second IGDH is the square described on IG, or BC; for since we have AC=AE and AB=AF, the difference AC—AB must be equal to the difference AE—AF, which gives BC=EF; but IG is equal to BC, and DG to EF, since the lines are parallel; therefore IGDH is equal to a square described on BC. And those two squares being taken away from the whole square, there remains the two rectangles BCGI, EFIH, each of which is measured by AB×BC: hence, the large square is equivalent to the two small squares, together with the two rectangles.

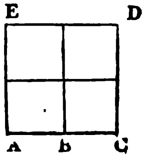
Cor. 1. If the line AC were divided into two equal parts, the two rectangles EI, IC, would become squares, and the square described on the whole line would be equivalent to four times the square on half the line.

Cor. 2. Hence it may be inferred, that the square of the whole line is equal to the square of one of the parts *plus* twice the rectangle of the whole line and the other part, *minus* the square of that part. For AD is the square of the whole line AC, and AI is the square of one of the parts, FD+DB twice the rectangle of the whole line and the other part; in which HG, the square of BC, is taken twice.

Cor. 3. We may likewise infer, that the square of the whole line, *minus* the square of one of the parts, *minus* twice the rectangle of its parts, is equal to the square of the other part.

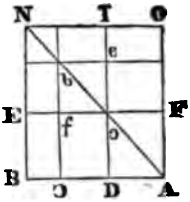
Scholium. This property is equivalent to the property demonstrated in algebra, in obtaining the square of a binomial, which is expressed thus:

$$(a+b)^2=a^2+2ab+b^2.$$



Scholium 2. If a right line AC is bisected, the square described on the whole line is equivalent to four times the square described on half the line. This explains the apparent paradox in arithmetic, that the square of a half is equal to a fourth. For the square of AB, the half of AC, is equivalent to a fourth of the square of AC.

Cor. 4. If a line be divided into three or any number of parts, the square described on the whole line will be equal to the sum of the squares of each of the parts, together with twice the rectangles of the parts into each other.



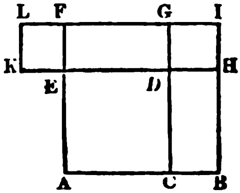
Let AB be the line, and D and C the points of division; then will $AB^2 = AD^2 + CB^2 + DC^2 + 2AD.DC + 2AD.CB + 2DC.CB$.

For FD is the square of the part AD, and *bc* the square of the part DC, and N*b* the square of the part BC, and the complements of these several squares are D*f* =F*e*, each equal to AD×DC, and O*e*= CE, each equal to AD×CB, and I*b*=E*b*, each equal to CB×DC. Now these several squares and their complements make up the square OB. The same may be shown for any number of parts into which a line may be divided. This is equivalent to the algebraical expression $(a+b+c)^2 = a^2 + 2ab + 2ac + b^2 + 2bc + c^2$.

PROPOSITION XI. THEOREM.

The square described on the difference of two lines is equivalent to the sum of the squares described on the lines, minus twice the rectangle contained by the lines.

Let AB and BC be two lines, AC their difference; then is AC^2, or $(AB-BC)^2 = AB^2 + BC^2 - 2AB \times BC$.



Describe the square ABIF; take AE =AC; draw CG parallel to BI, HK parallel to AB, and complete the square EFLK.

The two rectangles CBIG, GLKD, are each measured by AB×BC; take them away from the whole figure ABILKEA, which is equivalent to $AB^2 + BC^2$, and there will evidently remain the square ACDE; hence the theorem is true.

Scholium. This proposition is equivalent to the algebraical formula, $(a-b)^2 = a^2 - 2ab + b^2$.

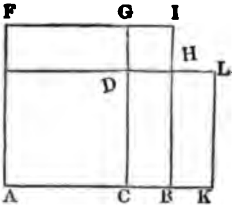
PROPOSITION XII. THEOREM.

The rectangle contained by the sum and the difference of two lines, is equivalent to the difference of the squares of those lines.

Let AB, BC, be two lines; then will

$$(AB+BC) \times (AB-BC) = AB^2 - BC^2.$$

On AB and AC, describe the squares ABIF, ACDE; produce AB till the produced part BK is equal to BC; and complete the rectangle AKLE.



The base AK of the rectangle EK, is the sum of the two lines AB, BC; its altitude AE is the difference of the same lines; therefore the rectangle AKLE is equal to (AB+BC)×(AB—AC). But this rectangle is composed of the two parts ABHE+BHLK; and the part BHLK is equal to the rectangle EDGF, because BH is equal to DE, and BK to EF; hence AKLE is equal to ABHE+EDGF. These two parts make up the square ABIF *minus* the square DHIG, which latter is equal to a square described on BC: hence we have

$$(AB+AC)\times(AB-BC)=AB^2-BC^2.$$

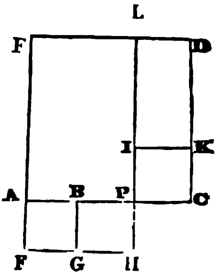
Scholium. This proposition is equivalent to the algebraical formula, $(a+b)\times(a-b)=a^2-b^2$.

PROPOSITION XIII. THEOREM.

The sum of the squares on two lines is equivalent to half the square on their sum, together with half the square on their difference.

The squares on the two lines AB, AC, are together equivalent to half the square on their sum, and half the square on BC their difference.

Let AG be the square on AB, and AD the square on CD, and through P, the middle of BC, let LH, parallel to EA, be drawn, meeting FG produced in H; make PI equal to PH, and draw IK parallel to PC.



Since the angles at A are right, EAF is a straight line, so that EH is a rectangle, and it is contained by lines equal to the sum of AC, AB, and half that sum AP; it is therefore equal to half the square of the sum AC, AB, (Prop. V. Cor. 1.) Again, IP, PC, being equal, by construction, to PH, PB, the rectangles PK, PG, are equal; hence the two squares AD, EG, are together equivalent to the two rectangles EH, LK; now, CK being equal to BG or AB, KD is equal to BC, for EC is a square; the rectangle LK is thus contained by lines equivalent to BC, and the half thereof PC, and is consequently equal to half the square on BC; it therefore follows that the squares AD, AG, are together equivalent to half the square on the sum of AC, AB, and half the square on BC, the difference of AC, AB.

Cor. Hence, twice the sum of the squares of two lines is equivalent to the squares of their sum and difference.

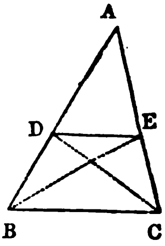
Scholium. This proposition is equivalent to the algebraical formula, $\frac{(a+b)^2}{2}+\frac{(a-b)^2}{2}=a^2+b^2$.

PROPOSITION XIV. THEOREM.

If a line be drawn parallel to the base of a triangle, it will divide the sides proportionally.

Let ABC be a triangle, and DE a straight line drawn parallel to the base BC; then will

AD : DB::AE : EC.



Draw BE and DC. The two triangles BDE, DEC, having the same base DE, and the same altitude, since both their vertices lie in a line parallel to the base, are equivalent (Prop. IV. Cor. II.)

The triangles ADE, BDE, whose common vertex is E, have the same altitude, and are to each other as their bases (Prop. VIII. Cor.) Hence we have

ADE : BDE::AD : DB.

The triangles ADE, DEC, whose common vertex is D, have also the same altitude, and are to each other as their bases; hence ADE : DEC::AE : EC.

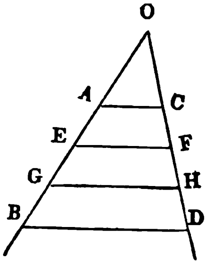
But the triangles BDE, DEC, are equivalent; and therefore we have (Prop. XIX. Cor. B. I.),

AD : DB::AE : EC.

Cor. 1. Hence, by composition, we have AD+DB : AD:: AE+EC : AE, or AB : AD::AC : AE; and also AB : BD:: AC : CE.

Cor. 2. If between two straight lines AB, CD, any number of parallels AC, EF, GH, BD, &c. be drawn, those straight lines will be cut proportionally, and we shall have AE : CF : EG : FH : GB : HD.

For, let O be the point where AB and CD meet. In the triangle OEF, the line AC being drawn parallel to the base EF, we shall have OE : AE::OF : CF, or OE : OF::AE : CF. In the triangle OGH, we shall likewise have OE : EG::OF : FH, or OE : OF::EG : FH. And by reason of the common ratio OE::OF, those two proportions give AE : CF :: EG : FH. It may be proved in the same manner, that EG : FH::GB : HD, and so on; hence the lines AB, CD, are cut proportionally by the parallels AC, EF, GH, &c.

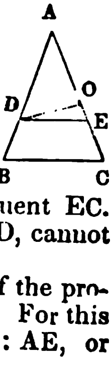


PROPOSITION XV. THEOREM.

Conversely, if two sides of a triangle are cut proportionally by a right line, this line will be parallel to the third side.

In the triangle ABC, let the line DE be drawn, making AD : DB::AE : EC; then will DE be parallel to BC.

For, if DE is not parallel to BC, draw DO parallel to it. Then, by the preceding theorem, we shall have AD : DB::AO : OC. But, by hypothesis, we have AD : DB::AE : EC; hence we must have AO : OC::AE : EC, or AO : AE::OC : EC: an impossible result, since AO, the one antecedent, is less than its consequent AE, and OC, the other antecedent, is greater than its consequent EC. Hence the parallel to BC, drawn from the point D, cannot differ from DE; hence DE is that parallel.



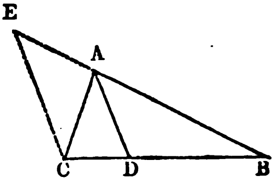
Scholium. The same conclusion would be true, if the proportion AB : AD::AC : AE were the proposed one. For this proportion would give AB—AD : AD::AC—AE : AE, or BD : AD::CE : AE.

PROPOSITION XVI. THEOREM.

The line which bisects any angle of a triangle divides the opposite sides into portions, which are to each other as the adjacent sides.

In the triangle ACB, let AD be drawn, bisecting the angle CAB; then will

BD : CD::AB : AC.



Through the point C, draw CE parallel to AD till it meets BA produced.

In the triangle BCE, the line AD is parallel to the base CE; hence we have the proportion (Prop. XIV.),

BD : DC::AB : AE.

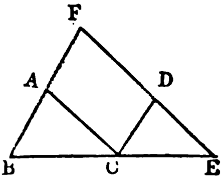
But the triangle ACE is isosceles : for, since AD, CE, are parallel, we have the angle ACE=DAC, and the angle AEC =BAD (Prop. XXIII. Cor. 3. B. II.) ; but, by hypothesis, DAC=BAD; hence the angle ACE=AEC, and consequently AE=AC (Prop. XV. B. II.) In place of AE in the above proportion, substitute AC, and we shall have BD : DC::AB : AC.

Scholium. The converse of this proposition is true, viz: If a line from the vertex of any angle of a triangle divide the side opposite into portions which are to each other as the sides adjacent, the line so drawn bisects the angle.

For, let the line AD divide BC so that BD : DC::BA : AC, then is the angle BAD=the angle CAD. Draw CE parallel to DA, meeting BA produced in E. Then (Prop. XIV.) BD : DC::BA : AE ; but, by hypothesis, BD : DC::BA : AC, therefore BA : AE::BA : AC ; consequently AE=AC, and therefore the angle ACE is equal to the angle AEC ; but because of the parallels AD, EC, the angles ACE, AEC, are respectively equal to the angles DAC, DAB; these angles are therefore equal, and consequently AD bisects the angle BAC.

PROPOSITION XVII. THEOREM.

Two equiangular triangles have their homologous sides proportional and are similar.



Let ABC, CDE, be two triangles which have their angles equal each to each, viz: BAC=CDE, ABC=DCE and ACB= DEC ; then the homologous sides, or the sides adjacent to the equal angles, will be proportional, so that we shall have BC : CE::AB : CD::AC : DE.

Place the homologous sides BC, CE, in the same straight line, and produce the sides BA, ED, till they meet in F.

Since BCE is a straight line, and the angle BCA is equal to CED, it follows that AC is parallel to DE (Prop. XXII. Cor. 2. B. II.) In like manner, since the angle ABC is equal to DCE, the line AB is parallel to DC. Hence the figure ACDF is a parallelogram.

In the triangle BFE, the line AC is parallel to the base FE; hence we have BC : CE::BA : AF (Prop. XIV.); or putting CD in the place of its equal AF,

BC : CE::BA : CD.

In the same triangle BEF, CD is parallel to BF which may be considered as the base: and we have the proportion BC : CE::FD : DE; or putting AC in the place of its equal FD,

BC : CE::AC::DE.

And finally, since both these proportions contain the same ratio BC : CE, we have

AC : DE::BA : CD.

Thus the equiangular triangles BAC, CED, have their homologous sides proportional. But two figures are similar when they have their angles equal, each to each, and their homologous sides proportional (Def. 1.); consequently the equiangular triangles BAC, CED, are two similar figures.

Scholium. Observe, that in similar triangles, the homologous sides are opposite to the equal angles; thus the angle ACB being equal to DEC, the side AB is homologous to DC; in like manner, AC and DE are homologous, because they are opposite to the equal angles ABC, DCE. When the homologous sides are determined, it is easy to form the proportions:

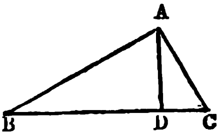
AB : DC::AC : DE::BC : CE.

Cor. 1. For the similarity of two triangles, it is enough that they have two angles equal, each to each, since then, the third will also be equal in both, and the two triangles will be equiangular.

Cor. 2. Hence, if from the right angle of a right angled triangle a perpendicular be let fall on the hypothenuse, the two partial triangles thus formed, will be similar to each other and to the whole triangle, since (Prop. XXIII. Cor. 2. B. III.) they are equiangular.

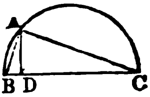
Cor. 3. Hence, also, we have BD : BA : : BA : BC, or BA^2=BD.BC, and DC : AC::AC : BC, or AC^2=DC.BC.

Therefore either side adjacent to the right angle of a right angled triangle is a mean proportional between the hypothenuse and the adjacent segment.



Cor. 4. And since the triangles BDA, ADC, are similar, by comparing their homologous sides we have BD : AD::AD :

DC, or BD.DC=AD². Hence the perpendicular let fall from the right angle of a right angled triangle on the hypothenuse is a mean proportional between the segments of the hypothenuse.



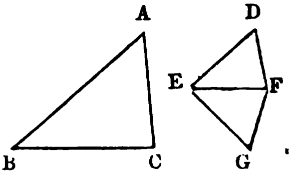
Cor. 5. If from a point A, in the circumference of a circle, two chords AB, AC, be drawn to the extremities of a diameter BC, the triangle BAC will be right angled at A (Prop. XIX. Cor. 2. B. III.); hence, first, the perpendicular AD is a mean proportional between the two segments BD, DC, of the diameter, or what is the same, AD^2=BD.DC.

Hence also, in the second place, the chord AB is a mean proportional between the diameter BC and the adjacent segment BD, or, what is the same, AB^2=BD.BC.

Cor. 6. Since the homologous sides of similar triangles are proportional, the sum of their sides or their perimeters are proportional also as their homologous sides.

PROPOSITION XVIII. THEOREM.

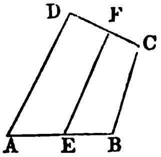
Conversely, two triangles, which have their homologous sides proportional, are equiangular and similar.



In the two triangles BAC, DEF, suppose we have BC : EF::AB : DE::AC : DF; then will the triangles ABC, DEF, have their angles equal, namely, A=D, B=E, C=F.

At the point E, make the angle FEG=B, and at F, the angle EFG =C; the third G will be equal to the third A, and the two triangles ABC, EFG, will be equiangular. Therefore, by the last theorem, we shall have BC : EF::AB : EG; but, by hypothesis, we have BC : EF::AB : DE; hence EG=DE. By the same theorem, we shall also have BC : EF::AC : FG; and, by hypothesis, we have BC : EF::AC : DF; hence FG =DF. Hence the triangles EGF, DEF, having their three sides equal, each to each, are themselves equal (Prop. XIII. B. II.) But, by construction, the triangles EGF and ABC are equiangular: hence DEF and ABC are also equiangular and similar.

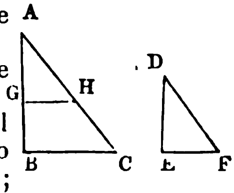
Scholium 1. By the last two propositions, it appears that in triangles, equality among the angles is a consequence of proportionality among the sides, and conversely; so that either of those conditions sufficiently determines the similarity of two triangles. The case is different with regard to figures of more than three sides: even in quadrilaterals, the proportion between the sides may be altered without altering the angles, or the angles may be altered without altering the proportion between the sides; and thus proportionality among the sides cannot be a consequence of equality among the angles of two quadrilaterals, or *vice versa.* It is evident, for example, that by drawing EF parallel to BC, the angles of the quadrilateral AEFD, are made equal to those of ABCD, though the proportion between the sides is different; and, in like manner, without changing the four sides AB, BC, CD, AD, we can make the point B approach D, or recede from it, which will change the angles.



Scholium 2. The two preceding propositions, which in strictness form but one, together with that relating to the squares described on the sides of a right angled triangle, (see Prop. XXIV. following) are the most important and fertile in results of any in geometry: they are almost sufficient of themselves for every application to subsequent reasoning, and for solving every problem. The reason is, that all figures may be divided into triangles, and any triangle into two right angled triangles. Thus the general properties of triangles include, by implication, those of all figures.

PROPOSITION XIX. THEOREM.

Two triangles, which have an angle of the one equal to an angle of the other, and the sides containing those angles proportional, are similar.



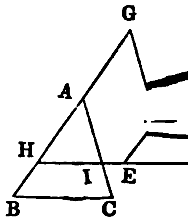
In the two triangles ABC, DEF, let the angles A and D be equal; then, if AB : DE::AC : DF, the two triangles will be similar.

Take AG=DE, and draw GH parallel to BC. The angle AGH will be equal to the angle ABC (Pr. XXIII. Cor. 3. B. II.); and the triangles AGH, ABC, will be equiangular: hence we shall have AB : AG::AC : AH. But, by hypothesis, we have AB : DE::AC : DF; and, by construction, AG=DE; hence HA=DF. The two triangles AGH, DEF, have an equal angle included between equal sides; therefore they are equal; but the triangle AGH is similar to ABC: therefore DEF is also similar to ABC.

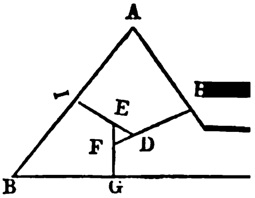
PROPOSITION XX. THEOREM.

Two triangles, which have their homologous sides paralle perpendicular to each other, are similar.

Let BAC, EDF, be two triangles.



First. If the side AB is parallel to DE, and BC to FF, the angle ABC will be equal to DEF (Prop. XXVI. B. II.); if AC is parallel to DF, the angle ACB will be equal to DFE, and also BAC to EDF; hence the triangles ABC, DEF, are equiangular; and consequently they are similar (Prop. XVII.)



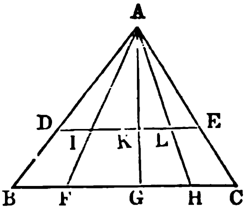
Secondly. If the side DE is perpendicular to AB, and the side DF to AC, the two angles I and H of the quadrilateral AIDH will be right angles; and since all the four angles are together equal to four right angles (Prop. XXVIII. Cor. 1. B. II.), the remaining two IAH, IDG, will be together equal to two right angles. But the two angles E
IDH, are also equal to two right angles: hence the a
EDF is equal to IAH or BAC. In like manner, if the t
side EF is perpendicular to the third side BC, it ma
shown that the angle DFE is equal to C, and DEF t
hence the triangles ABC, DEF, which have the sides of
one perpendicular to the corresponding sides of the ot
are equiangular and similar.

Scholium.—In the case of the sides being parallel, the
mologous sides are the parallel ones: in the case of thei
ing perpendicular, the homologous sides are the perpen
lar ones. Thus in the latter case DE is homologous
AB, DF with AC, and EF with BC.

The case of the perpendicular sides might present a
tive position of the two triangles different from that exh
ed in the diagram. But we might always conceive a t
gle DEF to be constructed within the triangle ABC, and
that its sides should be parallel to those of a triangle c
pared with ABC; and then the demonstration given in
text would apply.

PROPOSITION XXI. THEOREM.

In any triangle, if a line be drawn parallel to the base, all lines drawn from a vertex will divide the base an parallel into proportional parts.



Let DE be parallel to the base BC, and the other lines drawn as in the figure; then will DI : BF::IK : FG :: KL : GH.

For, since DI is parallel to BF, the triangles ADI and ABF are equiangular; and we have DI : BF::AI : AF: and since IK is parallel to FG, we have in like manner AI : AF::IK : FG; hence the ratio AI : AF being common, we shall have DI : BF::IK : FG. In the same manner we shall find IK : FG::KL :GH; and so with the other segments: hence the line DE is divided at the points I, K, L, in the same proportion, as the base BC, at the points F, G, H.

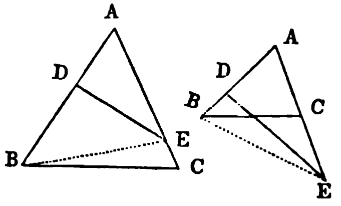
Cor. Therefore if BC were divided into equal parts at the points F, G, H. the parallel DE would also be divided into equal parts at the points I, K. L.

PROPOSITION XXII. THEOREM.

Two triangles, having an angle in each equal, are to each other as the rectangles of the sides which contain the equal angles.

In the two triangles ABC, ADE, let the angle A be equal to the angle A; then will the triangle

ABC : ADE::AB.AC : AD.AE.



Draw BE. The triangles ABE, ADE, having the common vertex E, have the same altitude, and consequently are to each other as their bases (Prop. VIII. Cor.): that is

ABE : ADE::AB : AD.

In like manner,

ABC : ABE::AC : AE.

Multiply together the corresponding terms of these proportions, omitting the common term ABE, we have

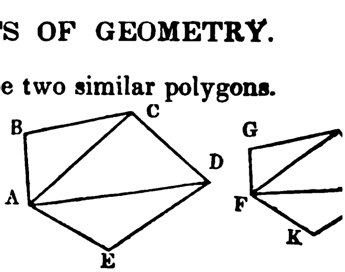
ABC : ADF::AB.AC : AD.AE.

Cor. Hence the two triangles would be equivalent, if the rectangle AB.AC were equal to the rectangle AD.AE, or if we had AB : AD::AE : AC; which would happen if DC were parallel to BE.

PROPOSITION XXIII. THEOREM.

Two similar polygons are composed of the same number of triangles, similar each to each and similarly situated.

Let ABCDE, FGHIK, be two similar polygons.



From any angle A, in the polygon ABCDE, draw diagonals AC, AD, to the other angles. From the homologous angle F, in the other polygon FGHIK, draw diagonals FH, FI, to the other angles.

These polygons being similar, the angles ABC, which are homologous, must be equal, and the side BC, must also be proportional to FG, GH, that is, A ::BC : GH (Def. 1.) Wherefore the triangles ABC, have each an equal angle, contained between propor sides; hence they are similar (Prop. XIX.); therefo angle BCA is equal to GHF. Take away these equ gles from the equal angles BCD, GHI, and there re ACD=FHI. But since the triangles ABC, FGH, are si we have AC : FH::BC : GH; and, since the polygo similar, BC : GH::CD : HI; hence AC : FH::CD : H the angle ACD, we already know, is equal to FHI; the triangles ACD, FHI, have an equal angle in each, i ed between proportional sides, and are consequently si (Prop. XIX.) In the same manner it might be show all the remaining triangles are similar, whatever be the ber of sides in the polygons proposed: therefore two si polygons are composed of the same number of triangl milar and similarly situated.

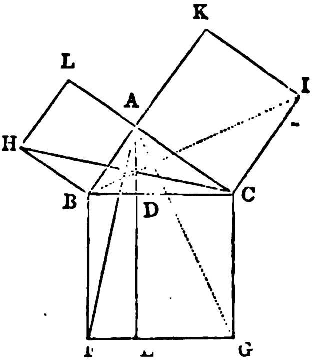
Scholium. 1. The converse of the proposition is eq true: If two polygons are composed of the same num triangles similar and similarly situated, those two pol will be similar.

For, the similarity of the respective triangles will gi angles ABC=FGH, BCA=GHF, ACD=FHI: hence B GHI, likewise CDE=HIK, &c. Moreover we shall ha : DG::BC : GH::AC : FH::CD : HI, &c.; hence th polygons have their angles equal and their sides proport consequently they are similar.

Corollary. The contours or perimeters of similar gons are to each other as their homologous sides, sinc perimeters of similar triangles of which they are com are in that ratio, (Prop. XVII. Cor. 6.)

PROPOSITION XXIV. THEOREM.

The square described on the hypothenuse of a right angled triangle is equivalent to the sum of the squares described on the other two sides.



Let the triangle ABC be right angled at A. Having described squares on the three sides, let fall from A, on the hypothenuse, the perpendicular AD, which produce to E: and draw the diagonals AF, CH.

The angle ABF is made up of the angle ABC, together with the right angle CBF: the angle CBH is made up of the same angle ABC, together with the right angle ABH; hence the angle ABF is equal to HBC. But we have AB=BH, being sides of the same square; and BF=BC, for the same reason: therefore the triangles ABF, HBC, have two sides and the included angle in each equal; therefore they are themselves equal (Prop. VIII. B. II.)

The triangle ABF is half of the rectangle BE, because they have the same base BF, and the same altitude BD (Prop. IV. Cor. 1.) The triangle HBC is in like manner half of the square AH; for the angles BAC, BAL, being both right angles, AC and AL form one and the same straight line parallel to HB (Prop. IV. B. II.): and consequently the triangle HBC, and the square AH, which have the common base BH, have also the common altitude AB; hence the triangle is half of the square.

The triangle ABF has already been proved equal to the triangle HBC; hence the rectangle BDEF, which is double of the triangle ABF, must be equivalent to the square AH, which is double of the triangle HBC. In the same manner it may be proved, that the rectangle CDEG is equivalent to the square AI. But the two rectangles BDEF, CDEG, taken together, make up the square BCGF: therefore the square BCGF, described on the hypothenuse, is equivalent to the sum of the squares ABHL, ACIK, described on the two other sides; in other words, $BC^2=AB^2+AC^2$.

Cor. 1. Hence the square of one of the sides of a right angled triangle is equivalent to the square of the hypothenuse diminished by the square of the other side; which is thus expressed: $AB^2 = BC^2 - AC^2$.

Cor. 2 It has just been shown that the square AH is equivalent to the rectangle BDEF; but by reason of the common altitude BF, the square BCGF is to the rectangle BDEF as the base BC is to the base BD; therefore we have

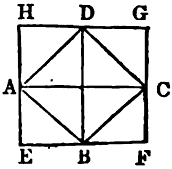
$$BC^2 : AB^2 :: BC : BD.$$

Hence the square of the hypothenuse is to the square of one of the sides about the right angle, as the hypothenuse is to the segment adjacent to that side. The word segment here denotes that part of the hypothenuse, which is cut off by the perpendicular let fall from the right angle: thus BD is the segment adjacent to the side AB; and DC is the segment adjacent to the side AC. We might have in like manner,

$$BC^2 : AC^2 :: BC : CD.$$

Cor. 3. The rectangles BDFE, DCGE, having likewise the same altitude, are to each other as their bases BD, CD. But these rectangles are equivalent to the squares AH, AI; therefore we have $AB^2 : AC^2 :: BD : DC$.

Hence the squares of the two sides containing the right angle, are to each other as the segments of the hypothenuse which lie adjacent to those sides.

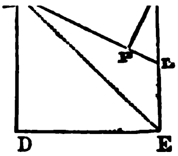


Cor. 4. Let ABCD be a square, and AC its diagonal: the triangle ABC being right angled and isosceles, we shall have $AC^2 = AB^2 + BC^2 = 2AB^2$; hence the square described on the diagonal AC, is double of the square described on the side AB.

This property may be exhibited more plainly, by drawing parallels to AC, through the points B and D. A new square EFGH will thus be formed, equal to the square of AC. Now EFGH evidently contains eight triangles each equal to ABE; and ABCD contains four such triangles: hence EFGH is double of ABCD.

Since we have $AC^2 : AB^2 :: 2 : 1$; by extracting the square roots, we shall have $AC : AB :: \sqrt{2} : 1$.

That the square described on the hypothenuse of a right angled triangle is equivalent to the sum of the squares described on the other two sides, may be otherwise demonstrated as follows:



ed on the hypothenuse in the t L; produce KC one side of quare erected on the side AC, eet the line BL in P. Pro- also DB to meet the line AG, side of the square FA, in N. n the angle HAC draw AM parallel to BC. Then will riangle ACM, which is equal to half the rhomboid BA- (Prop. IV.), be equal to the triangle PBC, which is l to half the rhomboid BAPC, since the triangle ABC is non to both rhomboids and is the half of each. Also in homboid BNCL, the triangles BCL and BCN are equal p. IV.) each to each; and if from each we take the ls BCP and BCA respectively, we have the triangle equal to the triangle CLP. Hence the triangle BPC+ triangle CPL is equal to the two triangles ACM+BAN; since in the triangle BCL the perpendicular CP on the BL divides the triangle into two similar triangles BPC CPL (Prop. XXI. Cor. 2.), which are similar also to M and BAN (Prop. XX.), since their homologous sides parallel, hence we have BA : AN::AC : CM::BC : CL op. XVI.) Draw the diagonals BG, AK and BE; then the ngles formed by these lines with the sides of the squares ectively will be to each other as the former triangles ectively, viz: BAG : BAN::ACK : ACM::BCE : BCL; by composition, BAG+ACK : BAN+ACM::BCE : ; hence if the triangle BAN+the triangle ACM=the ngle BCL, then also the triangle BAG+the triangle ACK e triangle BCE. Hence the squares FA+HC being douif similar triangles are described on the three sides of a right angled triangle, if the sides of the right angled triangle form homologous sides of the similar triangles, that described on the hypothenuse will be equivalent to the sum of the other two.

Cor. B. Hence, also, if similar rhomboids are described on the three sides of a right angled triangle, the one described on the hypothenuse will be equivalent to the sum of the other two, since similar rhomboids are the doubles of similar triangles, having the same bases and altitudes.

Cor. C. And since the similar triangles BCL, ACM, BAN, described on the sides BC, AC, AB, are proportional to the squares CD, CH, AF, described on the same sides, being homologous sides of the similar triangles; we may further enunciate that similar triangles are proportional to the squares described on their homologous sides.

Cor. D. It having been shown that the sum of the triangles ACK+BAG=the triangle BCE, and that the sum ACM +BAN=BCL, it follows that ACK—ACM+BAG—BAN= AMK+BNG=BLE, which are also similar triangles, and similarly related to their respective sides of the triangle ABC, since they intercept with those sides respectively the similar triangles ACM, BAN, BCL. And hence we may infer that of similar triangles similarly situated in reference to the sides of a right angled triangle, that referred to the hypothenuse is equivalent to the sum of those referred to the other two sides. And hence also, similar triangles similarly related to the sides of a right angled triangle, or similarly drawn in reference to their sides, are also proportional to the squares described on those sides respectively.

Cor. E. And since similar polygons are composed of similar triangles similarly placed (Prop. XXIII.), if similar polygons are described on the hypothenuse and on the other two sides of a right angled triangle, the sides of the triangle forming homologous sides of the similar polygons, the polygon described on the hypothenuse will be equivalent to the sum of the other two.

Cor. F. Hence also, as in similar triangles (Cor. C.), and for the same reasons, similar polygons are proportional to the squares described on their homologous sides.

Cor. G. Since it has been shown that of similar triangles similarly related to the sides of a right angled triangle, those similar triangles are proportional to the squares on the several sides of the triangle to which they are respectively referred, and since this is true, in whatever manner the triangles are drawn in reference to those sides, it follows that similar triangles, and also similar polygons, are proportional to the squares described on any lines similarly drawn in each.

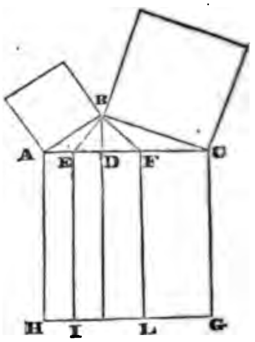
Scholium. Since BP : BC :: BC : BL, the product of the extremes will be equal to that of the means, or $BC^2=BP.BL$. For the same reason, we have $CL^2=PL.BL$; therefore $BC^2+CL^2=BP.BL+PL.BL=(BP+PL).BL=BL.BL=BL^2$; or the square described on the hypothenuse BL is equal to the two squares described on the two sides BC, CL, as we have previously shown in the proposition.

The property of the equality of the square of the hypothenuse with the sum of the squares of the other two sides of a right angled triangle, and that of the proportionality of the sides of equiangular triangles on which the former is perhaps dependent, are the two fundamental propositions of geometry, and those on which nearly all the others depend. No wonder, then, that Pythagoras, the discoverer of this, should have sacrificed a hundred oxen to the *Gods*, for his discovery, in those days of Mythology, as some historians have informed us that he did.

PROPOSITION XXV. THEOREM.

If a square be described on each of the sides of any triangle, as ABC, *and if the lines* BE *and* BF *be drawn in such manner that the angles* AEB *and* CFB *shall each be equal to the angle* B; *and lastly, if* EI *and* FL *be drawn parallel to* CG *or* AH, *the sides of the square; the square of* AB *will be equal to the rectangle* AI, *and the square of* BC= *the rectangle* CL; *consequently the sum of the squares on* AB *and* BC *will be equal to the square described on the base* AC, *less the rectangle* EL, *if the angle* B *be obtuse, and plus the same rectangle, if the angle* B *be acute.*

For the angles ABC and AEB are equal by hypothesis, and because the triangles ABC and AEB have the angle at A common and the angle at E equal to the angle at B, the remaining angles ABE and ACB are equal (Prop. XXVII. Cor. 2. B. II.); hence the triangles ABC and AEB are similar. Therefore AC : AB::AB : AE (Proposition XVII); whence it follows that the rectangle of AC×AE, or AE×AH, is equal to the square of AB.



In like manner it may be shown that the square of BC is equal to the rectangle CL.

But it is plain that, if the angle B be obtuse, the line BE will fall between the points A and D, and the line DF between C and D; the contrary of which is the case if the angle B be acute; and that these two lines coincide with the perpendicular BD, when the angle B is a right angle.

In the first case then, it is evident that the sum of the squares of the sides is less than the square of the base by the rectangle EL; and in the second case that they exceed it by the rectangle EL.

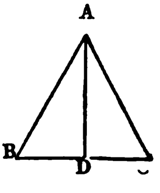
Lastly, that if the triangle be right angled at B, as the rectangle EL vanishes, the sum of the squares of the sides is equal to the square of the base.

Scholium. This theorem, it may be observed is a generalization of the XXIVth Proposition.

PROPOSITION XXVI. THEOREM.

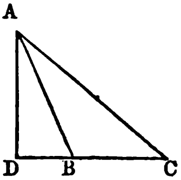
In every triangle, the square of a side opposite an acute angle is less than the sum of the squares of the other two sides, by twice the rectangle contained by the base and the part of the base, or the base produced, included between the perpendicular from the opposite angle and the acute angle on the base.

Let ABC be a triangle, and AD perpendicular to the base CB; then will $AB^2=AC^2+BC^2-2BC\times CD$.



There are two cases.

First. When the perpendicular falls within the triangle ABC, we have $BD=BC-CD$, and consequently $BD^2=BC^2+CD^2-2BC \times CD$ (Prop. XI.) Adding AD^2 to each, and observing that the right angled triangles ABD, ADC, give $AD^2+BD^2=AB^2$, and $AD^2+CD^2=AC^2$, we have $AB^2=BC^2+AC^2-2BC \times CD$.

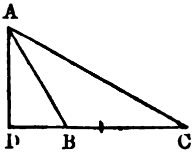


Secondly. When the perpendicular AD falls without the triangle ABC, we have $BD=CD-BC$; and consequently $BD^2=CD^2+BC^2-2CD \times BC$ (Prop. XI.) Adding AD^2 to both, we find, as before, $AB^2=BC^2+AC^2-2BC \times CD$.

PROPOSITION XXVII. THEOREM.

In any triangle having an obtuse angle, the square of the side opposite thereto exceeds the squares of the base and other side, by twice the rectangle of the base and the distance of the perpendicular from the vertex of the obtuse angle.

In the triangle ABC, let B be an obtuse angle, AD the perpendicular on the prolongation of the base BC, then will the square of AC be equivalent to the squares of AB, BC, together with twice the rectangle of BC, BD.



For the square of AC is equivalent to the squares of CD, DA (Prop. XXIV.), and the square of CD is equivalent to the squares of CB, BD, together with twice the rectangle of CB, BD (Prop. X.); therefore the square of AC is equivalent to the squares of the three lines CB, BD, DA, and twice the rectangle of CB, BD, that is, to the squares of CB, BA, and twice the rectangle of CB, BD.

Cor. 1. From the last two propositions, the converse of proposition XXIV. immediately follows, that is, it the square of any side of a triangle be equivalent to the sum of the squares of the other two sides, the angle opposite the former shall be right; for these propositions show, that if such equivalence exist, the angle can neither be acute nor obtuse.

Cor. 2. We may, moreover, readily infer the converse of these two propositions themselves, that is, first, if in the triangle ABC (see diagram to Prop. XXVI.) the square of AB is equivalent to the squares of AC, BC, diminished by twice the rectangle of BC, CD, the angle C shall be acute; for by the above proposition and proposition XXIV. if this angle were either obtuse or right, the said equivalence could not exist. Again, if in the triangle ABC the square of AC is equivalent to the squares of AB, BC, together with twice the rectangle of CB, BD, the angle B, opposite AC, shall be obtuse; for by last proposition, and proposition XXIV., this angle can neither be acute nor right.

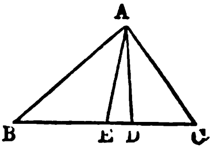
Scholium. The last corollary may obviously be expressed in a more unrestricted form, thus: If the square of any side of a triangle is less than the sum of the squares of the other two sides, the angle opposite the former side is acute, but if it is greater than that sum, the opposite angle is obtuse.

PROPOSITION XXVIII. THEOREM.

In any triangle, if a right line be drawn from the vertex to the middle of the base, twice the square of this line, together with twice the square of half of the base, is equivalent to the sum of the squares of the other two sides of the triangle.

Let ABC be the triangle, and AE a line drawn to the middle of the base BC; then will

$$2AE^2+2BE^2=AB^2+AC^2.$$



On BC, let fall the perpendicular AD. Then, by Prop. XXVI.

$$AC^2=AE^2+EC^2-2EC\times ED.$$

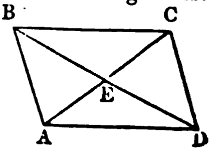
And by Prop. XXVII.

$$AB^2=AE^2+EB^2+2EB\times ED.$$

Hence, by adding, and observing that EB and EC are equal, we have

$$AB^2+AC^2=2AE^2+2EB^2.$$

Cor. Hence, in every parallelogram, the squares of the sides are together equivalent to the squares of the diagonals.



For the diagonals AC, BD, bisect each other (Prop. XXXV. B. II.); consequently the triangle ABC gives

$$AB^2+BC^2=2AE^2+2BE^2.$$

The triangle ADC gives, in like manner,

$$AD^2+DC^2=2AE^2+2DE^2.$$

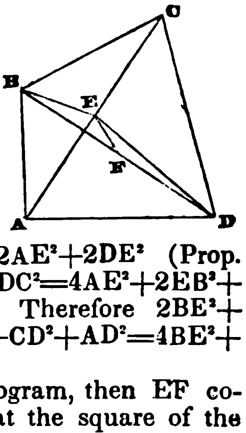
Adding the corresponding members together, and observing that BE and DE are equal, we shall have

$$AB^2+AD^2+DC^2+BC^2=4AE^2+4DE^2.$$

But $4AE^2$ is the square of $2AE$, or of AC; $4DE^2$ is the square of BD (Prop. X. Cor. 1.): hence the squares of the sides are together equivalent to the squares of the diagonals.

PROPOSITION XXIX. THEOREM.

In every quadrilateral, the sum of the sqares of the four sides is equal to the sum of the squares of the two diagonals, plus four times the square of the line that joins the middle of these diagonals.



Let ABCD be a quadrilateral figure, the two diagonals of which are AC and BD; and let us suppose them bisected in E and F; join EF: then will the sum of the squares of the four sides be equal to the sum of the squares of the two diagonals, plus four times the square of the line EF.

For, join E to B and D. Then $AB^2+BC^2=2AE^2+2EB^2$, and $AD^2+DC^2=2AE^2+2DE^2$ (Prop. XXVIII.); therefore $AB^2+BC^2+AD^2+DC^2=4AE^2+2EB^2+2ED^2$. But $BE^2+DE^2=2BF^2+2EF^2$. Therefore $2BE^2+2ED^2=4BF^2+4EF^2$. And $AB^2+BC^2+CD^2+AD^2=4BE^2+4BF^2+4EF^2=AC^2+BD^2+4FE^2$.

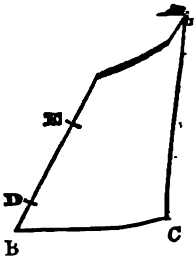
Cor. If the quadrilateral is a parallelogram, then EF coincides, and the proposition shows that the square of the sides of a parallelogram are together equal to the squares of the diagonals, as formerly shown in Cor. to Prop. XXVIII.

Scholium. The preceding proposition, therefore, is only a particular case of the present one.

PROPOSITION XXX. THEOREM.

The square of the difference of the hypothenuse and sum of the base and perpendicular of a right angled triangle, is equal to twice the rectangle of the difference of the hypothenuse and each of the other sides.

In the right angled triangle ABC, the square of $(AC+BC-AB)=$twice the rectangle of $(AB-BC)\times(AB-AC.)$



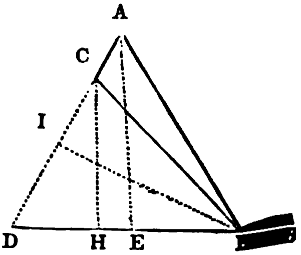
For $AC^2+CB^2=AB^2$ (Prop. XXIV.) Now if we set off from B toward A a distance $BE=BC$, and from A toward B a distance $AD=AC$, we shall have $BE^2+AD^2=AB^2$. Now $AD^2=AE^2+ED^2+2AE.ED$ (Prop. X.), and $BE^2=BD^2+ED^2+2ED.BD$. Therefore, BE^2+AD^2 or $AB^2=BD^2+AE^2+2ED^2+2BD.ED+2AE.ED$. But $AB^2=BD^2+AE^2+ED^2+2AE.DB+2ED.DB+2BD.ED$ (Prop. X. Cor. 4.); and this being put equal to the other expression for AB^2, and cancelling the common terms, there will remain $ED^2=2AE.DB$.

Cor. By an inverse process, it may be shown, that if twice the rectangle AE.DB be equal to the square DE, the line AB will be the hypothenuse of a right angled triangle ACB.

Scholium. By this proposition we are enabled to find rational numbers for the three sides of a right angled triangle. For since $AE.DB=\frac{1}{2}DE^2$, if DE be expressed by a whole number and $\frac{1}{2}DE^2$ be resolved into factors, then $AE+DE$, $DB+DE$ and $AE+ED+DB$ will be the two legs and hypothenuse of a right angled triangle. Let $DE=2n$, then $\frac{1}{2}DE^2=2n^2=AE.DB$; now take $AE=2n^2$ and $DB=1$; then the sides will be $2n^2+2n$, $2n+1$, and $2n^2+2n+1$; if $n=1$, the sides are 3, 4 and 5; if $n=2$, the sides are 5, 12 and 13 &c. If AE be taken$=2n$ and $DB=n$, then the sides will be $3n$, $4n$ and $5n$.

PROPOSITION XXXI. THEOREM.

The area of any triangle, referred to a base drawn from one of its angles to meet the opposite side produced, is equal to half the product of the base with the difference of perpendiculars from the other two angles to this base.

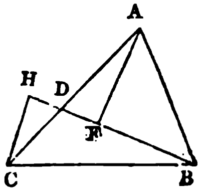


Let ABC be a triangle, and let BD be a base exterior to the triangle, drawn from the angle B to meet the opposite side AC produced in D; then will the area of the triangle ABC be equal to half the product of the base BD into the difference of the perpendiculars AE and CH drawn from the two angles A and C to the base BD.

For, from the angle B draw BI perpendicular to AD; then in the three triangles ABD, BCD and ABC, all of the same altitude BI, their areas are as their several bases AD, CD and AC (Prop. IV. Cor. 3.), and (Prop. VIII.) area $ABD=\frac{1}{2}AD\times BI$, area $BCD=\frac{1}{2}CD\times BI$, and area $ABC=(\frac{1}{2}AD-\frac{1}{2}DC)\times BI$ or $\frac{1}{2}AC\times BI$. And reciprocally on the base BD, the triangle $ABD=\frac{1}{2}BD\times AE$, triangle $BCD=\frac{1}{2}BD\times CH$, and triangle ABC=triangle ABD—triangle $BCD=\frac{1}{2}BD\times(AE-CH.)$

Scholium. If the base, instead of being exterior, should fall within the triangle, as BD, then the area of the triangle ABC is equal to half the product of the base BD into the sum of the perpendiculars AF+CH let fall from the angles A and C on this base or base produced.

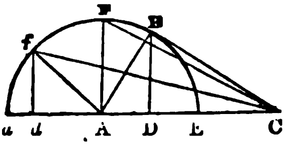
For the base BD divides the triangle ABC into the two triangles ABD and CBD, which are respectively equal to the base BD multiplied by one half their respective altitudes AF and CH on that base.



PROPOSITION XXXII. THEOREM.

The area of every triangle is equal to half the product of any two of its sides reduced in the ratio of either of those sides to the perpendicular drawn to the other side, or the other side produced from the opposite angle.

Let ABC be a triangle; from the angle B let fall the perpendicular BD; then will the area of the triangle be equal to half the product of the side AC multiplied by AB, reduced in the ratio of the side AB to the perpendicular BD.



Produce AD toward *d*, and from A as a centre, with the distance AB, describe the semicircle *a*FE; then from the point A draw FA perpendicular to AC, and join FC. It is here evident, that the two triangles FAC and BAC are respectively to each other as AF or AB is to BD; but the area of the triangle AFC is equal to half the product of the base AC multiplied by the altitude FA. Therefore the area of the triangle ABC is equal to the area of the triangle AFC, reduced in the ratio of AF or AB to BD. Thus AB : BD : : ½AB.BC : the area of the triangle ABC.

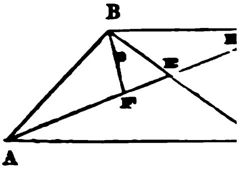
Again, draw A*f* and *f*C, making the triangle A*f*C; draw *fd* perpendicular to *a*C: then will the triangle AFC be to the triangle A*f*C as AF or A*f* to *fd*.

Thus, AF or $Af : fd :: \frac{1}{2}AC \times (AF$ or $Af)$: the tria AfC.

Scholium. The merits of this theorem will be appreci when we come to treat of trigonometry ; for it may the shown that it will not even be necessary to know the le of the perpendicular BD, but the area may be determine having the two sides and their included angle given, hal product of which must be multiplied by the sine of such gle; and the sine may be shown to be the ratio of the of a triangle to a perpendicular drawn from one of the a cent angles to the opposite side.

PROPOSITION XXXIII. THEOREM.

The area of a trapezium is equal to half the product o two diagonals reduced in the ratio of one of those di nals to the sum of the two perpendiculars from its extr ties to the other diagonals.



Let ABCD be a trapezium, and AC and BD its diagonals, and let BF and DH be perpendiculars from the extremities of the diagonal BD to the diagonal AC. Then will the area of the trapezium be equal to half the product or rectangle of the two diagonals reduced in the ratio of the diagonal B the perpendiculars BF and DH.

For, let the trapezium be divided into the four tria AEB, BEC, CED and AED: and it is evident by last pr sition, that

$BE : BF :: \frac{1}{2}AE.BE$: area of the triangle AEB,
and $BE : BF :: \frac{1}{2}EC.BE$: area of the triangle BEC,
and $ED : DH :: \frac{1}{2}EC.ED$: area of the triangle CED,
and $ED : DH :: \frac{1}{2}AE.ED$: area of the triangle AED.

Whence by addition we have

$2BE+2ED : 2BF+2DH :: \frac{1}{2}AE.BE+\frac{1}{2}EC.BE+\frac{1}{2}EC.E$ $\frac{1}{2}AE.ED$: (area $AEB+BEC+CED+AED$)=the t zium ABCD:

and $BE+ED : BF+DH :: (\frac{1}{2}AE+\frac{1}{2}EC)\times BE+(\frac{1}{2}EC+\frac{1}{2}$ $\times ED$:

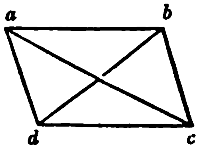
and $BE+ED : BF+DH :: (\frac{1}{2}AE+\frac{1}{2}EC)\times(BE+ED)$: pezium ABCD.

or $1 : BF+DH :: \frac{1}{2}(AE+EC)$: area ABCD.

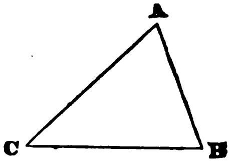
Hence, $\frac{1}{2}(AE+EC)\times(BF+DH)$=area of the trapeziu

Cor. 1. Hence it follows that if two quadrilateral fig have their diagonals equal, and intersecting each other at the same angle, whatever may be their difference in other respects, these quadrilaterals will be equal as to their area.

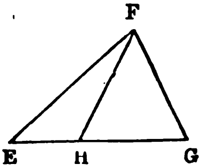
Thus the trapezium ABCD (see last figure) is equal to the parallelogram *abcd*, when its diagonals are equal to those of the trapezium and inclined toward each other at the same angle.



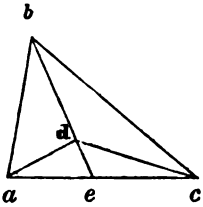
Cor. 2. Hence also the triangle ABC, if two of its sides AC and CB are equal to the two diagonals of the trapezium and inclined at the same angle, will be equal to the trapezium.



Cor. 3. And in the triangle EFG, if the base EG and the line FH equal the diagonals, and if FH is inclined to EG in the same angle, the area will be equal to that of the trapezium.

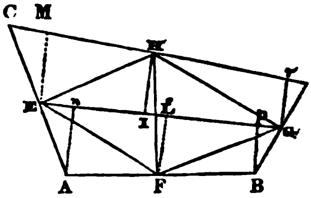


Cor. 4. And lastly in the quadrilateral *abcd*, the diagonals of which do not cut each other, if *ac* and *db* are equal to AC and DB, and if the angle *bec* is equal to the angle BEC, the areas of the two figures will be equal.



PROPOSITION XXXIV. THEOREM.

If the sides of any quadrilateral be bisected, and if lines be drawn from the points of bisection so as to form another quadrilateral within the former, the quadrilateral so formed will be a rhomboid, or the opposite sides will be equal and parallel.



Let EFGH be a quadrilateral formed by joining the bisections of the sides of the quadrilateral ABDC, and the quadrilateral EFGH will be a rhomboid.

For, draw the diagonals EG, FH, and on the diagonal EG let fall the perpendiculars An, ME, FL, HI, Bp, rG, and in tl triangle EGF, its altitude FL is equal to $\frac{1}{2}An+\frac{1}{2}Bp$, and tl altitude IH of the triangle EGH on this same base EG equal to $\frac{1}{2}EM+\frac{1}{2}rG$; and because AE=EC, and since An and EMC are similar triangles, An is equal to EM; hen also $Bp=Gr$. Therefore the altitude IH=the altitude Ll Hence the triangle EGF=the triangle EGH; and in tl same manner it may be shown that the triangle FHG equal to the triangle FHE. Hence the quadrilateral mustl a rhomboid.

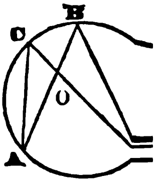
Cor. If from the points of bisection of the opposite sid of a quadrilateral, lines be drawn, those lines will bise each other.

PROPOSITION XXXV. THEOREM.

The segments of two chords which intersect each other in circle, are reciprocally proportional.

Let the chords AB and CD intersect at O: then will

AO : DO::OC : OB.



Draw AC and BD. In the triangles ACO, BOD, the angles at O are equal, being vertical; the angle A is equal to the angle D, because both are inscribed in the same segment (Prop. XIX. Cor. 1. B. III.); for the same reason the angle C=B; the triangles are therefore similar, and the homologous sides give the proportion

AO : DO::CO : OB.

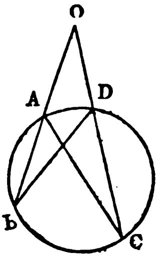
Cor. Therefore AO.OB=DO.CO; hence the rectangl der the two segments of the one chord is equal to the angle under the two segments of the other.

PROPOSITION XXXVI. THEOREM.

If from the same point without a circle, two secants be drawn terminating in the common arc, the whole secants will be reciprocally proportional to their external segments.

Let the secants OB, OC, be drawn from the point O: then will

OB . OC::OD : OA.



For, drawing AC, BD, the triangles OAC, OBD have the angle O common; likewise the angle B=C (Prop. XIX. Cor. 1. B. III.); these triangles are therefore similar; and their homologous sides give the proportion

OB : OC::OD : OA.

Cor. Hence the rectangle OA.OB is equal to the rectangle OC.OD.

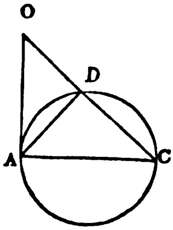
Scholium. This proposition, it may be observed, bears a great analogy to the preceding, and differs from it only as the two chords AB, CD, instead of intersecting each other within, cut each other without the circle. The following proposition may also be regarded as a particular case of the proposition just demonstrated.

PROPOSITION XXXVII. THEOREM.

If from the same point without a circle, a tangent and secant be drawn, the tangent will be a mean proportional between the secant and its external segment.

From the point O, let the tangent OA, and the secant OC, be drawn; then will

OC : OA::OA : OD, or $OA^2=OC.OD$.



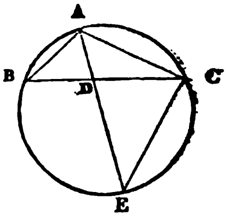
For, drawing AD and AC, the triangles OAD, OAC, have the angle O common; also the angle OAD, formed by a tangent and a chord, has for its measure half of the arc AD (Prop. XXI. B. III.); and the angle C has the same measure: hence the angle OAD=C; therefore the two triangles are similar, and we have the proportion OC : OA::AO : OD, which gives $OA^2=OC.OD$

PROPOSITION XXXVIII. THEOREM.

If either angle of a triangle be bisected by a line terminating in the opposite side, the rectangle of the sides including the bisected angle, is equivalent to the square of the bisecting line together with the rectangle contained by the segments of the third side.

In the triangle BAC, let AD bisect the angle A; then will

$$AB.AC=AD^2+BD.DC.$$



Describe a circle through the three points A, B, C; produce AD till it meets the circumference, and draw CE.

The triangle BAD is similar to the triangle EAC; for, by hypothesis, the angle BAD=EAC; also the angle B=E, since they are both measured by half of the arc AC; hence these triangles are similar, and the homologous sides give the proportion BA : AE::AD : AC; hence BA.AC=AE.AD; but AE=AD+DE, and multiplying each of these equals by AD, we have AE.AD=AD²+AD.DE; now AD.DE=BD.DC (Prop. XXXV. Cor.); hence, finally,

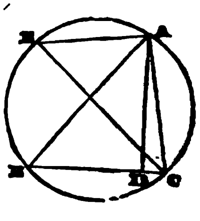
$$BA.AC=AD^2+BD.DC.$$

PROPOSITION XXXIX. THEOREM.

In every triangle, the rectangle contained by two sides is equivalent to the rectangle contained by the diameter of the circumscribed circle, and the perpendicular let fall upon the third side.

In the triangle ABC, let AD be drawn perpendicular to BC; and let EC be the diameter of the circumscribed circle: then will

$$AB.AC=AD.CE.$$



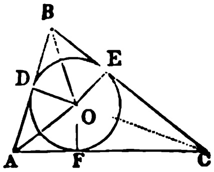
For, drawing AE, the triangles ABD, AEC, are right angled, the one at D, the other at A: also the angle B=E; these triangles are therefore similar, and they give the proportion AB : CE::AD : AC; hence AB.AC=CE.AD.

Cor. If these equal quantities be multiplied by the same quantity BC, there will result AB.AC.BC=CE.AD.BC; now AD.BC is double of the area of the triangle (Prop. VIII.); therefore the product of three sides of a triangle is equal to its area multiplied by twice the diameter of the circumscribed circle.

The product of three lines is sometimes called a *solid*, for a reason that will be given in another place. Its value is easily conceived, by imagining that the lines are reduced into numbers, and multiplying these numbers together.

Scholium. It may also be demonstrated, that the area of a triangle is equal to its perimeter multiplied by half the radius of the inscribed circle.

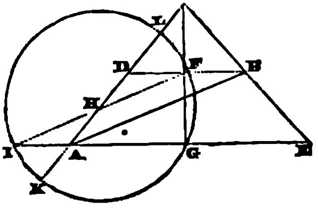
For, the triangles AOB, BOC, AOC, which have a common vertex at O, have for their common altitude the radius of the inscribed circle; hence the sum of these triangles will be equal to the sum of the bases AB, BC, AC, multiplied by half the radius OD; hence the area of the triangle ABC is equal to the perimeter multiplied by half the radius of the inscribed circle.



PROPOSITION XL. THEOREM.

If from half the sum of the several sides of any triangle each of the sides are severally taken, and their remainders are multiplied into each other, and their product into the half sum, then the area of the triangle is equal to the square root of their product.

For, let ABC be the given triangle. From the angle B draw BD meeting the side AC, making CD=CB, and from the angle A draw AE parallel to DB to meet the side CB produced in E, and CE will be equal to CA. Draw also CFG bisecting DB and AE perpendicularly in F and G, and from F draw FHI parallel to the side BA meeting the side AD in H, and AE produced in I. Then with the centre H and radius HF describe a circumference meeting AC produced in K, which will also pass through G, since G is a right angle, and through I, because AI is equal and parallel to DF; therefore HD=HA, and HF =HI=½AB.



Hence HA or HD is half the difference of the sides AC, CB, and HC=half their sum or$=\frac{1}{2}AC+\frac{1}{2}CB$; also $HK=HI=\frac{1}{2}IF$ or $\frac{1}{2}AB$; consequently, $CK=\frac{1}{2}AC+\frac{1}{2}CB+\frac{1}{2}AB$, half the sum of all the three sides of the triangle ABC; or CK= $\frac{1}{2}$ that sum, which sum we will call S. Again, $HK=HI=\frac{1}{2}IF=\frac{1}{2}AB$, or $KL=AB$; therefore $CL=CK-KL=\frac{1}{2}S-AB$, and $AK=CK-CA=\frac{1}{2}S-AC$, and $AL=DK=CK-CD=\frac{1}{2}S-CB$.

Now AG.CG=the triangle ACE, and AG.FG=the triangle ABE; therefore AG.CF=triangle ACB. Also AG : CG : : DF or IA : CF; therefore AG.CF=triangle ACB=CG.IA =CG.DF. Consequently, AG.CF.CG.DF=square of the triangle ACD.

But $CG.CF=CK.CL=\frac{1}{2}S\times(\frac{1}{2}S-AB)$, and $AG.DF=AK.AL=(\frac{1}{2}S-AC)\times(\frac{1}{2}S-BC.)$ Therefore AG.CF.CG.DF=triangle $ACB^2=\frac{1}{2}S\times(\frac{1}{2}S-AB)\times(\frac{1}{2}S-AC)\times(\frac{1}{2}S-BC.)$ Hence $\sqrt{(\frac{1}{2}S\times(\frac{1}{2}S-AB)\times(\frac{1}{2}S-AC)\times(\frac{1}{2}S-BC))}$=the area of the triangle ABC.

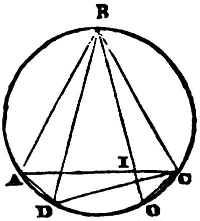
Cor. Hence, to find the radius of a circle inscribed in a triangle whose sides are given, all that is necessary is to multiply together the difference of each side and the half sum, and divide the square root of this product by half the sum of the sides, since it is shown (Prop. XXXIX. Schol.) that the area of a triangle is equal to its perimeter multiplied by half the radius of its inscribed circle.

PROPOSITION XLI. THEOREM.

In every quadrilateral inscribed in a circle, the rectangle of the two diagonals is equivalent to the sum of the rectangles of the opposite sides.

In the quadrilateral ABCD, we shall have

$$AC.BD=AB.CD+AD.BC.$$



Take the arc CO=AD, and draw BO meeting the diagonal AC in I.

The angle ABD=CBI, since the one has for its measure half of the arc AD, and the other, half of CO, equal to AD; the angle ADB=BCI, because they are both inscribed in the same segment AOB; hence the triangle ABD is similar to the triangle IBC, and we have the proportion AD : IC : : BD : BC; hence AD.BC=CI.BD. Again, the triangle ABI is similar to the triangle BDC ; for the arc AD being equal to CO, if OD be added to each of them, we shall have the arc AO =DC; hence the angle ABI is equal to DBC; also the angle BAI to BDC, because they are inscribed in the same segment: hence the triangles ABI, DBC, are similar, and the homologous sides give the proportion AB : BD::AI : CD; hence AB.CD=AI.BD.

Adding the two results obtained, and observing that

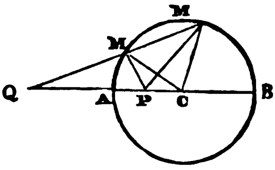
$$AI.BD.CI.BD=(AI+CI).BD=AC.BD,$$

we shall have

$$AD.BC+AB.CD=AC.BD.$$

PROPOSITION XLII. THEOREM.

If a point be taken on the radius of a circle, and this radius be then produced and a second point taken on it without the circumference of the circle; these points being so situated, that the radius of the circle shall be a mean proportional between their distances from the centre; if lines be drawn from those points to any points of the circumference, the ratio (of these lines) will be constant.



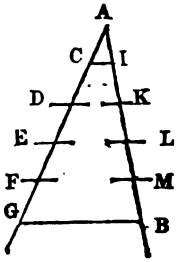
Let P be the point within the circumference, and Q the point without; then if CP : CA::CA : CQ, the ratio of QM and MP will be the same, for all positions of the point M.

For by hypothesis, CP : CA:: CA : CQ; or substituting CM for CA, CP : CM::CM : CQ; hence the triangles CPM, CQM, have each an equal angle C contained by proportional sides; hence they are similar (Prop. XIX.); and hence the third side MP is to the third side MQ, as CP is to CM or CA. But by division, the proportion CP : CA::CA : CQ gives CP : CA::CA—CP : CQ—CA, or CP : CA::AP : AQ ; therefore MP : MQ::AP : AQ.

PROBLEMS RELATING TO THE FOURTH BOOK.

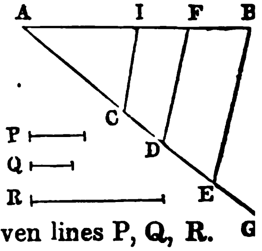
PROBLEM I.

To divide a given straight line into any number of equal parts, or into parts proportional to given lines.



First. Let it be proposed to divide the line AB into five equal parts. Through the extremity A, draw the indefinite straight line AG; and taking AC of any magnitude, apply it five times upon AG; join the last point of division G, and the extremity B, by the straight line GB; then draw CI parallel to GB: AI will be the fifth part of the line AB; and thus, by applying AI five times upon AB, the line AB will be divided into five equal parts.

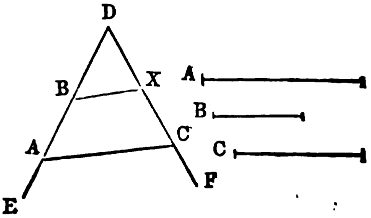
For, since CI is parallel to GB, the sides AG, AB, are cut proportionally in C and I (Prop. XIV.) But AC is the fifth part of AG, hence AI is the fifth part of AB.



Secondly. Let it be proposed to divide the line AB into parts proportional to the given lines P, Q, R. Through A, draw the indefinite line AG; make AC=P, CD=Q, DE=R; join the extremities E and B; and through the points C, D, draw CI, DF, parallel to EB; the line AB will be divided into parts AI, IF, FB, proportional to the given lines P, Q, R.

PROBLEM II.

To find a fourth proportional to three given lines, A, B, C.

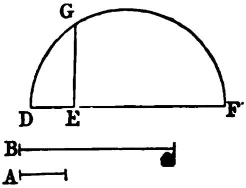


Draw the two indefinite lines DE, DF, forming any angle with each other. Upon DE take DA=A, and DB=B: upon DF take DC=C; draw AC; and through the point B, draw BX parallel to AC; DX will be the fourth proportional required; for, since BX is parallel to AC, we have the proportion DA : DB::DC : DX; now the first three terms of this proportion are equal to the three given lines; consequently DX is the fourth proportional required.

Cor. A third proportional to two given lines A, B, may be found in the same manner, for it will be the same as a fourth proportional to the three lines A, B, B.

PROBLEM IV.

To find a mean proportional between two given lines A *and* B.

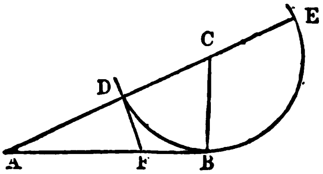


Upon the indefinite line DF, take DE=A, and EF=B; upon the whole line DF, as a diameter, describe the semicircle DGF; at the point E, erect upon the diameter the perpendicular EG meeting the circumference in G; EG will be the mean proportional required.

For, the perpendicular EG, let fall from a point in the circumference upon the diameter, is a mean proportional between DE, EF, the two segments of the diameter (Prop. XVII. Cor. 4.); and these segments are equal to the given lines A and B.

PROBLEM IV.

To divide a given line into two parts, such that the greater part shall be a mean proportional between the whole line and the other part.



Let AB be the given line.

At the extremity B of the line AB, erect the perpendicular BC equal to the half of AB; from the point C, as a centre, with the radius CB, describe a semicircle; draw AC cutting the circumference in D; and take AF=AD: the line AB will be divided at the point F in the manner required; that is, we shall have AB : AF::AF : FB.

For, AB being perpendicular to the radius at its extremity, is a tangent; and if AC be produced till it again meets the circumference in C, we shall have AE : AB::AB : AD (Prop. XXVII.); hence, by division, AE—AB : AB::AB—AD : AD.

But since the radius is the half of AB, the diameter DE equal to AB, and consequently AE—AB=AD=AF; al because AF=AD, we have AB—AD=FB; hence AF : A ::FB : AD or AF; whence, by exchanging the extremes the means, AB : AF::AF : FB.

Cor. Since AB=DE, the proportion AE : AB::AB : A enables us to perform the following somewhat singular pr blem, viz: *To increase a given line, so that the given li may be a mean proportional between the whole line so increas and the part added.* All that is necessary after performi the above construction is to add DA to AB.

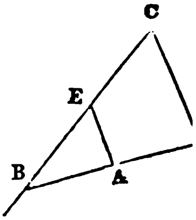
Scholium. This sort of division of the line AB is call division in extreme and mean ratio: the use of it will be pe ceived in a future part of the work. It may further be c served, that the secant AE is divided in extreme and me ratio at the point D; for, since AB=DE, we have AE : I ::DE : AD.

PROBLEM V.

Through a given point, in a given angle, to draw a line that the segments comprehended between the point and t two sides of the angle shall be equal.

Let BCD be the given angle, and A the given point.

Through the point A, draw AE parallel to CD, make BE=CE, and through the points B and A draw BAD; this will be the line required.

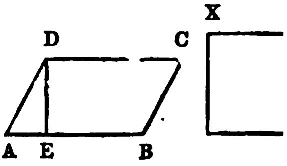


For, AE being parallel to CD, we have BE : EC :: BA : AD; but BE=EC; therefore BA=AD.

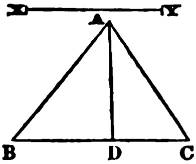
PROBLEM VI.

To describe a square that shall be equivalent to a given par lelogram, or to a given triangle.

First. Let ABCD be the given parallelogram, AB its base, DE its altitude: between AB and DE find a mean proportional XY; then will the square described upon XY be equivalent to the parallelogram ABCD.



For, by construction, $AB : XY :: XY : DE$; therefore, $XY^2 = AB.DE$; but $AB.DE$ is the measure of the parallelogram, and XY^2 that of the square; consequently, they are equivalent.



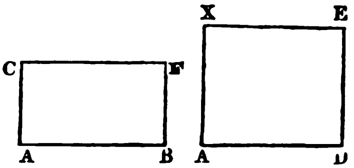
Secondly. Let ABC be the given triangle, BC its base, AD its altitude: find a mean proportional between BC and the half of AD, and let XY be that mean; the square described upon XY will be equivalent to the triangle ABC.

For, since $BC \,.\, XY :: XY : \frac{1}{2}AD$, it follows that $XY^2 = BC.\frac{1}{2}AD$: hence the square described upon XY is equivalent to the triangle ABC.

PROBLEM VII.

Upon a given line, to describe a rectangle that shall be equivalent to a given rectangle.

Let AD be the line, and ABFC the given rectangle.



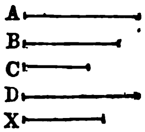
Find a fourth proportional to the three lines AD, AB, AC, and let AX be that fourth proportional; a rectangle constructed with the lines AD and AX will be equivalent to the rectangle ABFC.

For, since $AD : AB :: AC : AX$, it follows that $AD.AX = AB.AC$; hence the rectangle ADEX is equivalent to the rectangle ABFC.

PROBLEM VIII.

To find two lines whose ratio shall be the same as the ratio of two rectangles contained by given lines.

Let A.B, C.D, be the rectangles contained by the given lines A, B, C, and D.



Find X, a fourth proportional to the three lines B, C, D; then will the two lines A and X have the same ratio to each other as the rectangles A.B and C.D.

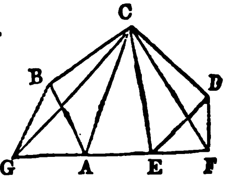
For, since $B : C :: D : X$, it follows that $C.D = B.X$; hence $A.B : C.D :: A.B : B.X :: A : X$.

Cor. Hence, to obtain the ratio of the squares described upon the given lines A and C, find a third proportional X to the lines A and C, so that A : C::C : X; you will then have

$$A.X \quad C^2, \text{ or } A^2.X \quad A.C^2; \text{ hence}$$
$$A^2 : C^2 :: A : X.$$

PROBLEM IX.

To find a triangle that shall be equivalent to a given polygon.



Let ABCDE be the given polygon. Draw first the diagonal CE cutting off the triangle CDE ; through the point D, draw DF parallel to CE, and meeting AE produced; draw CF : the polygon ABCDE will be equivalent to the polygon ABCF, which has one side less than the original polygon.

For, the triangles CDE, CFE, have the base CE common, they have also the same altitude, since their vertices D and F, are situated in a line DF parallel to the base : these triangles are therefore equivalent (Prop. IV. Cor. 2.) Add to each of them the figure ABCE, and there will result the polygon ABCDE, equivalent to the polygon ABCF.

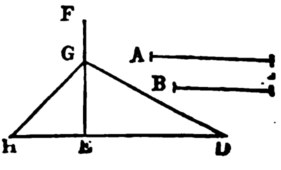
The angle B may in like manner be cut off, by substituting for the triangle ABC the equivalent triangle AGC, and thus the pentagon ABCDE will be changed into an equivalent triangle GCF.

The same process may be applied to every other figure; for, by successively diminishing the number of its sides, one being retrenched at each step of the process, the equivalent triangle will at last be found.

Scholium. We have already seen that every triangle may be changed into an equivalent square (Prob. VI.); and thus a square may always be found equivalent to a given rectilineal figure, which operation is called *squaring* the rectilineal figure, or finding the *quadrature* of it.

PROBLEM X.

To find the side of a square which shall be equivalent to the sum or the difference of two given squares.



Let A and B be the sides of the given squares.

First. If it is required to find a square equivalent to the sum of these squares, draw the two indefinite lines ED, EF, at right angles to each other; take ED=A, and EG

=B; draw DG: this will be the side of the square required.

For, the triangle DEG being right angled, the square described upon DG is equivalent to the sum of the squares upon ED and EG.

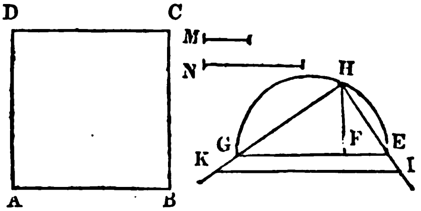
Secondly. If it is required to find a square equivalent to the difference of the given squares, form in the same manner the right angle FEH; take GE equal to the shorter of the sides A and B; from the point G as a centre, with a radius GH, equal to the other side, describe an arc cutting EH in H: the square described upon EH will be equivalent to the difference of the squares described upon the lines A and B.

For, the triangle GEH is right angled, the hypothenuse GH=A, and the side GE=B; hence the square described upon EH, is equivalent to the difference of the squares A and B.

Scholium. A square may thus be found, equivalent to the sum of any number of squares; for a similar construction which reduces two of them to one, will reduce three of them to two, and these two to one, and so of others. It would be the same, if any of the squares were to be subtracted from the sum of the others.

PROBLEM XI.

To find a square which shall be to a given square as a given line to a given line.



Let AC be the given square, and M and N the given lines.

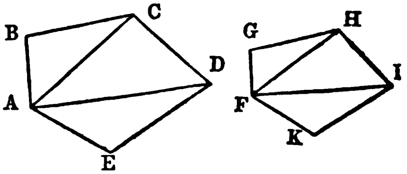
Upon the indefinite line EG, take EF=M, and FG=N; upon EG as a diameter describe a semicircle, and at the point F erect the perpendicular FH. From the point H, draw the chords HG, HE, which produce indefinitely: upon the first, take HK equal to the side AB of the given square, and through the point K draw KI parallel to EG; HI will be the side of the square required.

For, by reason of the parallels KI, GE, we have $HI : HK :: HE : HG$; hence, $HI^2 : HK^2 :: HE^2 : HG^2$: but in the right angled triangle EHG, the square of HE is to the square of HG as the segment EF is to the segment FG (Prop. XXIV. Cor. 3.), or as M is to N; hence $HI^2 : HK^2 :: M : N$. But

HK=AB; therefore the square described upon HI is to the square described upon AB as M is to N.

PROBLEM XII.

Upon a given line, to describe a polygon similar to a given polygon.



Let FG be the given line, and AEDCB the given polygon.

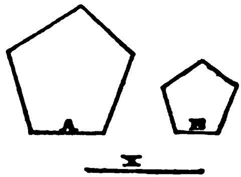
In the given polygon, draw the diagonals AC, AD; at the point F make the angle GFH =BAC, and at the point G the angle FGH ABC; the lines FH, GH, will cut each other in H, and FGH will be a triangle similar to ABC. In the same manner upon FH, homologous to AC, describe the triangle FIH similar to ADC; and upon FI, homologous to AD, describe the triangle FIK similar to ADE. The polygon FGHIK will be similar to ABCDE, as required.

For, these two polygons are composed of the same number of triangles, which are similar and similarly situated (Prop. XXIII. Sch.)

PROBLEM XIII.

Two similar figures being given, to describe a similar figure which shall be equivalent to their sum or their difference.

Let A and B be two homologous sides of the given figures.



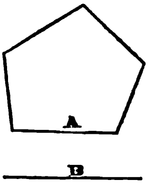
Find a square equivalent to the sum or to the difference of the squares described upon A and B; let X be the side of that square; then will X in the figure required, be the side which is homologous to the sides A and B in the given figures. The figure itself may then be constructed on X, by the last problem.

For, the similar figures are as the squares of their homologous sides; now the square of the side X is equivalent to the sum or to the difference of the squares described upon the homologous sides A and B; therefore the figure described upon the side X is equivalent to the sum or to the difference of the similar figures described upon the sides A and B.

PROBLEM XIV.

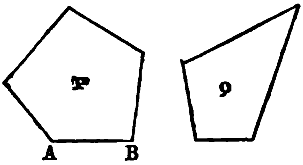
To describe a figure similar to a given figure, and bearing to it the given ratio of M *to* N.

Let A be a side of the given figure, B the homologous side of the figure required. The square of B must be to the square of A, as M is to N: hence B will be found by (Prob. XI.), and knowing B, the rest will be accomplished by (Prob. XII.)



PROBLEM XV.

To construct a figure similar to the figure P, *and equivalent to the figure* Q.



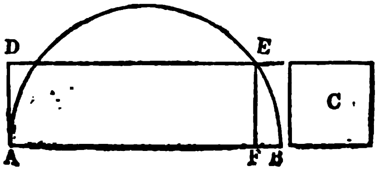
Find M, the side of a square equivalent to the figure P, and N, the side of a square equivalent to the figure Q. Let X be a fourth proportional to the three given lines, M, N, AB; upon the side X, homologous to AB, describe a figure similar to the figure P; it will also be equivalent to the figure Q.

For, calling Y the figure described upon the side X, we have $P : Y :: AB^2 : X^2$; but, by construction, $AB : X :: M :$ or $AB^2 : X^2 :: M^2 : N^2$; hence $P : Y :: M^2 : N^2$. But, by construction also, $M^2 = P$ and $N^2 = Q$; therefore $P : Y :: P :$; consequently $Y = Q$; hence the figure Y is similar to the figure P, and equivalent to the figure Q.

PROBLEM XVI.

To construct a rectangle equivalent to a given square, and having the sum of its adjacent sides equal to a given line.

Let C be the square, and AB equal to the sum of the sides of the required rectangle.



Upon AB as a diameter describe a semicircle; draw the line DE parallel to the diameter, at a distance AD from it, equal to the side of the given square C; from the point E, where the parallel cuts the circumference, draw EF perpendicular to the diameter; AF and FB will be the sides of the rectangle required.

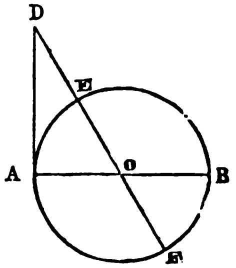
For their sum is equal to AB; and their rectangle AF.FB is equivalent to the square of EF, or to the square of AD; hence that rectangle is equivalent to the given square C.

Scholium. To render the problem possible, the distance AD must not exceed the radius; that is, the side of the square C must not exceed the half of the line AB.

PROBLEM XVII.

To construct a rectangle that shall be equivalent to a given square, and the difference of whose adjacent sides shall be equal to a given line.

Suppose C equal to the given square, and AB the difference of the sides.

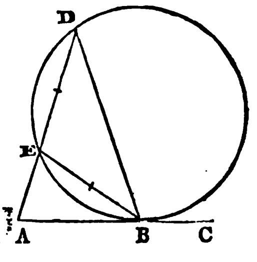


Upon the given line AB as a diameter, describe a semicircle; at the extremity of the diameter draw the tangent AD, equal to the side of the square C; through the point D and the centre O draw the secant DF; then will DE and DF be the adjacent sides of the rectangle required.

For, first, the difference of these sides is equal to the diameter EF or AB; secondly, the rectangle DE, DF, is equal to AD^2 (Prop. XXXVII.); hence that rectangle is equivalent to the given square C.

PROBLEM XVIII.

Upon a given base AB *to construct an isosceles triangle, having each of the angles at the base double the vertical angle.*



Produce AB till the rectangle AC.BC may be equivalent to the square of AB (Prob. IV.), then, with the base AB and sides each equal to AC, construct the isosceles triangle DAB, and the angle A will be double the angle D.

For make DE=AB, or make AE=BC, and join EB.

Then, by construction, AD : AB : : AB : AE, for AE=BC; consequently the triangles DAB, BAE, have a common angle A contained by proportional sides: hence (Prop. XIX.) they are similar, and, therefore, these triangles are both isosceles, for DAB is isosceles by construction, so that AB=EB; but AB=DE; consequently DE=EB, and therefore, the angle D is equal to the angle EBD: hence the exterior angle AEB is equal to double the angle D, but the angle A is equal to the angle AEB; therefore the angle A is double the angle D.

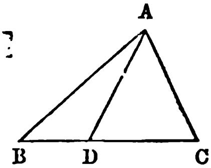
Scholium. It is obvious that, in a triangle so constructed, the vertical angle is a fifth part of two right angles, and each angle at the base is two-fifths of two right angles, or one fifth of four right angles.

PROBLEM XIX.

To divide a triangle ABC *into two parts by a line from* A, *the vertex of one of its angles, so that the parts may be to each other as a straight line* M *to another straight line* N.

Divide BC into parts BD, DC, proportional to M, N; draw the line AD, and the triangle ABC will be divided as required.

For since triangles of the same altitude are to each other as their bases, we have ABD : ADC::BD : DC::M : N.



Scholium. A triangle may evidently be divided into any number of parts proportional to given lines, by dividing the base in the same proportion.

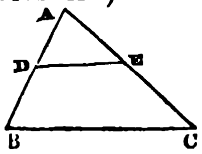
PROBLEM XX.

To divide a triangle into the two parts by a line drawn parallel to a side BC, *so that these parts may be to each other as two straight lines* M, N.

As M+N is to N, so make AB^2 to AD^2 (Prob. XI.)

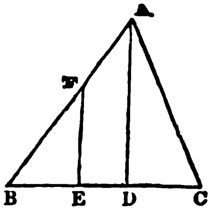
Draw DE parallel to BC, and the triangle is divided as required.

For the triangles ABC, ADE, being similar, ABC : ADE::$AB^2 : AD^2$; but M+N : N::$AB^2 : AD^2$; therefore ABC : ADE ::M+N : N ; consequently (Prop. XV. Cor. 3. B. I.) BDEC : ADE::M : N.



PROBLEM XXI.

To divide a triangle ABC *into two parts by a line perpendicular to the base, so that these parts may be to each other as two given lines* M, N.



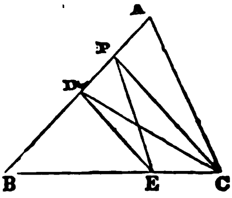
From the vertical angle A draw the perpendicular AD, let M′+N′ represent the triangle ABC, and let Q′ represent the triangle ADC. Then, since BD : DC::ABD : ADC, and BC : DC::ABC : ADC, therefore BC : DC::M′+N′ : Q′; or M+N : Q.^{2}

Draw EF parallel to DA, so that BD shall be to BE^2 as M—Q to N, and the line EF will divide the triangle as required, since similar triangles are proportional to the squares described on their homologous sides.

PROBLEM XXII.

To divide a triangle into two parts by a line drawn from a given point P *in one of its sides, so that the parts may be to each other as two given lines* M, N.

Draw PC, and divide AB in D, so that AD is to DB as M is to N; draw DE parallel to PC, join PE, and the triangle will be divided by the line PE into the proposed parts.



For, join DC; then because PC, DE, are parallel, the triangles PDE, CDE, are equal; to each add the triangle DEB, then PEB=DCB; and consequently, by taking each from the triangle ABC, there results the quadrilateral ACEP equivalent to the triangle ACD.

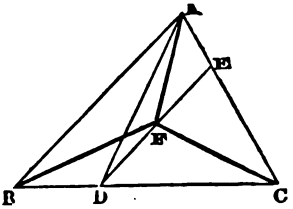
Now ACD : BCD::AD : DB::M : N; consequently,

ACEP : PEB::M : N.

Scholium. The above operation suggests the method of dividing a triangle into any number of equal parts by lines drawn from a given point in one of its sides: for if AB be divided into equal parts, and lines be drawn from the points of division parallel to PC, they will intersect BC and AC; and if from these several points of intersection lines be drawn to P, they will divide the triangle into equal parts.

PROBLEM XXIII.

To divide a triangle into three equivalent parts by lines drawn from the vertices of the angles to the same point within the triangle.

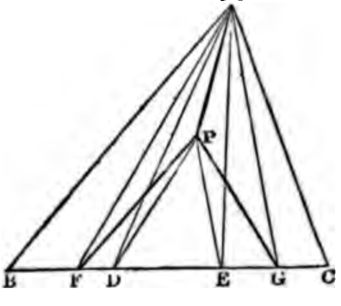


Make BD equal to a third part of BC, and draw DE parallel to BA, the side to which BD is adjacent. From F, the middle of DE, draw the straight lines FA, FB, FC, and they will divide the triangle as required.

For, draw DA, then since BD is one third of BC, the triangle ABD is one third of the triangle ABC; but ABD=ABF (Prop. III. Cor. 2.); therefore ABF is one third of ABC; also since DF=FE, BDF=AFE, likewise CFD=CFE: consequently the whole triangle FBC is equal to the whole triangle FCA; and FBA has been shown to be equal to a third part of the whole triangle ABC: consequently the triangles FBA, FBC, FCA, are each equal to a third part of ABC.

PROBLEM XXIV.

To divide a triangle ABC *into three equivalent parts by lines drawn from* P, *a given point within it.*



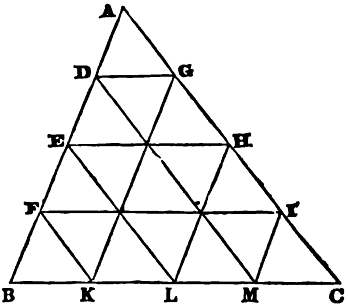
Divide BC into three equal parts in the points D, E, and draw PD, PE; draw also AF parallel to PD, and AG parallel to PE: then if the lines PF, PG, PA be drawn, the triangle ABC will be divided by them into three equivalent parts.

For join AD, AE; then because AF, PD, are parallel, the triangle AFP is equivalent to the triangle AFD; consequently, if to each of these there be added the triangle ABF, there will result the quadrilateral ABFP, equivalent to the triangle ABD; but since BD is a third part of BC, the triangle ABD is a third part of the triangle ABC (Prop. IV. Cor.): consequently the quadrilateral ABFP is a third part of the triangle ABC. Again, because AG, PE, are parallel, the triangle AGP is equivalent to the triangle AGE; add to each the triangle ACG, and there results the quadrilateral ACGP equivalent to the triangle ACE; and this triangle is one third of ABC; hence the quadrilateral ACGP is one third of the triangle ABC; consequently, the spaces ABFP, ACGP, FPG, are each equal to a third part of the triangle ABC.

PROBLEM XXV.

To divide a triangle into any square number of equal triangles similar to each other and to the original triangle.

Let it be required to divide the triangle ABC into sixteen equal triangles similar to it.

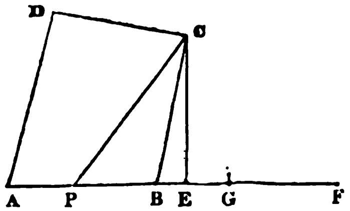


Divide one side AB into four equal parts, and from the points of division draw DG, EH, FI, parallel to BC, and DM, EL, FK parallel to AC. Through the intersections of these parallels draw GK, HL, IM, and the triangle ABC will be divided as required.

For the triangles whose bases are AD, DE, EF, FB, have their sides parallel to those of the triangle ABC, and are therefore similar to it; and as the bases of these triangles are equal, the triangles themselves are equal, and have equal altitudes, so that a single straight line GK passes through the vertices of all, and is parallel to AB. The rhomboid DK is divided into equal rhomboids by the parallels from E and F, and these again are divided into equal triangles by their diagonal. In a similar manner the rhomboid AM is divided into equal triangles, as also the rhomboids DL, EK, BI and FC, and each of these rhomboids contain one or more of the triangles contained in DK; hence the triangles are all equal, and they are similar to ABC.

PROBLEM XXVI.

To divide a quadrilateral into two parts by a straight line drawn from C, the vertex of one of its angles, so that the parts may be to each other as a line M *to another line* N.



Draw CE perpendicular to AB, and construct a rectangle equivalent to the given quadrilateral, of which one side may be CE; let the other side be EF; and divide EF in G, so that M : N::GF : EG; take BP equal to twice EG, and join PC, then the quadrilateral will be divided as required.

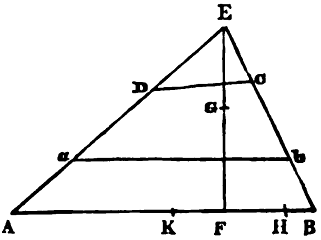
For by construction the triangle CPB is equivalent to the rectangle CE.EG; therefore the rectangle CE.GF is to the triangle CPB as GF is to EG. Now CE.GF is equivalent to the quadrilateral DP, and GF is to EG as M is to N, therefore

DP : CPB::M : N;

that is, the quadrilateral is divided as required.

PROBLEM XXVII.

To divide a quadrilateral AC *into two parts by a line parallel to* AB *one of its sides, so that these parts may be to each other as the line* M *is to the line* N.



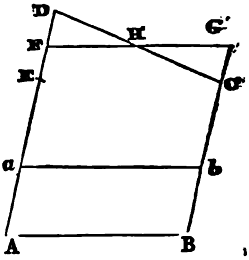
Produce AD, BC, till they meet in E; draw the perpendicular EF and bisect it in G. Upon the side GF construct a rectangle equivalent to the triangle EDC, and let HB be equal to the other side of this rectangle. Divide AH in K, so that AK : KH::M : N, and as AB is to KB, so make EA^2 to Ea^2; draw *ab* parallel to AB, and it will divide the quadrilateral into the required parts.

For since the triangles EAB, E*ab*, are similar, we have the proportion EAB : E*ab*::EA^2 : Ea^2; but by construction, EA^2 : Ea^2::AB : KB, so that EAB : E*ab*::AB : KB::AB.GF : KB.GF; and consequently, since by construction EAB=AB.GF, it follows that E*ab*=KB.GF, and therefore AK.GF=A*b*; and since by construction AH.GF=AC, it follows that KH.GF=*a*C. Now AK.GF : KH.GF::AK : KH; but by construction, AK : KH::M : N; consequently,

A*b* : *a*C::M : N;

that is, the quadrilateral is divided as required.

If the sides AD, BC, are parallel, then make AE=BC, bisect ED in F, and divide AF in *a*, so that A*a* : *a*F : : M : N, then the parallel *ab* will divide the quadrilateral as required.

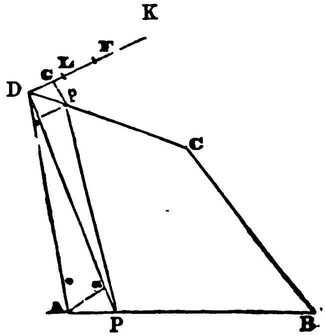


For draw FG parallel to *ab* meeting the production of BC, then since FD =CG, the triangles DFH, GCH, are equal; so that the quadrilateral *a*C is equivalent to the rhomboid *a*G, and by construction M : N : : A*a* : *a*F : : A*b* : *a*G; consequently,

M : N : : A*b* : *a*C.

PROBLEM XXVIII.

To divide a quadrilateral into two parts by a line drawn from P, *a point in one of its sides, so that the parts may be to each other as a line* M *is to a line* N.



Draw PD, upon which construct a rectangle equivalent to the given quadrilateral, and let DK be the other side of this rectangle; divide DK in L, so that DL : LK : : M : N; make DF=2DL, and FG equal to the perpendicular A*a*; draw G*p* parallel to DP, join the points P, *p*, and the quadrilateral will be divided as required.

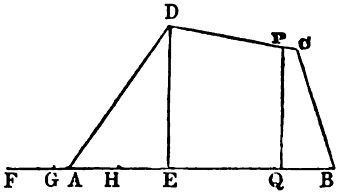
For draw the perpendicular *po*; then, by construction, PD.DK= AC, and PD.DF=PD.A*a*+PD.*po*, that is, PD.DF is equivalent to twice the sum of the triangles APD, *p*PD; consequently since DL is half DF, PD.DL=AP.*p*D; and therefore PD.LK=PBC*p*; but PD.DL : PD.LK : : DL : LK : : M : N; consequently,

AP*p*D : PBC*p* : : M : N;

hence the quadrilateral is divided as required.

PROBLEM XXIX.

To divide a quadrilateral ABCD *by a line perpendicular to one of the sides* AB, *so that the two parts may be to each other as a line* M *is to a line* N.



Construct on DE, perpendicular to AB, a rectangle DE.EF, equivalent to the quadrilateral AC, and divide FE in G, so that FG: GE::M : N. Bisect AE in H, and (Prob. XXVI.) divide the quadrilateral EC into two parts by a line PQ parallel to the side DE, so that those parts may be to each other as FG is to GH, then PQ will also divide the quadrilateral as required.

For, by construction, DE.EF=AC, and DE.EH=DAE; ence DE.HF=EC, and consequently, since the quadrilateal EC is divided in the same proportion as the base FH of s equivalent rectangle, it follows that QC=DE.FG, and P=DE.GH, also AE=DE.GE; consequently,

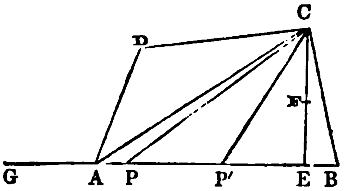
QC : AP::FG : GE::M : N;

at is, the quadrilateral is divided as required.

PROBLEM XXX.

o *divide a quadrilateral* ABCD *into three equivalent parts by lines drawn from* C, *the vertex of one of its angles.*

Draw the diagonal AC, and the perpendicular CE, which sect in F; construct upon FE a rectangle equivalent to the angle DAC, and let AG be equal to the other side of this ctangle.



Then the remaining conruction of this problem will ary accordingly as AB exeeds twice AG, is less than alf AG, or is between these vo. Let us first suppose that B>2AG.

Take AP equal to a third part of AB—2AG; bisect PB in ; then draw the lines CP, CP′, and they will divide the adrilateral into three equivalent parts.

For PB=AB—AP, and since AP=$\frac{1}{3}$AB—$\frac{2}{3}$AG, by contruction, it follows that PB=AB—($\frac{1}{3}$AB—$\frac{2}{3}$AG)=$\frac{2}{3}$AB+ AG, that is to say, PB=$\frac{2}{3}$GB, and therefore PB.FE=$\frac{2}{3}$GB. E, that is, the triangle CPB is two thirds of the quadrilatral AC; but the triangles CPP′, CP′B, having equal bases, P′, P′B, are each half the triangle CPB, and consequently one third of the quadrilateral: hence the spaces DAPC, CPP′, CP′B, are equivalent.

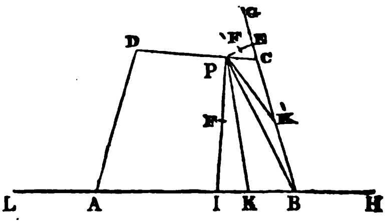
Scholium. In this case of the problem the lines of division must necessarily fall within the triangle CAB; for AB being greater than 2AG, $AB.FE > 2AG.FE$, that is, the triangle CAB exceeds twice the triangle DAC, and is therefore greater than two-thirds of the quadrilateral. In the third case of the problem (provided AG is not equal to $\frac{1}{2}AB$, nor to 2AB) this triangle will be less than two-thirds of the quadrilateral, but greater than one-third; consequently one line of division only will fall within the triangle CAB, and the other within the triangle DAC; in this case, therefore, after having determined the line CP′, by making $AP' = \frac{1}{3}(2AB - AG)$, it will only be necessary to divide the remaining quadrilateral AP.CD, by Problem XXVI., into two equal parts, by a line from the point C. In the second case of this problem, that is to say, when AB is less than half AG, the triangle CAB will be less than one-third of the proposed quadrilateral, and consequently both lines of division will fall within the other triangle DAC; and therefore this case is virtually the same as the first: the perpendicular from C being to the side AD instead of AB. When we happen to have $AG = \frac{1}{2}AB$, or $AG = 2AB$, then the diagonal CA will be one line of division.

A mere inspection of the figure will always enable us to determine upon which side the perpendicular is to fall; for as one triangle will be always at least double the other in surface, the greatest will be at once recognized.

PROBLEM XXXI.

To divide a quadrilateral AC *into three equivalent parts by lines drawn from a given point* P *in one of its sides.*

The construction of this problem will be modified accordingly as the triangle CPB, cut off by the line PB, is less or greater than one third of the quadrilateral AC, which will be discovered by constructing upon a common base, rectangles equivalent to the triangle CPB, and to the quadrilateral ADPB :—If the altitude of the former be less than half the altitude of the latter, the triangle CPB will obviously be less than one third of the quadrilate-



ral AC; and if it be greater, then the same triangle will exceed one-third of the quadrilateral. Suppose then $CPB < \frac{1}{3}AC$. Draw PI perpendicular to AB, and upon FI, the half of PI, construct a rectangle equivalent to ADPB; and let the other side of this rectangle be equal to LB: construct also on FI a rectangle equivalent to the triangle CPB, and let BH be the other side of this rectangle.

Take $BK = \frac{1}{3}LB - \frac{2}{3}BH$; join PK, then PK will be one of the lines of division.

For the triangle PKB is equal to the rectangle BK.FI, and the triangle $CPB = BH.FI$ by construction, therefore, $PKBCP = (KB + BH).FI$; but by construction, $KB = \frac{1}{3}LB - \frac{2}{3}BH$; therefore $KB + BH = \frac{1}{3}LB + \frac{1}{3}BH = \frac{1}{3}LH$; hence $PKBCP = \frac{1}{3}LH.FI = \frac{1}{3}ABCD$.

If $CPB > \frac{1}{3}AC$, then, instead of the perpendicular PI to AB, draw PE perpendicular to BC or its production, and upon F′E, the half thereof, construct a rectangle equivalent to ADPB, and let GB be equal to its base. Take $BK' = \frac{2}{3}BC - \frac{1}{3}BG$, and join PK′, then PK′ will be one of the lines of division.

For $K'C = BC - BK'$, and by construction $BK' = \frac{2}{3}BC - \frac{1}{3}BG$; therefore $K'C = BC - (\frac{2}{3}BC - \frac{1}{3}BG) = \frac{1}{3}BC + \frac{1}{3}BG$, so that the triangle $PK'C = \frac{1}{3}BC.F'E + \frac{1}{3}BG.F'E$, which by construction is equivalent to one third of the whole quadrilateral ABCD.

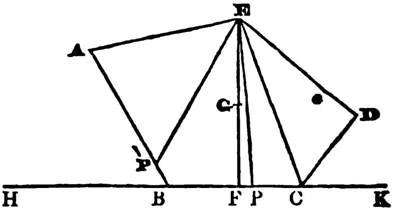
The other line of division will be readily found by Prob. XXVI.

Scholium. If $LB = \frac{1}{2}BH$, or $BC = \frac{1}{2}BG$, then the value of BK or BK′ is 0, which must necessarily be the case, since the line PB must then cut off a third part from the quadrilateral AC. This case of the problem will be intimated by the preliminary construction employed as above directed, to ascertain whether the part cut off by this line be greater or less than a third of the quadrilateral. It may be necessary to remark that the common base upon which the rectangle equivalent to these two parts are constituted, may as well be one of the perpendiculars PI or PE; for should the one chosen happen to be that which is to be drawn in the construction of the problem, it is plain that this construction will thus be forwarded.

PROBLEM XXXII.

To divide the irregular pentagon ABCDE *into three equivalent surfaces by lines drawn from* E, *the vertex of one of the angles.*

The construction of this problem, like that of the preceding, will vary accordingly as the diagonal EC cuts off a portion EDC, less or greater than the third part of the whole polygon, which may be ascertained by performing the same preliminary operation as was directed in the preceding problem. Suppose that the triangle EDC is less than a third of the polygon, in which case both lines of division must necessarily fall to the left of the diagonal EC.



Draw EF perpendicular to BC, and bisect it in G. Upon GF construct a rectangle GF.HC equivalent to the quadrilateral AC; construct also upon GF a rectangle GF.CK equivalent to the triangle EDC. Take CP=$\frac{1}{3}$HC—$\frac{2}{3}$CK, and draw EP, which will cut off a portion PD equivalent to a third part of the polygon.

For the triangle ECP is equivalent to the rectangle GF.PC, and by construction the quadrilateral AC is equivalent to GF.HC; hence the quadrilateral AP is equivalent to GF.(HC—PC), but PC=$\frac{1}{3}$HC—$\frac{2}{3}$CK; consequently HC—PC=$\frac{1}{3}$HC+$\frac{1}{3}$CK=$\frac{1}{2}$HK: therefore AP=$\frac{1}{3}$GF.HK=$\frac{1}{3}$ABCDE.

Having found one line of division, the other, EP, may be found by Problem XXVI.

If one of the lines of division EP fall to the right of EC, then the perpendicular from E must be drawn to CD, and the remaining construction will suggest itself from what has been done above and in the preceding problem.

Scholium. It appears to be unnecessary to extend these problems upon the *division of polygons* to any greater length; a sufficient number has been given on this subject to afford the student an opportunity of applying the principles established in the preceding books to an interesting and useful class of problems. We have been thus particular in treating this branch of the subject, in order to prepare the student for one of the most difficult subjects in practical surveying, viz: the dividing and laying out of fields according to any proposed conditions. We shall merely add the three following by way of exercise, and then terminate this part of the subject, by adding one curious problem in reference to forming a square from a rectangle by a transposition of its parts; for the ingenious construction of which the author is indebted to Mr. C. H. Hinckley.

1. To divide a pentagon by a line drawn from the vertex

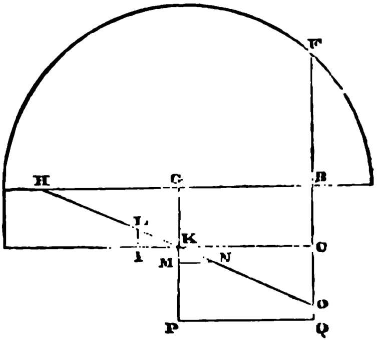
of one of its angles, so that the parts may be to each other as a line M is to a line N.

2. To divide a pentagon into three equivalent surfaces by lines drawn from a given point in one of its sides.

3. To divide a pentagon into three equivalent surfaces by lines drawn from two given points in one of its sides.

PROBLEM XXIII.

Let it be required to form a rectangle AC *into a square, by the transposition of its parts.*

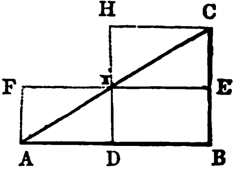


Let the side AB of the rectangle be produced to E, so that BE shall be= the side BC; on AE as a diameter describe a semicircle, and draw BF perpendicular to AE, and BF will be a mean proportional between AB and BE or BC (Prop. XVII. Cor. 5.), which will be evidently equal to a side of the square equivalent to the rectangle. On AB set off BF as often as possible. viz: BG, GH, each to BF, and from D on the line DC take DI=BF. From each of the divisions on the line AB, except the last one at H, draw lines, as GK, parallel to BC. Join the last division H on the line AB to the point on the line DC, where the line parallel to BC from the next previous division of AB terminates; draw IL perpendicular to DC to meet the line HK. On GB construct the square BP.

Then since HG is equal to GB, and GK is parallel to BC, and HG is also equal to KC. Hence, if the triangle HGK, which is right angled at G, be brought into the position KCO the side HG will coincide with KC, and GK will coincide with CO which forms with BC one and the same continued right line, since KCB and KCO are each right angles. Now, because DK is=AG, and DI=HG, then also IK=AH; and since the interior angles formed by a line which meets two parallel lines are together equal to two right angles (Prop. XXII. B. II.), it follows that KHA+KHG is=to two right angles, and because the meeting of the lines HL and DI produced in K forms together with HKG a right angle, if the pentagon DAHLI be brought in the position PMNOQ, the line OQ will be a production of the right line BO, and since Q is a right angle, PQ must be parallel to GB, and it is also equal to GB by construction; and since D or P is a right angle by hypothesis, PMKG will be found in the same right line; and since we have previously shown that IK is equal to AH or MN, and since $HL+LK=HK$ or KO, it follows that $LK=KN$; and since PMN is a right angle, and also LIK, it follows that the triangle LIK may be made to coincide with the triangle KMN, so that the line IK shall coincide with the line MN, the line LK with the line KN, and the line LI with the line KM. And hence the rectangle ABCD is all included in the square BP. It follows that the construction is perfect.

Scholium. It will be obvious that any rectilinear figure may be so divided as that, by a transposition of its parts, it may be formed into a square; since any polygon may be divided into right angled triangles, and right angled triangles may when necessary be formed into rectangles by drawing a line parallel to one of the sides including the right angle, bisecting the other two sides, and by placing the vertical angle so cut off so that its hypothenuse shall coincide with the remaining portion of the hypothenuse of the original triangle, placing the two acute angles of the given triangle in contact.

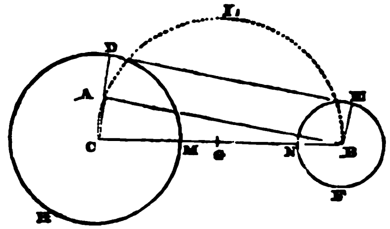
Thus if the triangle CEI be cut off from the triangle ABC by the line EI parallel to AB, bisecting the sides BC, AC, in the points E and I, and if this triangle be brought in the position IFA, so that the angle C shall be adjacent to the angle A, and the angle EIC shall be in the position of the angle FIA, then the triangle will be formed into the rectangle AFEB. And in the same manner may the rectangle DHCB be formed, by placing the triangle ADI in the position CHI.



2. It is not always necessary that a polygon shall be divided into triangles in order to prepare them for a susceptibility of being formed into a square, but in many cases a rectangle may be formed from a polygon merely by a transposition of a few angular portions, and as we have shown, a rectangle may always be formed into a square.

PROBLEM XXXIV.

To draw a right line, that shall be a tangent to two given circles.



Let DH and EF be two circles, to which the required tangent is to be drawn. On the distance BC of their centres, as a diameter, describe the semicircle CLB; take the distance CA equal to the difference of the radii of the two given circles, viz: CD—BN, and through the point A where this distance meets the arc of the semicircle draw the radius CD; draw also AB, and from the centre B draw the radius BE perpendicular to AB; join DE, which will be the required tangent to the two given circles.

For the angle CAB, being an angle in the semicircumference, is a right angle (Prop. XIX. Cor. 2. B. III.), and since CD is one right line, DAB is also a right angle, and the angle ABE is a right angle by construction. Moreover, since we have taken CA=CD—BE; CD—CA, or AD, must be equal to BE; hence we have the quadrilateral ABED, two of whose opposite sides are equal and parallel. Therefore the figure is a parallelogram or rhomboid; and since two of its angles, A and B, are right, the remaining two, D and E, are also right (Prop. XXVII. Cor. 1. B. II.) Hence (Prop. IX. B. III.), the line DE is tangent to the two circles.

BOOK V.

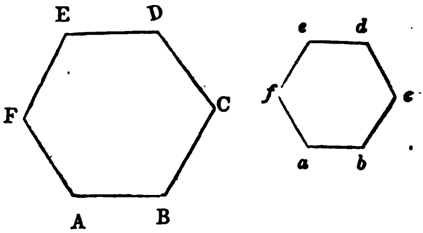
REGULAR POLYGONS AND THE MEASUREMENT OF THE CIRCLE.

Definition.

A polygon which is at once equilateral and equiangular is called a *regular* polygon. Regular polygons may have any number of sides: the equilateral triangle is one of three sides; the square is one of four.

PROPOSITION I. THEOREM.

Two regular polygons of the same number of sides are similar figures.



Suppose, for example, that ABCDEF, *abcdef*, are two regular hexagons. The sum of all the angles is the same in both figures, being in each equal to eight right angles (Prop. XXVIII. Cor. 3. B. II.) The angle A is the sixth part of that sum; so is the angle *a*: hence the angles A and *a* are equal; and for the same reason, the angles B and *b*, the angles C and *c*, &c., are equal.

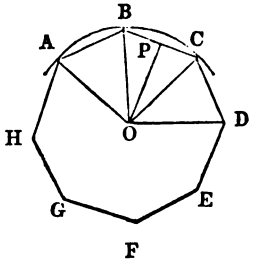
Again, since the polygons are regular, the sides AB, BC, CD, &c., are equal, and likewise the sides *ab*, *bc*, *cd*, &c., (Def.); it is plain that AB : *ab* :: BC : *bc* :: CD : *cd*, &c.; hence the two figures in question have their angles equal, and their homologous sides proportional; consequently they are similar (Def. 1. B. IV.)

Cor. The perimeters of two regular polygons of the same number of sides, are to each other as their homologous sides (Prop. XXIII. Cor. B. IV.), and their surfaces are to each other as the squares of those sides (Prop. XXIV. Cor. F. B. IV.)

Scholium. The angle of a regular polygon, like the angle of an equiangular polygon, is determined by the number of its sides (Prop. XXVIII. B. II.)

PROPOSITION II. THEOREM.

Any regular polygon may be inscribed in a circle and circumscribed about one.



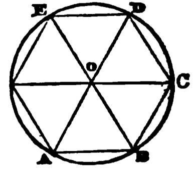
Let ABCDE, &c., be a regular polygon; describe a circle through the three points A, B, C, the centre being O, and OP the perpendicular let fall from it, to the middle point of BC: draw AO and OD.

If the quadrilateral OPCD be placed upon the quadrilateral OPBA, they will coincide; for the side OP is common; the angle OPC=OPB, each being a right angle; hence the angle PC will apply to its equal PB, and the point C will fall on B: besides, from the nature of the polygon, the angle PCD=PBA; hence CD will take the direction BA; and since CD=BA, the point D will fall on A, and the two quadrilaterals will entirely coincide. The distance OD is therefore equal to AO; and consequently the circle which passes through the three points A, B, C, will also pass through the point D. By the same mode of reasoning, it might be shown, that the circle which passes through the three points B, C, D, will also pass through the point E; and so of all the rest: hence the circle which passes through the points A, B, C, passes also through the vertices of all the angles in the polygon, which is therefore inscribed in this circle.

Again, in reference to this circle, all the sides AB, BC, CD, &c., are equal chords; they are therefore equally distant from the centre (Prop. VIII. B. III.): hence, if from the point O with the distance OP, a circle be described, it will touch the side BC, and all the other sides of the polygon, each in its middle point, and the circle will be inscribed in the polygon, or the polygon described about the circle.

Scholium. 1. The point O, the common centre of the inscribed and circumscribed circles, may also be regarded as the centre of the polygon; and upon this principle the angle AOB is called *the angle at the centre,* being formed by two radii drawn to the extremities of the same side AB.

Since all the chords AB, BC, CD, &c., are equal, all the angles at the centre must evidently be equal likewise; and therefore the value of each will be found by dividing four right angles by the number of sides of the polygon.

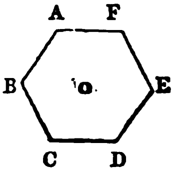


Scholium 2. To inscribe a regular polygon of a certain number of sides in a given circle, we have only to divide the circumference into as many equal parts as the polygon has sides: for the arcs being equal, the chords AB, BC, CD, &c., will also be equal; hence likewise the triangles AOB, BOC, COD, must be equal, because the sides are equal each to each; hence all the angles ABC, BCD, CDE, &c., will be equal; hence the figure ABCDEH will be a regular polygon.

PROPOSITION III. THEOREM. (*Converse of Prop. II.*)

If from a common centre, circles can be inscribed within and circumscribed aboat a polygon, that polygon is regular.

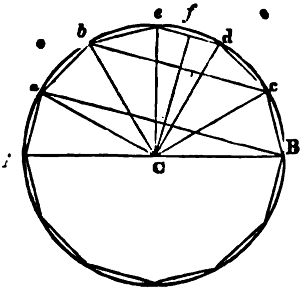
Suppose that from the point O, as a centre, circles can be described in, and about the polygon in the margin; this polygon is regular.



For, supposing these circles to be described, the inner one will touch all the sides of the polygon; these sides are therefore equally distant from its centre (Prop. IX. B. III.), and consequently, being chords of the outer circle they are equal, and, therefore, include equal angles (Prop. XIX. Cor. 1. B. III.) Hence the polygon is at once equilateral and equiangular, that is (Def.), it is regular.

PROPOSITION IV. THEOREM.

A diagonal cutting off three, five, or any odd number of sides, from a regular polygon, is parallel to the middle one of those sides.



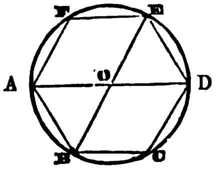
Let AB be a duodecagon, or polygon of twelve sides, and let us conceive a circle to be circumscribed about it. From the centre C draw the radial line C*f* bisecting one of the sides *ed* in *f*, and this line will be perpendicular to *ed* (Prop. VI. B. III.) From the angles *b* and *c* draw the diagonal *bc*, cutting off the three sides *be*, *ed* and *dc*. Now, because the side *be* is equal to the side *dc* (Def.), they each subtend equal arcs (Prop. IV. B. III.); the arcs subtended by the chords *be* and *dc* are equal; and because parallel lines intercept equal arcs of the circumference (Prop. X. B. III.), the lines *ed* and *bc* are parallel. Hence, also, for the same reasons, *a*B is parallel to *bc* or *ed*.

Cor. Hence, if the diagonal cuts off an even number of sides, a perpendicular from the centre of such diagonal will bisect the vertical angle. Thus the line C*e* perpendicular to a diagonal supposed to be drawn from *b* to *d*, bisects the vertical angle *bed*.

PROPOSITION V. THEOREM.

The side of a regular hexagon inscribed in a circle is equal to the radius of that circle.

Let ABCDEF be a regular hexagon inscribed in a circle, the centre of which is O, then a side as BC will be equal to the radius OD.

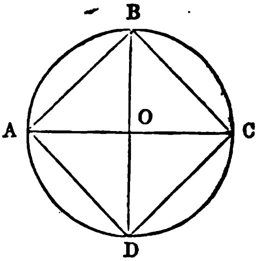


For draw AD. Then, since the arcs subtended by equal chords are equal, it follows that the three arcs AF, FE, ED, are together equal to the three arcs AB, BC, CD; that is, the diagonal AD is a diameter of the circle. For similar reasons the diagonal BE is also a diameter. Now, AD is parallel to BC, and BE to CD (Prop. IV.); so that BD is a rhomboid, and, consequently, BC is equal to OD.

Scholium. Hence, in order to inscribe a regular hexagon in a given circle, nothing more is required than to repeat the radius of the circle round the circumference. By joining the alternate points A, C; C, E; E, A, an equilateral triangle will be inscribed in the circle.

PROPOSITION VI. PROBLEM.

To inscribe a square in a given circle.

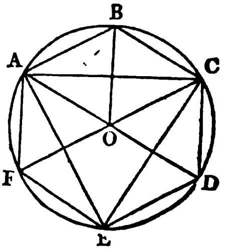


Draw two diameters AC, BD, cutting each other at right angles; join their extremities A, B, C, D: the figure ABCD will be a square. For the angles AOB, BOC, &c. being equal, the chords AB, BC, &c. are also equal; and the angles ABC, BCD, &c. being in semicircles, are right angles.

Scholium. Since the triangle BCO is right angled and isosceles, we have $BC : BO :: \sqrt{2} : 1$ (Prop. XXIV. Cor. 4. B. IV.); hence *the side of the inscribed square is to the radius, as the square root of 2, is to unity.*

PROPOSITION VII. PROBLEM.

In a given circle to inscribe a regular hexagon and an equilateral triangle.



From any point B in the circumference apply the radius OB six times, (Prop. V. Sch.) to the circumference, and the figure ABCDEF, thus formed, is an incribed hexagon.

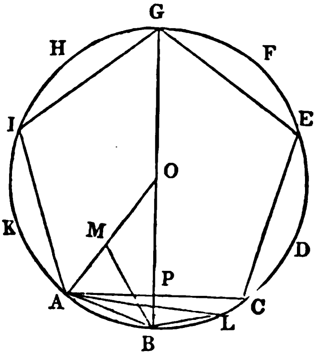
Join the alternate angles AE, EC and CA, and the figure AEC thus formed is an equilateral triangle, since the three sides subtend equal arcs of the circumference.

Scholium. The figure ABCO is a parallelogram and even a rhombus, since $AB=BC=CO=AO$; hence the sum of the squares of the diagonals AC^2+BO^2 is equivalent to the sum of the squares of the sides, that is, to $4AB^2$, or $4BO$ (Prop. XXVIII. Cor. B. IV.): and taking away BO^2 from both, there will remain $AC^2=3BO^2$; hence $AC^2 : BO^2 :: 3 : 1$, or $AC : BO :: \sqrt{3} : 1$; hence *the side of the inscribed equilateral triangle is to the radius as the square root of three is to unity.*

PROPOSITION VIII. PROBLEM.

In a given circle, to inscribe a regular decagon, then a penta-

gon, and also a regular pentedecagon or polygon of fifteen sides.



Divide the radius AO in extreme and mean ratio at the point M (Prop. IV. B. IV.); take the chord AB equal to OM the greater segment; AB will be the side of the regular decagon, and will require to be applied ten times to the circumference.

For, drawing MB, we have by construction, AO : OM :: OM : AM; or, since AB=OM, AO : AB::AB : AM; since the triangles ABO, AMB, have a common angle A, included between proportional sides, they are similar (Prop. XIX. B. IV.) Now the triangle OAB being isosceles, AMB must be isosceles also, and AB=BM; but AB=OM; hence also MB=OM; hence the triangle BMO is isosceles.

Again, the angle AMB being exterior to the isosceles triangle BMO, is double of the interior angle O (Prop. XXVII. Cor. 6. B. II.): but the angle AMB=MAB; hence the triangle OAB is such, that each of the angles OAB or OBA, at its base, is double of O, the angle at its vertex; hence the three angles of the triangle are together equal to five times the angle O, which consequently is the fifth part of the two right angles, or the tenth part of four; hence the arc AB is the tenth part of the circumference, and the chord AB is the side of the regular decagon.

2d. By joining the alternate corners of the regular decagon, the pentagon ACEGI will be formed, also regular.

3d. AB being still the side of the decagon, let AL be the side of a hexagon; the arc BL will then, with reference to the whole circumference, be $\frac{1}{6}-\frac{1}{10}$, or $\frac{1}{15}$; hence the chord BL will be the side of the regular polygon of fifteen sides, or pentedecagon. It is evident also, that the arc CL is the third of CB.

Scholium. Any regular polygon being inscribed, if the arcs subtended by its sides be severally bisected, the chords of those semi-arcs will form a new regular polygon of double the number of sides: thus it is plain, that the square will enable us to inscribe successively regular polygons of 8, 16, 32, &c. sides. And in like manner, by means of the hexagon, regular polygons of 12, 24, 48, &c. sides may be inscribed; by means of the decagon, polygons of 20, 40, 80, &c. sides; by means of the pentedecagon, polygons of 30, 60, 120, &c. sides.

It is further evident, that any of the inscribed polygons will be less than the inscribed polygon of double the number of sides, since a part is less than the whole.

And it is manifest that each polygon will exceed the preceding in surface and perimeter. It is obvious that a regular polygon of any number of sides whatever might be inscribed in a circle, if its circumference could be divided into any proposed number of equal parts; but this cannot be effected universally till we have the means of trisecting any given arc or angle, a problem which has not yet been brought to a complete solution. No method has yet been devised, of inscribing in a circle a regular heptagon, or which is the same thing, of dividing the circumference into 7 equal parts.

And indeed till quite recently it was supposed that no others, but such as we have noticed above, could admit of geometrical inscription in a circle. But by a curious work, published by M. Gaus, of Gottingen, in 1801, and afterward translated into French, under the title of *Recherches Arithmetiques*, it was shown that the circumference of a circle could be divided into any number of equal parts, capable of being expressed by the formula 2^n+1, provided it be a prime number, that is a number that cannot be resolved into factors.

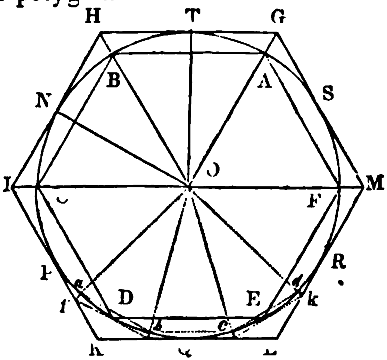
The number 3 is the simplest of this kind, it being the value of this formula when $n=1$; the next prime number is 5, and this also is expressed by the formula. But we have already inscribed polygons of three and five sides. The next prime number expressed by the formula is 17, so that a seventeen sided polygon may be inscribed in a circle.

M. Gaus's theorem, although it establishes the above geometrical fact, depends upon the theory of algebraical equations, and involves other considerations of a nature that do not enter into elements of geometry; we have therefore merely alluded to it.

PROPOSITION IX. PROBLEM.

A regular inscribed polygon being given, to circumscribe a similar polygon, about the circle.

Let CBAFED be a regular polygon.



At T, the middle point of the arc AB, apply the tangent GH, which will be parallel to AB (Prop. X. B. III.); do the same at the middle point of each of the arcs BC, CD, &c.; these tangents, by their intersections, will form the regular circumscribed polygon GHIK, &c., similar to the one inscribed.

Since T is the middle point of the arc BTA, and N the middle point of the equal arc BNC, it follows, that BT=BN; or that the vertex B of the inscribed polygon, is at the middle point of the arc NBT. Draw OH. The line OH will pass through the point B.

For, the right angled triangles OTH, OHN, having the common hypothenuse OH, and the side OT=ON, must be equal (Prop. XIX. B. II.), and consequently the angle TOH=HON, wherefore the line OH passes through the middle point B of the arc TN. For a like reason, the point I is in the prolongation of OC; and so with the rest.

But, since GH is parallel to AB, and HI to BC, the angle GHI=ABC (Prop. XXVI. B. II.); in like manner HIK=BCD; and so with all the rest: hence the angles of the circumscribed polygon are equal to those of the inscribed one. And further, by reason of these same parallels, we have GH : AB::OH : OB, and HI : BC::OH : OB; therefore GH : AB::HI : BC. But AB=BC, therefore GH=HI. For the same reason, HI=IK, &c.; hence the sides of the circumscribed polygon are all equal; hence this polygon is regular, and similar to the inscribed one.

In like manner, let the chords *ab*, *bc*, *cd*, &c., be drawn, forming a regular inscribed polygon *abcd*, &c., of twice as many sides as the former; and if we complete the construction as above, we shall have also a regular circumscribed polygon *fghk*, &c., similar to the inscribed one.

Cor. 1. Reciprocally, if the circumscribed polygon GHIK, &c. were given, and the inscribed one ABC, &c., were required to be deduced from it, it would only be necessary to draw from the angles G, H, I, &c. of the given polygon, straight lines OG, OH, &c., meeting the circumference in the points A, B, C, &c.; then to join those points by the chords, AB, BC, &c.; this would form the inscribed polygon. An easier solution of this problem would be simply to join the points of contact T, N, P, &c., by the chords TN, NP, &c., which likewise would form an inscribed polygon similar to the circumscribed one.

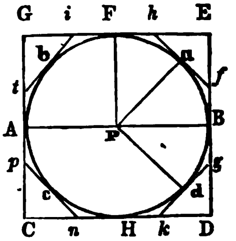
Cor. 2. Hence we may circumscribe about a circle any regular polygon, which can be inscribed within it, and conversely.

Cor. 3. It has been shown that $NH=HT$; hence it is plain that $NH+HT=HT+TG=HG$, one of the equal sides of the polygon.

PROPOSITION X. PROBLEM.

A circle and regular circumscribed polygon being given, it is required to circumscribe the circle by another regular polygon having double the number of sides.

Let the circle whose centre is P, be circumscribed by the square CDEG: it is required to find a regular circumscribed octagon.



Bisect the arcs AH, HB, BF, FA, and through the middle points c, d, a, b, draw tangents to the circle, and produce them till they meet the sides of the square: then will the figure $pnkg$, &c. be a regular octagon.

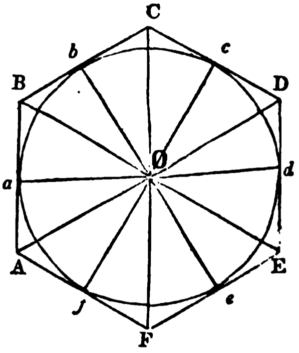
For, having drawn Pd, Pa, let the quadrilateral $PdgB$, be applied to the quadrilateral $PBfa$, so that PB shall fall on PB. Then, since the angle dPB is equal to the angl BPa, each being half a right angle, the line Pd will fall its equal Pa, and the point d on the point a. But the angl Pdg, Paf, are right angles (Prop. IX. B. III.); hence th line dg will take the direction af. The angles PBg, PB are also right angles; hence Bg will take the direction B therefore, the two quadrilaterals will coincide, and the poi g will fall at f; hence $Bg=Bf$, $dg=af$, and the angle dg $=Bfa$. By applying in a similar manner, the quadrilatera PBfa, PFha, it may be shown, that $af=ah$, $fB=Fh$, an the angle $Bfa=ahF$. But since the two tangents fa, f are equal (Prop. IX. Cor. 3.), it follows that fh, whic is twice fa, is equal to fg, which is twice fB.

In a similar manner it may be shown that $hf=hi$, and th angle $Fib=Fha$, or that any two sides or any two angles the octagon are equal: hence the octagon is a regular pol gon (Def.) The construction which has been made in the case of the square and the octagon, is equally applicable to other polygons.

Cor. It is evident that the circumscribed square is greater than the circumscribed octagon by the four triangles, *Cnp*, *k*D*g*, *h*E*f*, G*il*; and if a regular polygon of sixteen sides be circumscribed about the circle, we may prove in a similar way, that the figure having the greatest number of sides will be the least; and the same may be shown, whatever be the number of sides of the polygons: hence, in general, *any circumscribed regular polygon, will be greater than a circumscribed regular polygon having double the number of sides.*

PROPOSITION XI. THEOREM.

Every regular polygon is composed of as many equal isosceles triangles, as the polygon has sides, and if the sides of the polygon are considered as their bases, their vertices will all be in the centre of the inscribed circle.



If in the polygon ABCDEF lines be drawn from the centre of the inscribed circle at O, to the several vertices of the polygon, then will these lines divide the polygon into as many equal triangles AOB, BOC, COD, &c., whose vertices are all at O, as the polygon has sides.

For since the sides of a regular polygon are all tangents to the inscribed circle (Prop. II.), and all equal by definition, and since those sides enclose the entire circle inscribed in the polygon, which is the measure of four right angles, it being all the angular space about a point or about the centre O, it follows that the lines AO, BO, CO, DO, EO and FO, divide the whole surface of the polygon.

And because a circle may be described about the polygon from the centre O, touching all the vertices A, B, C, D, E, F, (Prop. II.), it follows that those lines drawn from the vertices to the centre are each equal to the radius of the circumscribing circle, and consequently equal to each other. Hence the bases being equal, and the other sides being equal each to each (Prop. XV. B. II.), the triangles will be equal and also isosceles.

Cor. Since the radii of the inscribed circle of the polygon, drawn to the centres of its sides, or to the points of contact of the sides with the circle, are perpendicular to those sides respectively (Prop. IX. B. III.), the radius of the inscribed circle is equal to the vertical heights of the several triangles.

Cor. 2. Hence the area of each of the triangles composing the polygon is equal to half the product of the side of the polygon, forming the base of the triangle, multiplied by the radius O*a*.

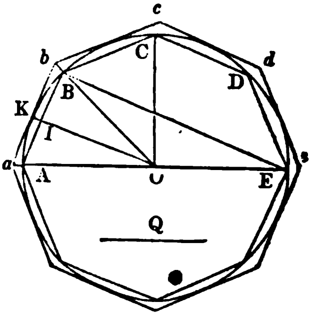
Cor. 3. Hence also, the polygon which is composed of all the triangles, is equal to the sum of all their bases, or the whole perimeter of the polygon, multiplied by half their common altitude, or half the radius of the inscribed circle, whatever be the number of the sides of the polygon.

Scholium. Since triangles are the most simple figures into which a polygon can be divided, and as polygons and all rectilinear figures may be divided into triangles (Prop. XXIII. B. IV.), therefore triangles may be considered as the elements of all rectilinear figures; and of each particular polygon, the triangles composing that polygon may be considered as the elementary triangles of that or a similar polygon.

PROPOSITION XII. THEOREM.

Two regular polygons, of the same number of sides, can always be formed, the one circumscribed about a circle, the other inscribed in it, which shall differ from each other by less than any assignable surface

Let Q be the side of a square less than the given surface. Bisect AC, a fourth part of the circumference, and then bisect the half of this fourth, and proceed in this manner, always bisecting one of the arcs formed by the last bisection, until an arc is found whose chord AB is less than Q. As this arc will be an exact part of the circumference, if we apply the chords AB, BC, CD, &c. each equal to AB, the last will terminate at A, and there will be



formed a regular polygon ABCDE, &c. in the circle.

Next, describe about the circle a similar polygon *abcde*, &c. (Prop. IX.): the difference of these two polygons will be less than the square of Q.

For, from the points *a* and *b*, draw the lines *a*O, *b*O, to the centre O: they will pass through the points A and B, as was shown in Prop. IX. Draw also OK to the point of contact K: it will bisect AB in I, and be perpendicular to it (Prop. VI. Sch. B. III.) Produce AO to E, and draw BE.

Let P represent the circumscribed polygon, and *p* the inscribed polygon: then, since the triangles *a*O*b*, AOB, are like parts of P and *p*, we shall have

$aOb : AOB :: P : p$ (Prop. XV. B. I.);

But the triangles being similar

$aOb : AOB :: Oa^2 : OA^2$, or OK^2.

Hence, $P : p :: Oa^2 : OK^2$.

Again, since the triangles O*a*K, EAB, are similar, having their sides respectively parallel,

$Oa^2 : OK^2 :: AE^2 : EB^2$, hence,

$P : p :: AE^2 : EB^2$, or by division,

$P : P-p :: AE^2 : AE^2-EB^2$, or AB .

But P is less than the square described on the diameter AE (Prop. X. Cor.); therefore $P-p$ is less than the square described on AB; that is, less than the given square on Q: hence the difference between the circumscribed and inscribed polygons may always be made less than a given surface.

Cor. 1. A circumscribed regular polygon, having a given number of sides, is greater than the circle, because the circle makes up but a part of the polygon; and for a like reason, the inscribed polygon is less than the circle. But by increasing the number of sides of the circumscribed polygon, the polygon is diminished (Prop. X. Cor.), and therefore approaches to an equality with the circle; and as the number of sides of the inscribed polygon is increased, the polygon is increased (Prop. VIII.), and therefore approaches to an equality with the circle.

Now, *if the number of sides of the polygons be indefinitely increased, the length of each side will be indefinitely small, and the polygons will ultimately become equal to each other, and equal also to the circle.*

For, if they are not ultimately equal, let D represent their smallest difference.

Now, it has been proved in the proposition, that the difference between the circumscribed and inscribed polygons, can be made less than any assignable quantity: that is, less than D: hence the difference between the polygons is equal to D, and less than D at the same time, which is absurd: therefore, the polygons are ultimately equal. But when they are equal to each other, each must also be equal to the circle, since the circumscribed polygon cannot fall within the circle, nor the inscribed polygon without it.

Cor. 2. Since the circumscribed polygon has the same number of sides as the corresponding inscribed polygon, and since the two polygons are regular, they will be similar (Prop. I.); and therefore when they become equal, they will exactly coincide, and have a common perimeter. But as the sides of the circumscribed polygon cannot fall within the circle, nor the sides of the inscribed polygon without it, it follows *that the perimeters of the polygons will unite on the circumference of the circle, and become equal to it.*

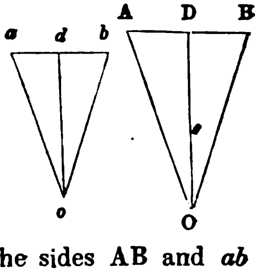
Cor. 3. When the number of sides of the inscribed polygon is indefinitely increased, and the polygon coincides with the circle, the line OI, drawn from the centre O, perpendicular to the side of the polygon, will become a radius of the circle, and any portion of the polygon, as ABCO, will become the sector OAKBC, and the part of the perimeter AB+BC, will become the arc AKBC.

Cor. 4. Hence also, as a regular polygon of an indefinite number of sides, coincides with a circle, whatever may be its radius or whatever its curvature, it may be inferred that if a polygon of an indefinite number of sides be described about a curve of any kind, the perimeter of the polygon, as also its area, may be regarded as equal to the length of the curve and to its area.

Scholium. The problem of the circle's quadrature consists in finding a square equivalent to a circle whose radius or diameter is given. Hence, if a polygon can be found equivalent to the circle, this polygon may be reduced to an equivalent square (Prob. IX. Sch. B. IV.), and in that case the quadrature of the circle would be obtained.

PROPOSITION XIII. THEOREM.

The perimeters of two regular polygons, having the same number of sides, are to each other as the radii of the circumscribed circles, and also, as the inscribed circles; and their areas are to each other as the squares of those radii.



Let AB be the side of one polygon, O, the centre, and consequently OA the radius of the circumscribed circle, and OD, perpendicular to AB, the radius of the inscribed circle; let *ab*, in like manner, be a side of the other polygon, *o* its centre, *oa* and *od* the radii of the circumscribed and the inscribed circles. The perimeters of the two polygons are to each other as the sides AB and *ab* (Prop. XXIII. Cor. B. IV.): but the angles A and *a* are equal, being each half ot the angle of the polygon; so also are the angles B and *b*; hence the triangles ABO, *abo*, are similar, as are likewise the right angled triangles ADO, *ado*; hence AB : *ab*::AO : *ao*::DO : *do*; hence the perimeters of the polygons are to each other as the radii AO, *ao*, of the circumscribed circles, and also, as the radii DO, *do*, of the inscribed circles.

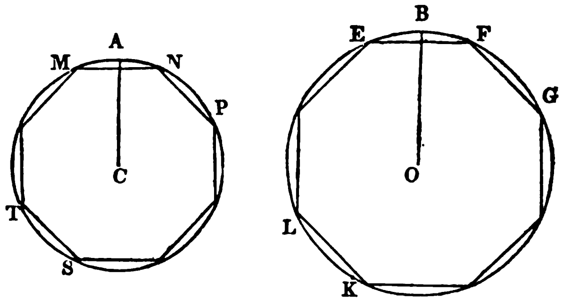
The surfaces of these polygons are to each other as the squares of the homologous sides AB, *ab*; they are therefore likewise to each other as the squares of AO, *ao*, the radii of the circumscribed circles, or as the squares of OD, *od*, the radii of the inscribed circles.

PROPOSITION XIV. THEOREM.

The circumferences of circles are to each other as their radii, and the areas are to each other as the squares of their radii.

Let us designate the circumference of the circle whose radius is CA by *circ.* CA; and its area, by *area* CA: it is then to be shown that

circ. CA : *circ.* OB::CA : OB, and that
area CA : *area* OB::CA^2 : OB^2.

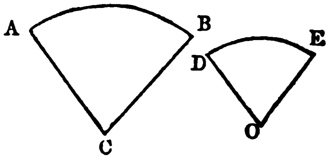


Inscribe within the circles two regular polygons of the same number of sides. Then, whatever be the number of sides, their perimeters will be to each other as the radii CA and OB (Prop. XIII.) Now, if the arcs subtending the sides of the polygon be continually bisected, until the number of sides of the polygons shall be indefinitely increased, the perimeters of the polygons will become equal to the circumferences of the circumscribed circles (Prop. XII. Cor. 2.), and we shall have

$$circ.\ CA : circ.\ OB :: CA : OB.$$

Again, the areas of the inscribed polygons are to each other as CA^2 to OB^2 (Prop. XIII.) But when the number of sides of the polygons is indefinitely increased, the areas of the polygons become equal to the areas of the circles, each to each (Prop. XII. Cor. 1.); hence we shall have

$$area\ CA : area\ OB :: CA^2 : OB^2.$$



Cor. The similar arcs AB, DE, are to each other as their radii AC, DO; and the similar sectors ACB, DOE, are to each other as the squares of their radii.

For, since the arcs are similar, the angle C is equal to the angle O (Def. 3. B. IV.); but C is to four right angles, as the arc AB is to the whole circumference described with the radius AC (Prop. XVIII. B. III.); and O is to the four right angles, as the arc DE is to the circumference described with the radius OD: hence the arcs AB, DE, are to each other as the circumferences of which they form part: but these circumferences are to each other as their radii AC, DO; hence

arc AB : *arc* DE::AC : DO.

For a like reason, the sectors ACB, DOE, are to each other as the whole circles; which again are as the squares of their radii; therefore

sect. ACB : *sect.* DOE::AC^2 : DO^2.

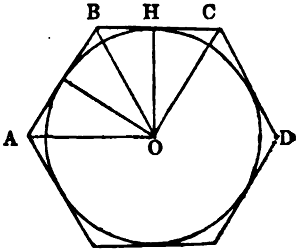
Cor. 2. It readily follows that ***similar segments are also as the squares of the radii***, for they result from similar sectors, by taking away from each the triangle formed by the chord and radii, which triangles being similar, are also to each other as the squares of the radii; therefore the sectors and triangles being proportional, it follows (Prop. XV. B. I.) that the segments also are as the sectors or as the squares of the radii, or indeed as the squares of their chords.

Cor. 3. Hence also, and in reference to (Prop. XIX. Cor. 3. and XXIV. Cor. E. B. IV.), the perimeters of any similar figures whatsoever are as their like sides, or as lines similarly drawn in each, and their areas are as the squares of their like parts.

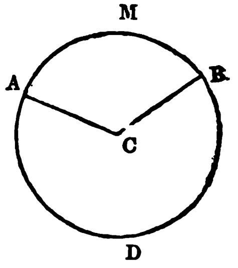
Scholium. From this proposition and its corollaries, we may infer that if on the several sides of a right angled triangle as diameters, circles be described, or if similar sectors or segments be formed on the sides, it will result that the figure on the hypothenuse will be equivalent to both those on the sides, since the same reasoning as employed in (Prop. XXIV. and Cor. B. IV) will also apply to this.

PROPOSITION XV. THEOREM.

The area of a circle is equal to the product of its circumference by half the radius.



For if about a circle whose radius is OH, we describe a regular polygon, ABCD, &c., the area of the polygon will be equal to the product of the perimeter ABCD, &c. into half OH, whatever be the number of the sides of the polygon (Prop. XI. Cor. 3.) Let the number of the sides of the polygon be indefinitely increased, and the perimeter of the polygon becomes the perimeter of the circle. Hence, the area of a circle is equal to the product of the circumference into half the radius.



Cor. 1. The area of a sector is equal to the arc of that sector multiplied by half its radius.

For, the sector ACB is to the whole circle as the arc AMB is to the whole circumference ABD (Prop. XVIII. Sch. B. III.), or as AMB×½AC is to ABD×½AC. But the whole circle is equal to ABD×½AC; hence the sector ACB is measured by AMB×½AC.

Cor. 2. Let the circumference of the circle whose diameter is unity, be denoted by π; then, because circumferences are to each other as their radii or diameters, we shall have the diameter 1 to its circumference π, as the diameter 2CA is to the circumference whose radius is CA, that is, $1 : \pi :: 2BA :$ *circ.* CA; therefore *circ.* CA$=\pi\times 2$CA. Multiply both terms by ½CA; we have ½CA×*circ.* CA$=\pi\times$CA^2, or *area* CA$=\pi\times$CA^2: hence the area of a circle is equal to the product of the square of its radius by the constant number π, which represents the circumference whose diameter is 1, or the ratio of the circumference to the diameter.

In like manner, the area of the circle, whose radius is OB, will be equal to $\pi\times$OB^2; but $\pi\times$CA$^2 : \pi\times$OB$^2 ::$ CA$^2 :$ OB^2; hence *the areas of circles are to each other as the squares of their radii,* which agrees with the preceding theorem.

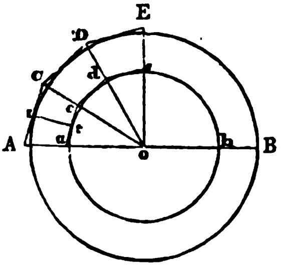
Scholium. We have already observed, that the problem of the quadrature of the circle consists in finding a square equal in surface to a circle, the radius of which is known. Now it has just been proved, that a circle is equivalent to the rectangle contained by its circumference and half its radius; and this rectangle may be changed into a square, by finding a mean proportional between its length and its breadth (Prob. III. B. IV.) To square the circle, therefore, is to find the circumference when the radius is given; and for effecting this, it is enough to know the ratio of the circumference to its radius, or its diameter.

Hitherto the ratio in question has never been determined except approximately; but the approximation has been carried so far, that a knowledge of the exact ratio would afford no real advantage whatever beyond that of the approximate ratio.

Archimedes showed that the ratio of the circumference to the diameter is included between $3\frac{10}{70}$ and $3\frac{10}{71}$; hence $3\frac{1}{7}$ or $\frac{22}{7}$ affords at once a pretty accurate approximation to the number above designated by π; and the simplicity of this first approximation has brought it into very general use. *Metius*, for the same number, found the ,much more accurate value $\frac{355}{113}$. At last the value π, developed to a certain order of decimals, was found by other calculators to be 3.14159265358-97932, &c.; and some have had patience enough to continue these decimals to the hundred and twenty-seventh place, and recently indeed to the hundred and fifty-fourth place. Such an approximation is evidently equivalent to perfect correctness: the root of an imperfect power is in no case more accurately known.

PROPOSITION XVI. THEOREM.

The area of the space included between two concentric circles, is equal to half the sum of the two circumferences multiplied by their distance from each other.



Let AB and *ab* be two concentric circles, and the area or space included between them will be equal to half the sum of the two circumferences of AB and *ab*, multiplied by the distance I*t* between the two circles.

For let two similar polygons ACDE, *acde*, be described about those circles, having their corresponding sides agreeing with or parallel to each other, viz: AC parallel to *ac*, CD parallel to *cd*, &c., and the spaces between the two polygons will be composed of the several trapeziums DC, *dc*, &c., each of which is equal to half the sum of the two corresponding sides of the two polygons, multiplied by the perpendicular distance between them (Prop. IX. B. IV.) Thus $\frac{1}{2}(AO+ac)\times tI$=area AC*ca*, and hence the area of all the trapeziums will be equal to half the sum of the perimeters of the two polygons, multiplied by the common distance *t*I. Now, let the number of the sides of the two concentric polygons be indefinitely increased, and the space between them will still be equal to half the sum of the two perimeters, multiplied by the distance between them. But (Prop. XII. Cor. 1.) when the number of sides of a polygon is indefinitely increased, the polygon becomes identical with the circle.

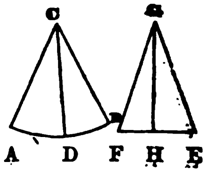
Cor. 1. We have from this and the preceding proposition, shown that the areas of circular spaces, as well as rectilineal spaces, are equal to their circumferences multiplied by certain right lines. We have, therefore, surfaces bounded by two orders of lines, viz: right lines and also circles, whose areas are equal to such right lines or circles, as bases, multiplied by their altitudes on such bases.

Cor. 2. The area of the sectoral segment E*eb*B is equal to half the sum of the two arcs EB, *eb*, multiplied by E*e*, the distance between the two arcs.

Scholium. 1. The space enclosed between two concentric circles is called a circular ring, or an annular space.

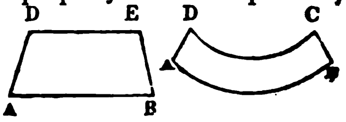
Scholium. 2. Let *ob* the diameter of the inner circumference equal 1, then may π represent the circumference of that circle; and if we call π' the circumference of the outer circle, and d=the difference of their radii, then the area of the ring will be measured by $\frac{\pi+\pi'}{2}\times d$, or $\frac{1}{2}d.(\pi+\pi')=\frac{1}{2}d\pi+\frac{1}{2}d\pi'$. And the area of any segment EB*be* of the ring will be expressed by the ratio of such portion, to the whole ring; or if the arc EB is $\frac{1}{n}$ part of the whole circumference AB, and $eb=\frac{1}{n}$ part of the circumference *ab*, then will the segment be expressed by $\frac{1}{2}d\ (\pi+\pi')\frac{1}{n}$.

Scholium. 1. Hence we have figures whose bases are of another order of lines than right lines, and whose areas are equal to those bases multiplied by certain other lines pertaining to those figures.



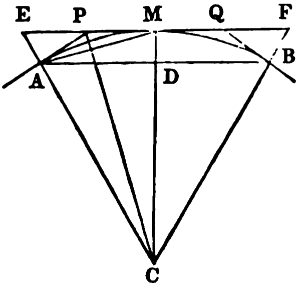
Thus (Prop. XV. Cor. 1.) we have the sector ABC, which is equal to its circular base AB multiplied by half the altitude. This is also the property of the triangle EFG, which is equal to its base EF multiplied by the altitude GH; but in order that a trilinear figure with a circular base should be equal to that base multiplied by the half its altitude, that altitude must be equal to the radius of the circular arc of the base.

We have also the sectoral segment ABCD, which is equal to half the sum of the two bases AB, DC, multiplied by AD the altitude, which is also the property of the trapezium, ABED.



PROPOSITION XVII. PROBLEM.

The surface of a regular inscribed polygon, and that of a similar polygon circumscribed, being given; to find the surfaces of the regular inscribed and circumscribed polygons having double the number of sides.



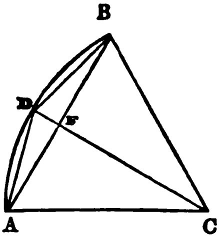
Let AB be a side of the given inscribed polygon; EF, parallel to AB, a side of the circumscribed polygon; C the centre of the circle. If the chord AM and the tangents AP, BQ, be drawn, AM will be a side of the inscribed polygon, having twice the number of sides; and $AP+PM=2PM$ or PQ, will be a side of the similar circumscribed polygon (Prop. IX. Cor. 3.) Now, as the same construction will take place at each of the angles equal to ACM, it will be sufficient to consider ACM by itself, the triangles connected with it being evidently to each other as the whole polygons of which they form part. Let A, then, be the surface of the inscribed polygon whose side is AB, B that of the similar circumscribed polygon; A' the surface of the polygon whose side is AM, B' that of the similar circumscribed polygon: A and B are given; we have to find A' and B'.

First. The triangles ACD, ACM, having the common vertex A, are to each other as their bases CD, CM; they are likewise to each other as the polygons A and A', of which they form part: hence $A : A' :: CD : CM$. Again the triangles CAM, CME, having the common vertex M, are to each other as their bases CA, CE; they are likewise to each other as the polygons A' and B, of which they form part; hence $A' : B :: CA : CE$. But since AD and ME are parallel, we have $CD : CM :: CA : CE$; hence $A : A' :: A' : B$; hence the polygon A', one of those required, is a mean proportional between the two given polygons A and B, and consequently $A' = \sqrt{A \times B}$.

Secondly. The altitude CM being common, the triangle CPM is to the triangle CPE as PM is to PE; but since CP bisects the angle MCE, we have $PM : PE :: CM : CE$ (Prop. XXI. B. IV.) $:: CD : CA :: A : A'$; hence $CPM : CPE :: A : A'$; and consequently $CPM : CPM+CPE$ or $CME :: A : A+A'$. But CMPA, or 2CMP, and CME are to each other as the polygons B′ and B, of which they form part; hence B′ : B::2A : A+A′. Now A′ has been already determined; this new proportion will serve for determining B′, and give us $B'=\frac{2A.B}{A+A'}$; and thus by means of the polygons A and B it is easy to find the polygons A′ and B′, which shall have double the number of sides.

PROPOSITION XVIII. THEOREM.

*A regular duodecagon is equivalent to ¾ of the square erected on the diameter of its circumscribed circle; or is equal to the square erected on the side of the equilateral triangle inscribed in the same circle.**



Let AC be the radius of a circle in which is inscribed the side AB of the hexagon; and if AD and DB be the sides of the regular duodecagon, it thence follows that by drawing the radius DC, it will cut the side AB perpendicularly (Prop. VI. B. III.) and divide it into two equal parts. Now it is evident, that the area of the duodecagon is equal to twelve times one of the triangles ADC or DCB; and as the triangle ADC is equal to the product of the radius by half of AF, or by the fourth part of the radius, since the chord AB is equal to the radius DC (Prop. V.) Hence the triangle ADC is equivalent to a fourth of the square of the radius; and 12 such triangles will be equal to three times the square of the radius, or to ¾ of the square of the diameter.

Again, the side of an equilateral triangle inscribed in a circle, the diameter being unity, is equal to $\sqrt{\frac{3}{4}}$; consequently, its square is also equal to ¾ of the square of the diameter, or equal to the duodecagon.

Scholium. Two of the inscribed polygons, viz: the square and the duodecagon, possess the property of having a simple numerical ratio to the square erected on the diameter of the circumscribed circle, for the inscribed square is exactly half of the circumscribed square.

PROPOSITION XIX. PROBLEM.

To find the approximate ratio of the circumference to the diameter.

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* First shown by *Suellius.*

Let the radius of the circle be 1; the side of the inscribed square will be $\sqrt{2}$ (Prop. VI. Sch.), that of the circumscribed square will be equal to the diameter 2; hence the surface of the inscribed square is 2, and that of the circumscribed square is 4. Let us therefore put $A=2$, and $B=4$; by proposition XVII. we shall find the inscribed octagon $A'=\sqrt{8}=2.8284271$, and the circumscribed octagon $B'=\frac{16}{2+\sqrt{8}}=3.3137085$. The inscribed and the circumscribed octagons being thus determined, we shall easily, by means of them, determine the polygons having twice the number of sides. We have only in this case to put $A=2.8284271$, $B=3.3137085$; we shall find $A'=\sqrt{A.B}=3.0614674$, and $B'=\frac{2\,A.B}{A+A'}=3.1825979$. These polygons of 16 sides will in their turn enable us to find the polygons of 32; and the process may be continued, till there remains no longer any difference between the inscribed and the circumscribed polygon, at least so far as that place of decimals where the computation stops, and so far as the 7th place, in this example. Being arrived at this point, we shall infer that the last result expresses the area of the circle, which, since it must always lie between the inscribed and the circumscribed polygon, and since those polygons agree as far as a certain place of decimals, must also agree with both as far as the same place.

We have subjoined the computation of those polygons, carried on till they agree as far as the seventh place of decimals.

Number of sides.	Inscribed polygon	Circumscribed polygon.
4	2.0000000	4.0000000
8	2.8284271	3.3137085
16	3.0614674	3.1825979
32	3.1214451	3.1517249
64	3.1365485	3.1441148
128	3.1403311	3.1422236
256	3.1412772	3.1417504
512	3.1415138	3.1416321
1024	3.1415729	3.1416025
2048	3.1415877	3.1415951
4096	3.1415914	3.1415933
8192	3.1415923	3.1415928
16384	3.1415925	3.1415927
32768	3.1415926	3.1415926

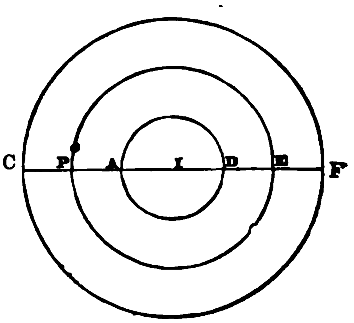
The area of the circle, we infer therefore, is equal to 3.1415926. Some doubt may exist perhaps about the last decimal figure, owing to errors proceeding from the parts omitted; but the calculation has been carried on with an additional figure, that the final result here given might be absolutely correct even to the last decimal place.

Since the area of the circle is equal to half the circumference multiplied by the radius, the half circumference must be 3.1415926, when the radius is 1; or the whole circumference must be 3.1415926, when the diameter is 1: hence the ratio of the circumference to the diameter, formerly expressed by π, is equal to 3.1415926. The number 3.1416 is the one generally used.

Scholium. We shall have occasion to recur to this subject again, in a subsequent volume, when from principles there demonstrated, we shall not only endeavor to show that the quadrature of the circle is possible; that is, that it is possible to give an expression for its value; but we shall develope some surprising properties of the circle and other curvilinear figures; which opens to Geometry a wider field of investigation.

PROPOSITION XX. THEOREM.

If there be any number of concentric circles, such that their several radii form a series in arithmetical progression whose common difference is equal to the radius of the first, then will the annular space between the first and second be equal to the circumference of the first multiplied by its distance from the second + the area of the inner circle; and the annular space between the second and third will be equal to the circumference of the second multiplied by its distance from the third + the area of the inner circle; and so on to any number.



Let AD, PE, CF, be three concentric circles, such that their several radii IA, IP, IC, shall form a series in arithmetical progression whose common difference is equal to IA; then will the annular space included between the circles AD and PE be equal to the circumference AD× IP+area AD. And the annular space between the circles PE and CF will be equal to the circumference PE×PC +area AD.

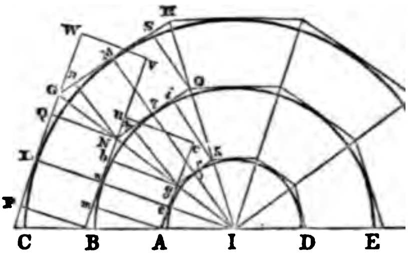
For (Prop. XVI.) the area of the annular space between the circumferences AD and PE, is equal to half the sum of their circumferences multiplied by the distance AP. The circumference PE=twice the circumference AD, its radius by hypothesis being double that of AD. Let circumference $AD=\pi$, then will circumference $PE=2\pi$; and let IA or AP= S, then will $\frac{\pi+2\pi}{2}\times 2S=\pi S+\frac{1}{2}\pi S=$the annular space; but the area AD is equal to $\frac{1}{2}\pi S$, hence the annular space between the circles AD and PE is equal to the circumference AD multiplied by the distance AP+the area of the circle AD.

Again, the area of the annular space between the circles PE and CF is equal to half the sum of their circumferences (PE+CF)×PC; and if circumference $AD=\pi$, circumference $PE=2\pi$, and circumference $CF=3\pi$, their radii being in that proportion by hypothesis, then will $\frac{3\pi+2\pi}{2}\times S$, or $1\frac{1}{2}\pi S+\pi S=2\pi S+\frac{1}{2}\pi S=$the annular space; but the inner circle $AD=\frac{1}{2}\pi S$, hence the annular space is equal to the circumference PE×PC+area of circle AD; and as the same may be shown for any number of concentric circles of the same series, it follows that the result is as enunciated in the proposition.

Cor. Hence the annular space included between any two concentric circles is equal to the circumference of the inner circle multiplied by the distance between the two + the area of a circle whose circumference is the difference of the two circumferences, and whose radius is equal to the distance between the two. Or it is equal to the circumference of the outer circle multiplied by the distance between the two—the area of a circle whose circumference is equal to the difference between the two circumferences, and whose radius is equal to the distance between the two circles.

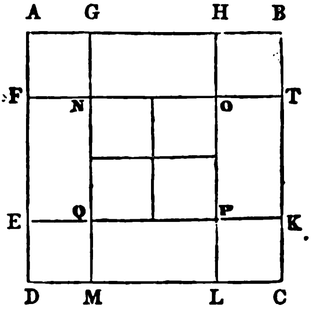
PROPOSITION XXI. THEOREM.

The angular space about a point and included within a given circumference, is equivalent to the angular space of a circular ring, the distance of whose inner and outer circumferences is equal to the radius of the given circle.



Let three regular, similar and concentric semi-polygons AD, BE, CF, be described about semicircles whose radii form a series in arithmetical progression, as in the last proposition. From the centre I, through the several a draw IG, IH, &c.; draw also the lines IL, IM, bisecti sides of the polygon. Then from the angles A and *g* inner polygon draw the lines A*m*, *gb*, perpendicular t and meeting BN in *m* and *b*; from the angles *g* and *h* *gk*, *hi*, perpendicular to *gh*, meeting NO in *k* and *i*; also from the angles B, N, O, similar lines BP, NQ, N perpendicular to the adjacent sides of the polygons will the triangles A*m*B, *gb*N, *gk*N, *hi*O, be each equ similar to the triangles I*e*A, I*eg*, I*fg*, I*fh*, their side angles being equal and similar; and the same is true triangles PBC, NQG, NRG, OSH. Hence, if we take all the above mentioned similar triangles from the BNO*hga*, there will remain the two rectangles A*gbm*, equal the rectangle AC*nm*; and if we take away the gles from the space BNOHGC, there will remain th rectangles BNQP, NOSR, equal the rectangle B Hence, the sum of the several similar triangles in each gonal space is equal, and is equal also to the angular about a point, with the radial distance equal to the di of the faces of the polygon from each other, and the r gles remaining in each portion is equal to the perim the inner polygon multiplied by the distance from th outer one. Let the number of the sides of the several gons be indefinitely increased, and the same will be but polygons of an indefinite number of sides are (Prop. XII. Cor. 1.), hence also the same is true with to the concentric circles, viz: the same angular space in each of the concentric circular rings, and it is equal angular space about the point I, the centre of the circle.

Cor. Hence, if to any rectangle a circle be added, radius is equal to one of the parallel sides of the rect their sum will be equal to the area of a circular rin distance of whose inner and outer circumferences is ec the radius of the circle added, and whose inner circ ence is equal to the length of one of the other parallel sides of the rectangle.

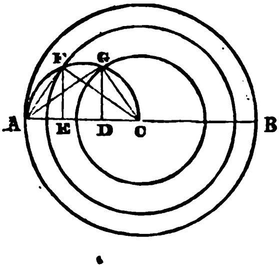


Scholium. The angular space between the two concentric squares, ABCD and QNOP, is equal to that of a square, one of whose sides is equal to twice the distance between the outer and inner ones. Thus, if the squares are described in such manner, that the side QP of the inner one shall be equal to 2QE, the distance between the two squares, then all the angular space, viz: AFNG, HOTB, KPLC, EQMD, will be equal to the inner square QNOP.

Hence, what has been shown to be true with regard to the angular space of the circular ring, and that of the space between two concentric squares, is also true of any angular space contained between any two concentric polygons, viz: whatever be the diameters of the polygons, the angular space is always the same, if the distance between them is the same.

PROPOSITION XXII. PROBLEM.

To divide a circle into any number of equal parts, by means of concentric circles.



Let it be proposed to divide the circle in the margin, whose centre is C, and diameter AB, into a certain number of equal parts, three for instance, by means of circles concentric with it.

Divide the radius AC into three equal parts, AE, ED, DC; draw the perpendiculars EF, DG, meeting the semi-circumference described upon AC, in the points F, G; draw CF, CG, and with these lines as radii from the centre C, describe circles: these circles will divide the proposed circle into the required number of equal parts.

For draw AF, AG; then the angle AGC being in a semicircle is a right angle; hence the triangles GAC, GDC, are similar, and consequently are to each other as the squares of their homologous sides, that is,

$$\begin{aligned} GAC : GDC &:: CA^2 : CG^2; \\ \text{but } GAC : GDC &:: CA : CD; \\ \text{hence } CA^2 : CG^2 &:: CA : CD; \end{aligned}$$

consequently, since circles are to each other as the squares of their radii, it follows that the circle whose radius is CA, is to that whose radius is CG, as CA is to CD, that is to say, the latter is one third of the former.

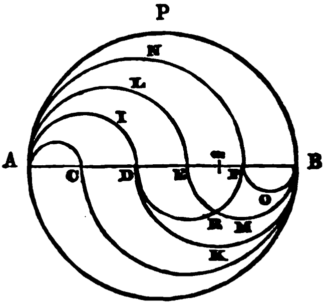
In like manner, by reasoning on the right angled triangles FAC, FEC, it may be proved that the circle whose radius is CF is two-thirds that whose radius is CA. Consequently the smaller circle and the two surrounding *annular* spaces are all equal.

PROPOSITION XXIII. PROBLEM.

To divide a circle into any number of equal parts, which shall be equal both in surface and boundary.

Let it be required to divide the circle whose diameter is AB into five parts, which shall be equal both in surface and boundary.

Divide the diameter into five equal parts, in the points C, D, E, F, and upon AC, AD, AE, AF, describe semicircles. Describe semicircles also upon BC, BD, BE, BF, but on the opposite side of the diameter AB: then the circle will be divided into the proposed number of curvilinear spaces equal to each other both in surface and boundary.



For the diameter AB is to the diameter AD, as the circumference on AB is to the circumference on AD, or as the semi-circumference on AB is to the semi-circumference on AD; also AB is to BD as the semi-circumference on AB is to that on BD. Consequently AB is to AD and BD together, as the semi-circumference APB is to the boundary AIDKB: therefore these two lines are equal. In a similar manner it may be shown that each of the other boundaries is also equal to the semi-circumference APB.

Again, the circle on AB is to the circles on AE, AF, as the square of AB is to the squares of AE, AF, respectively. Consequently,

$$circ.\ AB : circ.\ AF - circ.\ AE :: AB^2 : AF^2 - AE^2.$$

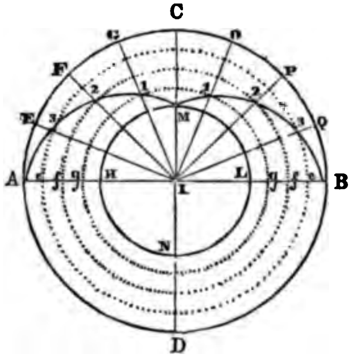
Now (Prop. XII. B. IV.) $AF^2 - AE^2 = (AE+AF).EF$: Let m be the middle of EF, then $AE+AF=2Am$. Hence,

$$circ.\ AB : \tfrac{1}{2}\ circ.\ AF - \tfrac{1}{2}\ circ.\ AE :: AB^2 : Am.EF;$$

that is to say, the circle on AB is to the space included between semicircles on AE and AF, as the square of AB is to the rectangle $Am.EF$. In exactly the same way it is proved that the circle on AB is to the space between the semicircles on BE and BF, as the square of AB is to the rectangle of $Bm.EF$. It follows therefore (Prop. XXI. B. I.) that the circle on AB is to the whole space ALEMBOFN, as the square of AB to the sum of the rectangles $Am.EF$, $Bm.EF$, that is, to the rectangle AB.EF, and this rectangle is one-fifth of AB^2; consequently the space ALEMBOFN is one-fifth of the circle, and the same may, in like manner, be shown of the other spaces.

PROPOSITION XXIV. PROBLEM.

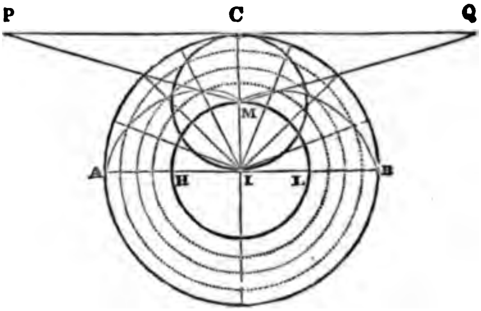
To divide a circular ring in such manner, that one portion may be equal to a rectangle formed on a line equal to the inner circumference of the ring, and whose altitude is equal to the distance of the two circamferences.



With the radius IH describe a circle HL; then with any other radius AI, which suppose equal 2IH, describe the circle AB, and the space included between the two circles is a circular ring; take the distance ADB on the outer circumference equal the inner circumference; divide the remaining portion of the outer circumference into any number of equal parts at the points E, F, G, C, O, P, Q, and draw the several radii

IE, IF, IG, &c. Then divide the distance AH between the two circumferences into half the number of equal parts at the points *e*, *f*, *g*, and through those points describe circles *gg*, *ff*, *ee*; from the point M, through the points 1, 2, 3, where those circles cut the radial lines, describe the curves M123A, M123B, and because the circumferences of circles are as their radii, the portions of those circles cut off towards D will all be equal.

Let the number of the concentric circles, and also the radial divisions be indefinitely increased, and the area is a function of those circular arcs cut off, and the sum of those circular arcs so cut off may be regarded as the area, since they will evidently agree with, and be equivalent to a similar number of parallel lines of equal lengths, drawn in the rectangle whose base is equal to the inner circumference, and altitude the distance AH.

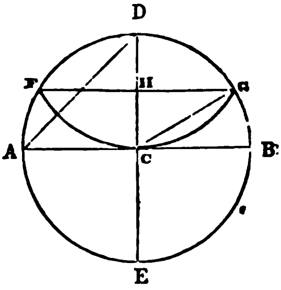


Cor. If a triangle PQM be formed, whose base PQ equals the semicircumference ACB=the circumference of the circle HL or CI, and whose altitude CM is the radius CM of the circle CI = the distance between the circumferences AB, HL, then the triangle wi[illegible] be equal to the circle CI—the portion of the ring AMBC[illegible] cut off by the curves AM.MB; since the triangle is equal to i[illegible] base PQ×CM, and the circle CI is equal to circumferenc[illegible] CI×radius CM.

Scholium. Whatever may be the distance between the tw[illegible] circumferences of the ring, it is still true that if a distance [illegible] set off on the outer circumference equal to the inner one, an[illegible] if portions of all the other concentric circles are taken equa[illegible] to these, the space cut off by the curve through their termi[illegible]nations is equal to the rectangle formed from a line equal t[illegible] the inner circumference, and the distance between the inne[illegible] and outer circumference; and when the distance between th[illegible] two circumferences is equal to the radius of the inner circl[illegible] the portion AFCBM cut off by the curve, is equal to the in[illegible]scribed circle HL.

PROPOSITION XXV. PROBLEM.

To draw a right line nearly equal to the circumference of a given circle.

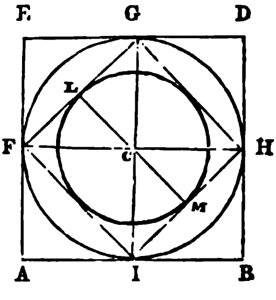


Through the centre C of the circle draw the two diameters AB, DE, perpendicular to each other. Join AD. Then with the radius CD and any point D in the circumference as a centre, describe an arc FCG, and from the points F, G, where this arc cuts the circumference of the circle, draw the line FG, and FG+AD will be nearly equal to the semicircumference ADB; hence the double of that sum, or 2FG+2AD=the circumference nearly.

For, the length of the chord AD being the hypothenuse of a right angled triangle, whose two sides AC, CD, are equal, it is equal to $\sqrt{2AC^2}$; and if AC=1, then AD=1.41421, &c. And GH=½FG is the side of the right angled triangle GHC, whose hypothenuse CG is the radius, and the side HC =half the radius; hence HG=$\sqrt{(CG^2—CH^2)}$=,86602, &c., and FG=1,73204, &c., and AD+FG=3,14625=the semicircumference, which is true to the first three figures, but the fourth should be 1 instead of 6.

PROPOSITION XXVI. PROBLEM.

To describe a circle equal to half a given circle.



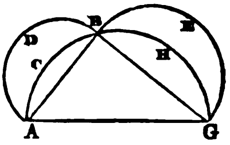
Let FGHI be the given circle; inscribe in the circle the square FLGHMI. If in this square a circle, whose radius is CL, be described, the circle so described will be equal to half the circle whose radius is CF.

For, circumscribe about the given circle the square ABDE, and this square will evidently be equal to twice the square FGHI, and the two squares will be squares described on the diameters of those circles respectively; but circles are as the squares erected on their diameters. Hence the circle inscribed in the square FGHI is equal to half the circle inscribed in the square ABDE.

Cor. Hence if a circle be circumscribed about a square and another circle be inscribed, the circumscribed circle will be double the inscribed one. And also, if a square circumscribe a circle, and another square is inscribed in the same circle, the circumscribing square will be double the inscribed square.

PROPOSITION XXVII. PROBLEM.

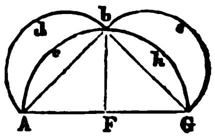
If on the sides of a right angled triangle as diameters, semicircles be described, the one described on the hypothenuse will be equal to the sum of the other two.



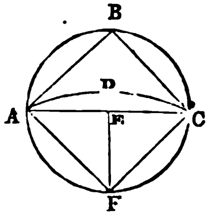
Let ABG be a right angled triangle, right angled at B. Describe on the hypothenuse AG, as a diameter, the semicircle ACHG, and on AB and BG as diameters, the semicircles ADB, BEG; and the semicircle ACHG described on the hypothenuse will be equivalent to the other two.

For, since it has been shown that circles are proportional to the squares of their diameters, it follows that if the sides of a triangle are made the diameters of circles, those circles will be as the squares of those sides respectively; but the square on the hypothenuse of a right angled triangle is equivalent to the other two (Prop. XXIV. B. IV.); hence the semicircle described on the hypothenuse must also be equivalent to the two semicircles described on the other two sides.

Cor. Hence, the two lunes ADBCA and BEGHB are together equivalent to the triangle ABG. For since the semicircle ACHG=ADB+BEG, if the segments ACB, BHG, which are common to both, be taken away, there will remain the spaces ADBCA+BEGHB=the triangle ABG.



Scholium. If the sides A*b*, *b*G, are equal, the two lunes A*dbc*A, *bo*G*hb*, will evidently be equal, and each will be equal to half the triangle A*b*G, or equal to A*b*F or G*b*F.



Scholium 2. Let ABC be a semicircle on the diameter AC, and AFC an isosceles right angled triangle. If from the point F as a centre, there be described through A and C the arc of a circle ADC on the base AC, the lunule ABCDA will be equivalent to the triangle CAF.

Since the square of FC is double the square of EC or of EF (Prop. XXVI. Cor.), the circle described with the radius FC will be double that described with the radius EC; consequently a fourth part of the former, or the quadrant FADC, will be equivalent to the half of the second, or to the semicircle ABC. If the common segment ADCEA therefore be taken away, the remainders, that is to say, the triangle AEC on the one hand, and the lunule ABCDA on the other, will be equivalent.*

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* These are the lunules of *Hippocrates*, of *Chios*. Those who wish to pursue this subject farther, will find their curiosity amply gratified by referring to *Riddle's* edition of *Hutton's Mathematical Recreations*.

BOOK VI.

ISOPERIMETRY.

Definitions.

1. When the value of one quantity is dependent on the relative values of two or more other quantities, in such manner that any variation in the ratio of those other quantities produces a variation in the ratio of this to those others, the relative value of such quantity is called a *maximum*, when it has reached the greatest relative magnitude it can possibly attain; and on the contrary, when it has arrived at the least possible relative magnitude, it is called a *minimum*.

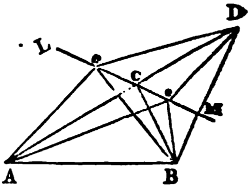
2. *Isoperimeters*, or *Isoperimetrical figures*, are those which have equal perimeters.

3. The *locus* of any point or intersection, &c., is the right line or curve in which these are always situated.

PROPOSITION I. THEOREM.

Of all triangles of the same base, and whose vertices fall in a right line given in position, the perimeter of the one whose other two sides form equal angles with this line, is a minimum.

Let AB be the common base of a series of triangles ABc, ABC, &c., whose vertices C, *c*, &c. are all in the right line LM, given in position; then is the triangle, whose sides AC, BC, are inclined in equal angles to the line LM, the least in perimeter.



For let BM be drawn from B perpendicular to LM, and produced to D, making MD=BM; join AD, and from the point C, where AD cuts LM, draw BC; also, from any other point *c* assumed in LM, draw *c*A, *c*B, *c*D. Then the triangles DMC, BMC, having the angle DCM=angle ACL (Prop. VI. B. II.) =MCB; by hypothesis, DMC=BMC, and DM

$=BM$, and MC common to both; hence also $DC=BC$ (Prop. VIII. B. II.)

So also we have $cD=cB$. Hence, $AC+CB=AC+CD=AD<(Ac+cD)$ (Prop. X. B. II.), or less than its equal $Ac+cB$. And consequently, $AB+BC+AC$ is less than $AB+Bc+Ac$.

Cor. 1. Of all equivalent triangles, having the same base, the isosceles triangle has the smallest perimeter.

For the locus of the vertices of all triangles of the same altitude will be in a right line parallel to the base; and when LM in the above figure becomes parallel to AB, then, since in such case the angle $MCB=ACL$, MCB also equals CBA (Prop. XXIII. Cor. 2. B, II.), $ACL=CAB$; it follows that $CAB=CBA$, and consequently $AC=CB$ (Prop. XV. B. II.)

Cor. 2. Of all triangles of the same surface the *equilateral triangle* has the minimum perimeter.

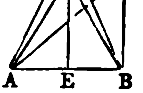
For we have just shown that the triangle of the least perimeter with the same base and altitude must be isosceles, whichever of the sides are considered as the base; therefore the triangle of the smallest perimeter has each pair of its sides equal and is hence equivalent.

Cor. 3. Of all rectilinear figures with a given surface and a given number of sides, that which is equilateral has the smallest perimeter.

For so long as any two adjacent sides are not equal, we may draw a diagonal to become a base to those two sides, and then draw an isosceles triangle equivalent to the triangle so cut off, but of less perimeter; hence the corollary is manifest.

PROPOSITION II. THEOREM.

Of all triangles of the same base and equal perimeters, the isosceles triangle has the greatest surface.



Let ABC, ABD, be two triangles of the same base AB, and with equal perimeters, of which the one ABC is isosceles, the other is not; then the triangle ABC has a surface, and consequently an altitude, greater than the surface or altitude of the triangle ABD.

Draw DF parallel to BA, to cut CE perpendicular to AB in F; then it is to be proved that CE is greater than FE.

The triangles AFB, ADB, are equal both in base and altitude; but the triangle AFB is isosceles, while ADB is scalene; therefore the triangle AFB has a smaller perimeter than the triangle ADB (Prop. I. Cor. 1.), or than ACB, by hypothesis; consequently, AF<AC; and in the right angled triangles AEF, AEC, having AE common, we have FE<CE.

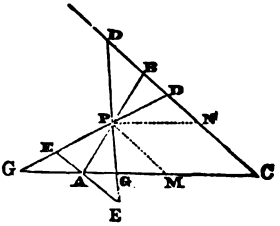
Cor. Of all isoperimetrical triangles, that whose surface is a maximum, is equilateral. And generally, of all isoperimetrical figures, of which the number of sides is given, that which has the greatest surface, has all its sides equal.

For so long as any two adjacent sides are not equal, the surface may be augmented without increasing the perimeter.

Scholium. By a parity of reasoning may it be proved that of all triangles of equal altitudes, and of which the sum of the two sides is equal, that which is isosceles has the greatest base. And of all triangles standing on the same base and having equal vertical angles, that which is isosceles is the greatest.

PROPOSITION III. THEOREM.

Of all right lines that can be drawn through a given point between two converging right lines given in position, that which is bisected by the given point forms with the other two lines the least triangle.



Let any series of lines AB, GD, &c., be drawn through the point P, to meet the right lines CA, CD, given in position, and the line AB, which is bisected by the point P, forms with CA, CD, the least triangle ABC.

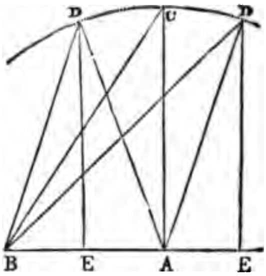
For through A draw EE parallel to CD, meeting DG and DG produced; then the triangles PBD, PAE, are manifestly equiangular; and since the corresponding sides PB, PA, are equal, the triangles are equal also. Hence PBD will be less or greater than PAG, according as CG is less or greater than CA. In the former case, let PACD be added to both; then we shall have BAC<DGC (Ax. 2.). In the latter case, if PGCB be added, we shall have DCG >BAC; and consequently, in this case also, BAC<DCG.

Cor. If PM and PN be drawn parallel to CB and CA respectively, the two triangles PAM, PBN, will be equal, and since AM in such case will be equal to PN=MC, these two triangles will be equivalent to the rhomboid PMCN, which rhomboid is equal to half ABC, but less than half DGC.

Whence it follows, that a rhomboid is always less than half a triangle in which it is inscribed, except when the base of the triangle is just double that of the rhomboid, this being the maximum rhomboid inscribed in the triangle.

PROPOSITION IV. THEOREM.

Of all triangles, in which two sides are given in length, the greatest is that in which the two given sides are perpendicular to each other.

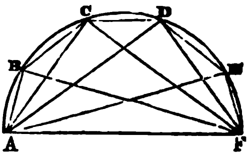


Let BAC, BAD, be two triangles in which the side AB is common, and the side AC=the side AD; if the angle BAC is right, the triangle BAC will be greater than any triangle BAD, of which the angle A is acute or obtuse.

For the base AB, being the same in each, the two triangles BAC, BAD, are to each other as their altitudes AC, DE; but the perpendicular DE is shorter than AD or AC; hence, the triangle BAD is less than the triangle BAC.

PROPOSITION V. THEOREM.

Of all rectilinear figures, in which all the sides except one are known, the greatest is that which may be inscribed in a semicircle whose diameter is that unknown side.



Let ABCDEF be the greatest polygon which can be formed from the sides AB, BC, CD, DE, EF, and one indeterminate side. This figure will be a maximum when the polygon may be inscribed in a semicircle of which that unknown side is the diameter.

Draw the diagonals AD, DF; and if the angle ADF is a right angle, the triangle ADF is a maximum; but if not, then by making it right, the triangle would be augmented, (Prop. IV.), and with it the whole polygon would be augmented, because the remaining parts ADC, CBA, DFE, would continue exactly the same. But since the angle ADF is a right angle, it being an angle in the semicircle by hypothesis, the triangle ADF is already a maximum, and hence the same may be said of ABF, ACF, AEF; hence when all the angles A, B, C, D, E, F, are in the semi-circumference of a circle of which the unknown side AF is the diameter, the figure is a maximum.

Scholium 1. The area of a polygon inscribed in a semicircle will not be altered by varying the order of the sides.

The sides AB, BC, CD, DE and EF, are the chords of so many arcs: the sum of these arcs, in whatever order they are arranged, will evidently be equal to the semi-circumference.

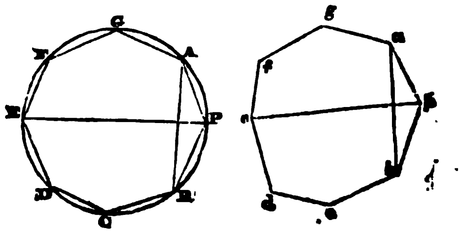
And the segments included between the given sides and their arcs, will be the same in whatever part of the circumference they are situated. But the area of the polygon is equal to the area of the semicircle, diminished by the sum of these segments.

Scholium 2. If a polygon, of which all the sides except one are given, be inscribed in a semicircle whose diameter is that unknown side; a polygon having the same given sides cannot be inscribed in any other semicircle which is either greater or less than this, and whose diameter is the undetermined side.

The given sides AB, BC, CD. DE and EF, are the chords of arcs, which together are equal to the semi-circumference, but in a larger circle each would be the chord of a smaller portion of the whole semi-circumference; and therefore the sum of the arcs would in this case be less than a semi-circumference; and in a smaller circle, each would be the chord of a greater part of the whole, and the sum of the arcs would be greater than a semi-circumference.

PROPOSITION VI. THEOREM.

Of all polygons whose sides are given, that which can be inscribed in a circle is a maximum.



Let ABCDEFG be the polygon inscribed, and *abcdefg* a polygon with equal sides but not inscribable in a circle, but AB=*ab*, BC=*bc*, &c.; yet the former polygon is greater than the latter.

Draw the diameter EP; join AP, PB; upon ab=AB make the triangle abp equal in all respects to ABP; and join ep. Then of the two figures $edcbp$, $pagfe$, one at least is not by hypothesis inscribable in the semicircle of which ep is the diameter. Consequently, one at least of those figures is smaller than the corresponding part of the figure PABCDEFG (Prop. V.); therefore the polygon APBCDEFG is greater than the figure $apbcdefg$; and if from these there be taken the respective triangles APB, apb, which are equal by construction, there will remain the polygon ABCDEFG greater than $abcdefg$.

Scholium. A polygon of which all the sides are given in length cannot be inscribed in circles of different diameters (Prop. V. Sch. 2.), neither will the area of the polygon be altered, although the order of the sides may be changed.

PROPOSITION VII. THEOREM.

The regular polygon is the greatest of all the polygons which have equal perimeters and the same number of sides.

For the greatest figure, under the given conditions, has all its sides equal (Prop. II. Cor.); and if the sum of the sides and also their number is given, the sides themselves become known. Hence (Prop. VI.), the figure is inscribable in a circle; consequently, it is a regular polygon.

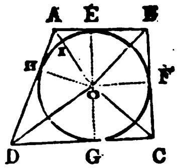
PROPOSITION VIII. THEOREM. (*Converse Prop. VII.*)

A regular polygon has a smaller perimeter than an irregular one with an equivalent surface and an equal number of sides.

Let R and W be two equivalent figures, having the same number of sides, of which R is regular and W irregular; let also R′ be a regular figure similar to R, and having a perimeter equal to that of W. Then (Prop. VII.) R′>W; but W=R, therefore R′>R. But R′ and R are similar; consequently, *perimeter* of R′>*perim.* R, while *perim.* R′=*perim.* W by hypothesis. Hence *perim.* W>*perim.* R.

PROPOSITION IX. THEOREM.

The surfaces of polygons circumscribed about the same or equal circles, are respectively as their perimeters.



Let the polygon ABCD be circumscribed about the circle EFGH, and let this polygon be divided into triangles, by lines drawn from its several angles to the centre O of the circle, and draw the several radii OE, OF, OG, OH, perpendicular to the sides of the polygon respectively; and the triangles formed on those sides as bases will all have equal altitudes, equal to the radii of the circle; since those sides are all tangents of the circumference, the area of each triangle ABO, BCO. is equal to its base AB, BC, multiplied by half the radius OE or OF. Hence the area of the whole polygon ABCD is equal to the product of the whole perimeter AB, BC. CD, DA, of the polygon multiplied by half the radius of the inscribed circle.

But the area of the circle is equal to the product of the circumference multiplied by half the radius (Prop. IX. Cor. 3. B. V.) Therefore the area of the circle is to that of the polygon, as the circumference of the former to the perimeter of the latter.

Now let P and P′ be any two polygons circumscribing a circle C; then by the foregoing we have

$$\textit{area}\ C : \textit{area}\ P :: \textit{circum.}\ C : \textit{perim.}\ P,$$
$$\textit{area}\ C : \textit{area}\ P' :: \textit{circum.}\ C : \textit{perim.}\ P'.$$

But since the antecedents of both these proportions are alike, the consequents are proportional (Prop. XIX. B. I.), that is,

$$\textit{area}\ P : \textit{area}\ P' :: \textit{perim.}\ P : \textit{perim.}\ P'.$$

Cor. 1. Each of the triangular portions ABO, of a polygon circumscribing a circle, is to the corresponding circular sector, as the side AB of the polygon to the arc of the circle included between AO, BO.

Cor. 2. Every circular arc is greater than its chord, and less than the sum of the tangents drawn from its extremities and produced till they meet.

The first part of this corollary is evident, because a right line is the shortest distance between two given points. The second part follows at once from this proposition; for EA+AH being to the arc EIH, as the quadrangle AEOH to the circular sector HIEO; and the quadrangle being greater than the sector, because it contains it; it follows that EA+AH is greater than the arc EIH.

Cor. 3. Hence also, any tangent EA is greater than its corresponding arc EI.

PROPOSITION X. THEOREM.

If a circle and a polygon, circumscribable about another circle, are isoperimeters, the area of the circle is a geometrical mean between that polygon and a similar polygon circumscribed about that circle.

Let C be a circle, P a polygon isoperimetrical to that circle and circumscribable about some other circle G; and P′ a similar polygon circumscribable about the circle C; it is affirmed that $P : C :: C : P'$.

For $P : P' :: \textit{perim.}^2 P :: \textit{perim.}^2 P' :: \textit{circum.}^2 C : \textit{perim.}^2 P'$, (Prop. XIII. B. V.) But (Prop. IX.) $P' : C :: \textit{perim.} P' : \textit{circ.} C :: \textit{perim.}^2 P' : \textit{perim.} P' \times \textit{circ.} C$.

Therefore $P : C :: \textit{circ.}^2 C : \textit{perim.} P \times \textit{circ.} C :: \textit{circ.} C : \textit{perim.} P' :: C : P'$

PROPOSITION XI. THEOREM.

If the area of a circle is equivalent to that of a polygon circumscribable about another circle, the perimeter of that polygon is a mean proportional between the circumference of the first circle and the perimeter of a similar polygon circumscribed about it.

Let $C = P$, and let P′ be circumscribed about C, and similar to P: then it is affirmed that $\textit{circ.} C : \textit{per.} P :: \textit{per.} P : \textit{per.} P'$.

For $\textit{circ.} C : \textit{per.} P' :: C : P' :: P : P' :: \textit{per.}^2 P : \textit{per.}^2 P'$.

Also $\textit{per.} P' : \textit{per.} P :: \textit{per.}^2 P' : \textit{per.} P \times \textit{per.} P'$.

Therefore $\textit{circ.} C : \textit{per.} P :: \textit{per.}^2 P : \textit{per.} P \times \textit{per.} P' :: \textit{per.} P : \textit{per.} P'$.

PROPOSITION XII. THEOREM.

The circle is greater than any rectilinear figure of the same perimeter; and it has a less perimeter than any rectilinear figure of an equivalent area.

For in the proportion $P : C :: C : P'$ (Prop. X.), since $C > P'$, therefore $P < C$.

And in the proportion $\textit{circ.} C : \textit{per.} P :: \textit{per.} P : \textit{per.} P'$ (Prop. XI.), or $\textit{circ.} C : \textit{per.} P' :: \textit{circ.}^2 C : \textit{per.}^2 P$,

$\textit{circ.} C < \textit{per.} P'$;

therefore, $\textit{circ.} C < \textit{per.}^2 P$, or $\textit{circ.} C < \textit{per.} P$.

Cor. 1. It follows at once from this and the two preceding propositions, that rectilinear figures, which are isoperimeters and each circumscribable about a circle, are respectively in the inverse ratio of the perimeters, or of the surfaces, of figures similar to them, and both circumscribed about one and the same circle. And that the perimeters of equivalent rectilineal figures, each circumscribable about a circle, are respectively in the subduplicate ratio of the perimeters, or of the areas of figures similar to them, and both circumscribed about one and the same circle.

Cor. Therefore the comparison of the perimeters of regular figures, having a different number of sides, and that of the areas of regular isoperimetrical figures, is reduced to the comparison of the perimeters, or of the areas of regular figures respectively similar to them and circumscribable about one and the same circle.

NOTES.

GEOMETRY, in its most extended application, is not only applied to the measurement and comparison of magnitudes, but also to determine their local relations in reference to each other, or to other objects; I have therefore defined it to be the science which treats of the comparison and measurement of magnitude and the relations of locality.

Geometry admits of two general divisions, *plane* and SOLID. By the former, we investigate the relations and properties of lines, angles and figures, which lie in the same plane; by the latter, those of solids, lines and surfaces, which lie in different planes.

The science is also, by authors, divided into *elementary* and *higher* geometry. *Elementary geometry* treats of right lines, circles, solids bounded by plane surfaces, and solids formed by the revolution of right lined figures and of the circle, including the cylinder, cone, polyædroid and sphere.

Higher geometry treats of the properties of spherical triangles, conic sections and other curves, as well as the solids enclosed by surfaces bounded by such curves.

The author has thought it expedient to deviate from most others who have written on this subject, in the arrangement; especially in that of the book on proportions, which is made the first book; but in Euclid, it is the fifth, and also in Young's; and in most authors, it is made to follow in a more or less advanced position in the work. Legendre, indeed, chose to leave out this subject altogether, referring his readers to the subject as embraced in the common treatises on arithmetic and algebra; supposing the student to possess some knowledge of these subjects, before entering on the study of geometry. But however well the student may be versed in these sciences, there seems to be a necessity that the doctrine of the application of numbers to magnitudes, and of ratios and proportions, should be investigated in a more rigorous manner than is usually done in works of that kind, before being made the foundations for geometrical reasoning; and as the student progresses in the application of the principles therein contained, he finds frequent necessity to refer to the premises, as well as to the deductions on that subject. Hence, the doctrine of proportion being of general application to all kinds of magnitudes, as well as numbers, and quantities of all kinds, should have a place in works on elementary geometry.

One principal reason, that proportion is made to precede the other parts of the work, is, that this, being not so purely geometrical, should not interrupt the consecutive order of that which is more strictly so; but if, in teaching this science, it is thought expedient, the student may be allowed to commence on the second book, recur to this afterward, as he will have no occasion for the appli tion of the principles of the first book, till he arrives at the four

It has been regarded as essential, that the subject of proporti as applied to magnitudes, should be preceded by a general appli tion of number to magnitudes; since the mode of treating the s ject requires that the magnitudes should be expressed in numb or symbols representing number.

A definition of *Number* is adopted which is believed to be accordance with the more correct definitions usually given, a perhaps at the same time of more general application.

The definition of *Ratio*, as given by Euclid, has been regarde by Simpson, Playfair and others, as more of a metaphysical tha a mathematical definition. Dr. Simpson, indeed, thinks that th definition in Euclid was added by some unskilful editor. The de finition there given is: *Ratio is the mutual relation of two magn tudes of the same kind to each other in respect of quantity.* Bu Playfair remarks, there is no reason to suppose it to have been add ed by others, than what arises from the definition being of n mathematical use; supposing at the same time, that a certain ide of order and method induced Euclid to give some general defini tion of ratio, before he used the term in the definition of equal ra tios.

Euclid's definition is given in the text, with such explanatio and qualification as renders it definite and gives it mathematica accuracy.

In the illustration to the definition, the ratio of A and B is sai to be $\frac{A}{B}$. It will be observed that subsequently, when comparin quantities with each other in this book, the *consequent* has bee made the numerator, and the *antecedent* the denominator; but th has no other effect but to invert the order of the ratio; the reason ing founded on the application of the principles, not being affecte thereby.

INCOMMENSURBLE QUANTITIES.

When magnitudes are compared with magnitudes, or numbe with numbers, according to some general order or law, by which w express certain other magnitudes or numbers, having certain rel tions to the former, it is found impracticable under certain cond tions to develope or express in definite terms, the magnitudes numbers having such relations; which under such circumstanc are called incommensurable magnitudes or quantities. But sinc the law of their development is always manifest, and since th magnitudes or quantities are always some known function of ce tain others which are or may be known, they may by means such functions be expressed; and hence, under such expressio may be rendered as definite, as if capable of being developed in terms independent of such function.

Therefore any magnitudes, whether commensurable or incommensurable, may be compared with each other, through the medium of terms, or functions under which they are expressed; and no embarrassment will arise, in the application of ratio and proportion to magnitudes, under any conditions, or having any relation to each other, whether commensurable with each other or not; provided only that they are so far of such nature that they can properly be made the subject of comparison with each other. Thus, lines may be compared with lines, surfaces with surfaces, solids with solids; but a line cannot properly be compared with a surface or a solid, although the number representing the line may be compared with the number representing the surface or the solid, and its ratio obtained in reference to those numbers; but in applying ratios to magnitudes, those magnitudes must be of the same kind.

BOOK II.

Having previously defined the term Geometry in the first book, it only remains for us to give here the definitions of the various subjects on which it treats.

Extension is considered the principal subject, and *Magnitude* is the result of extension.

The definition of Geometry by Euclid is, that it is the science which has for its object the measurement of magnitude.

Legendre defines it to be the science which has for its object the measurement of extension.

Now, both of these definitions I consider faulty: the first, that it is not sufficiently general, and the second has the same fault in a certain degree; besides the term extension being more indefinite as commonly used, especially when spoken of in reference to two or three dimensions.

The definition of a *point* might perhaps not appear difficult; but no definition, that has yet appeared, is perfectly free from objection. Euclid defined it to be, *that which has no parts or which has no magnitude*. It was objected to this definition, that it contained only a negative; and Playfair gave as a definition, *that which has position but not magnitude*. This definition has received considerable favor and been adopted by many others, among whom is Hutton. But it may be objected to this definition, that it implies embodiment or substance, by the use of the demonstrative pronoun, although it is afterwards so qualified as to divest it of this quality; yet to the ordinary apprehension there would appear a degree of absurdity in the expression. For an illustration, let it be written thus, that a point is that *thing* which has position but not magnitude. Now, the idea of a thing possessing no magnitude is a manifest absurdity, and the idea conveyed by the first part of the definition, viz: that which has position, gives us an idea of embodiment, and consequently of magnitude.

Legendre's definition is perhaps still more faulty; he says, *the extremities of a line are points; a point therefore has no extension.* Here, it will be observed that he has not told us what a point is. It is true, he has told us the extremities of a line are points, and therefore a point has no extension; he has given us a particular case of a point, and forced a conclusion upon us from this particular case; and this conclusion, if examined, will be found to need proof, for the idea that is associated with the expression, *the extremities* of a line, is very naturally that the extremity is a portion of that line, and since the line is extension in one direction, then the extremity or point might be supposed to be a portion of that extension. So we might say the vertex of an angle is a point, and one might infer, with the same reasons, that a point therefore possesses no extension; but we might in this case be called upon to prove that a point did not possess extension in two dimensions, which might perhaps be somewhat difficult to do, to perfect satisfaction, from this definition.

The definition adopted in the text, it is believed, is not liable to any of the objections urged against those mentioned above. It is at once definite and explicit, without any necessary idea of embodiment, and the qualification is in perfect harmony with the affirmation.

After the definition of a point, a line is easily defined; it being the extension of a point or the extension of locality in one dimension: but it is much more difficult and inelegant to attempt to define a point from the properties of a line, as is done by Legendre.

I have in the text generally adopted the term *right line*, in preference to that of *straight line*, as being more in accordance with the analysis of other terms used in the science. Thus we have the term *rectilineal*; if this is used, there would seem to be a necessity that the simple words from which this is compounded should be used and defined. I have, however, in the subsequent parts of the work, used promiscuously the terms right or straight, synonymously; by the use of which in that way, the student will naturally acquire the idea of their perfect synonomy.

In definition 6, it is observed that the motion of a point generates a line. It is frequently objected, that motion should not be made the subject of geometrical science. To this it may be observed, that this is not a definition of what a line is, nor necessarily any part of such definition. It is only an illustration, and perhaps as obvious an illustration as can be produced. It is therefore thought to involve no impropriety, as here used.

DEFINITION XVII.

The definition of parallel lines, as usually given, involves an idea so indefinite and vague, that it seems one should be sought more strictly in accordance with mathematical accuracy and precision; one that will at once convey a definite idea, without conducting the mind into the mazes of infinity.

Playfair's definition is this: *parallel lines* are such as are in the same plane, and which, being produced ever so far both ways, do not meet. Legendre's definition is nearly similar.

What, I would ask, can convey a more confused and indefinite idea than this? The student, before he has fairly acquired a correct notion of right lines, is compelled to delineate in his mind an abstraction, which involves much consideration even in those that are more experienced in the subject

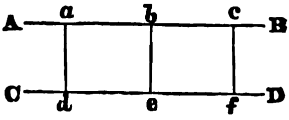
And what I would ask, is defined by this? Is the condition or mutual relation of the lines under consideration rendered definite even to the more advanced student?

It requires a degree of mental investigation, to deduce from this definition the conditions of the lines before us; by saying that their indefinite extension should never meet.

And their conditions are only made known as a corollary of the XXXIVth Proposition in Playfair's Euclid, and they are made the subject of the XXVth Proposition in Legendre, viz: that *parallel lines are equidistant from each other in all their parts.* This I have made the definition of parallel lines in the text. Hutton, indeed, gives the same definition, but coupled with the condition that they never meet, though ever so far produced This condition I conceive to be wholly unnecessary in the definition, and highly improper, since it is susceptible of proof from the premises, and not necessarily connected with the definition; and therefore should be made the subject of a rigorous demonstration, a course which is pursued in this work, and makes the subject of a scholium to Proposition XX. B. II.

The definition given in the text has frequently been given by others, by Boswick, and Simpson in the first edition of his elements made use of the same. Much fault, however, has been found by authors with this definition.

Playfair remarks concerning it, that "it contains an axiom, and takes for granted a property of straight lines that ought either to be laid down as self-evident, or demonstrated, if possible, as a theorem. Thus if," says he, "from the three points *d*, *e*, *f*, in the straight line CD, perpendiculars *da*, *eb*, *fc*, be drawn, all equal to one another, it is implied in the definition that the points *a*, *b* and *c*, are all in the same straight line; which, though it be true, it was not the business of the definition to inform us of. Two perpendiculars," he observes, "as *da*, *fc*, are alone sufficient to determine the position of the straight line AB, and therefore the definition ought to be, that two straight lines are parallel when there are two points in the one,



from which the perpendiculars drawn to the other are equal, and on the same side of it."

To all this I would answer, that in giving a definition of lines having certain relation to other lines, we should give such definitions as will definitely and most obviously express those relations, and if possible, in terms and under such conditions, that the mind will readily yield its assent; and then every property which is not necessarily included in the definition should be made the subject of a theorem.

Now, what more definitely expresses the relation of parallel lines, than to say they are equidistant in all their parts? The mind readily yields assent to the conditions of that relation thus expressed, and it is this definite relation that we wish to define. How, I ask, could we know *a priori*, or without analyzing the subject, that right lines, which are equidistant in two points, are equidistant in all their parts? Hence, what would be the idea conveyed to the mind by saying, parallel lines are those that, lying in the same plane, are equidistant in two points? The nature or meaning of parallel lines would not thereby be expressed. It is true that, by reasoning on the subject, we might ultimately come to a correct conclusion in relation to their properties or relations to each other, but definitions should be at once explicit and express plainly the ideas meant to be conveyed. In order to divest this definition of every degree of impropriety, I have subsequently, viz: in the XXth Proposition, demonstrated, that *right lines which lie in the same plane and are equidistant in two points, are equidistant in all their points, and hence are parallel*, according to definition. After once demonstrating this, it is sufficient, in speaking of any two right lines, to say they are parallel, if they are equidistant in two points and lie in the same plane.

In the second part of this work, it will be demonstrated, that if two lines cannot be included in the same plane, they cannot be parallel. I would here suggest another definition for parallel lines, viz: *Parallel right lines* are such as lie in the same direction and are not inclined toward each other. This perhaps would express the properties of such lines, in such manner as to convey a distinct idea of their relations without assuming anything in relation to the properties of such lines that needs to be proved, since nothing is affirmed of the lines but their position in reference to each other.

A new definition is given in the article Geometry, in the *Encyclopædia Metropolitana*, which is this: Parallei lines are those in which any point being taken in the one, and any point being taken in the other, the perpendicular distance of these points from the other line shall be equal to each other.

Whatever may be the definition employed, much difficulty is necessarily experienced at all times, in discussing the subject of parallel lines, and deducing the properties dependent thereon in a strictly rigorous manner. It is unnecessary to extend our remarks further on this subject. Those who wish to pursue it, may find

icious remarks in Notes to Playfair's Euclid, Young's
and Thompson's Abridgment of Legendre's Geometry.

PROPOSITION XXV.

demonst[illegible]ion we have conceived the production of a
generated by the motion of a point, according to the illus-
our definition of a right line. Now, if it be objected to
ccount, it may be observed that the production of a line
y supposes motion in that production ; and therefore there
impropriety, when speaking of the production of a line,
g of the means of that production also.

BOOK III.

bjects mostly discussed in this book are the circle and the
ent of angles. It may be observed, that angles have no
the lines by which they are subtended, and that an an-
no case be compared with a right line ; but since (Prop.
he angles are always proportional to the arcs of circles
about their vertices as centres, the arcs of circles so de-
cluded between any two lines forming an angle, may be
as the measure of such angle ; and if the circumference
is divided into number of equal parts, one of which is
s a unit, the number of such parts or units intercepted
o lines forming an angle in the centre of such circle,
e measure of such angle.

PROPOSITIONS XXIII AND XXIV.

se propositions we may obtain data of importance in tri-
; and as a corollary to the XXIIId. we are enabled to
the same result that is obtained by Prop. VIII. B. VI.
Euclid. But not yet having treated of similar figures,
e the application of this principle till we treat of similar
rtional figures in Book IV.

BOOK IV.

bject of the proportionality of the sides of similar figures,
ade to precede the celebrated proposition relating to the
described on the three sides of a right angled triangle ;
neans of the corollary to Proposition XXIII. B. III., we
led to deduce from the demonstration of Proposition

XVII. several corollaries, one of which is made the subj Prop. VIII. B. VI. Euclid, before referred to.

And by means of adopting this order in the arrangeme are enabled to bring the subject of similar and proportional f into the properties of the right angled triangl[illegible]and thereby new demonstration of that proposition, in reference to the p tionality of the sides of similar triangles, we are enabled i most direct and satisfactory manner possible, to show in the laries, that the areas of all similar rectilinear figures are t other as the squares described on their homologous sides. T in Euclid, Legendre, and other authors, are made the subj several propositions, but by means of the second mode of d stration in the XXIVth Proposition, the whole subject is ren much more satisfactory, and the results are more beautifull tained.

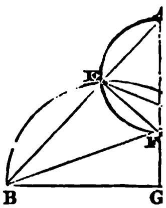
PROPOSITION XXV.

Is a generalization of the XXIVth. This is said to have first demonstrated by young *Clairault* when at the age of 16. The demonstration here is somewhat altered from the origin

Several curious propositions follow toward the close o book, some of which are wholly original, and some are take other sources; but generally with alteration so as to make correspond with the style of this work. Many of these pr tions are not generally found in elementary works.

To this book, the following proposition might be added :

If from the three vertices of a triangle perpendiculars be dra the opposite sides, they will intersect each other in the same and the rectangle of the parts will be the same in each.



If in the triangle ABC, perpendiculars BD, CE and AG, be drawn from the vertices B, C and A, to the opposite sides, they will all intersect each other in some point F, such that the rectangle of the parts of each perpendicular, viz: BF.FD=CF.FE =AF.FG.

Draw DE ; then in the quadrilateral AEFD, since the opposite angles AEF, ADF, are equal to two right angles, a circle may be described it (Prop. XIX. Cor. B. III.) Also, since the angles BEC, are right angles, their opposite sides, viz: BC, which is co to both, will be the diameter of the semicircle BEDC whic cumscribes both triangles. Now the angles BCE, BDE, su the same arc BE, and hence are equal; but the angles FDE, which are subtended by the arc EF, are also equal; hence th gle BCE is equal to the angle BAG; and since the triangles BCE, BAG, have a common angle B, it follows that the third angles BEC, BAG, are also equal.

And in a similar manner it may be shown that the angle CBD = the angle CAG, and hence each of the angles of the triangle BCR are equal and similar to the angles of the triangle ACG.

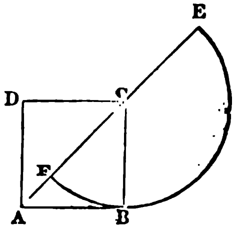
And (Prop. XXXV. Cor. B. IV.) BF.FD=CF.FE; also since the triangles CFG, AFE, are similar, CF : FG : : AF : FE; therefore CF.FE=AF.FG. Consequently,

BF.FD=CF.FE=AF.FG.

INCOMMENSURABLE MAGNITUDES.

From the known properties of certain figures, it is discovered that certain lines are incommensurable with certain other lines drawn in the same figure, which may be shown from the following proposition.

The diagonal of a square is incommensurable with its side.



In any square ABCD, the diagonal AC is incommensurable with its side AB.

From the angle C as a centre, with the radius CB, describe the semicircle FBE. Then since the angle CBA is a right angle, AB is a tangent to the circumference, and (Prop. XXXVII. B. IV.) AE : AB : : AB : AF, and hence (Prop. XXV. B. I.) AE, AB, are incommensurable; consequently AC, AB, are also incommensurable, for if these had a common measure, the same would also measure their sum AE (Prop. I. B. I.), which has been proved to be incommensurable with AB. Hence the diagonal of a square is incommensurable with its side.

Scholium. Although the line AC cannot be accurately developed in the same series of numbers, or any multiple or submultiple thereof, yet it is evident that its value is definite and may be definitely expressed in view of its known relation to those other quantities, viz: by $\sqrt{(AB^2+BC^2)}$, or $\sqrt{2AB}$.

The foregoing preposition as well as the XXVth proposition B. I., were first given in this form by Young. Legendre has the same, but under a different form.

The variety of problems introduced at the close of this book enables the student to apply the principles in such a manner, as to give them much practical utility. That part relating to the division of surfaces is much more copious than is generally found in works of this kind. Most of these problems have been taken pr miscuously from Legendre, Young and Euclid. Some, howeve have been added.

BOOK V.

By means of circumscribed and inscribed polygons, which treated of in this book, we are enabled by a process of reason i somewhat indirect, but not wholly unsatisfactory, to arrive at m of the useful properties of the circle; and that perhaps by the o means susceptible of being introduced into an elementary work.

When it is proposed that a polygon shall consist of an *indefi* *number* of sides, it is to be understood that the number is exceedi ly great, unlimited; the term *indefinite* is used by mathematici in the same sense as *infinite*. It is evident that whatever may affirmed of a polygon in all its variations, from a limited numbe sides to an indefinite number, the conditions remaining the sa may also be affirmed of the circle, to which the polygon infini approaches.

The equilateral triangle may be regarded as the most sim regular polygon, it being one with the least number of sides; hence it may be regarded as one of the extremes, and the ci may be regarded as the other extreme, and whatever is true of ery term from the first to the last, must also be true of the last.

Some have endeavored to demonstrate the properties of the cle, and its relation to rectilinear figures, in a more rigorous m ner, but when examined strictly, their demonstrations rest on same foundation as that here used.

PROPOSITION XVI.

This proposition might be deduced from proposition xv., taking one circle from another, and then reasoning by an algebr cal process; but it was thought best that the principle should established geometrically, more particularly as it was designed th this principle and mode of demonstration should be applied to ot or curvilinear figures in the subsequent parts of the work.

BOOK VI.

This book treats of the isoperimetry of plane surfaces.

Isoperimetry, or the science in which it is required to find amon figures of the same or of different kinds, those which within equ perimeters shall comprehend the greatest surfaces, has long e gaged the attention of mathematicians. By means of the meth of fluxions or the calculus, the subject has been elegantly treated, especially the more difficult questions relating to the science.

The elements of the science were for a time neglected in pursuit of the most abstruse and intricate matters which the subject embraced. Simpson was the first to restore the science in its elementary form, by publishing in a geometrical work a chapter on maxima and minima of geometrical quantities, and some of the most simple problems concerning isoperimeters. The next who treated the subject in an elementary manner was Simon Lhuillien of Geneva, in 1782. His principal object in the publication of his work on the subject was to supply a deficiency in this respect, which he found in most elementary works, and to determine with regard to both the surfaces and solids, those which possessed the minimum of contour with the same capacity, and reciprocally, the maximum of capacity with the same perimeter.

M. Legendre, and also Hutton, have more recently considered the subject, in a manner different from either of the former authors. These selections are made chiefly from the authors last mentioned, with little variation. The subject might be farther extended, but sufficient is given to enable the student to apply the principles as opportunity or necessity occurs. The isoperimetry of solids is reserved for its proper place in connection with the subject.

THE APPLICATION OF THE FOREGOING PRINCIPLES TO THE MENSURATION OF SUPERFICIES.

The area or content of a surface is determined by finding how many times it contains some other surface which is assumed as the unit of measure. Thus, when we say that a square yard contains 9 square feet, we understand that one square foot is taken for the unit of measure, and that this unit is contained 9 times in the square yard.

The most convenient unit of measure for a surface is a square whose side is the linear unit in which the linear dimensions of the figure are estimated. Thus, if the linear dimensions are feet, it will be most convenient to express the area in square feet; if the linear dimensiens are yards, it will be most convenient to express the area in square yards, &c.

The rectangle, or product of two lines, designates the rectangle constructed on these lines as sides; and the numerical value of this product expresses the number of times the rectangle contains its unit of measure.

I. LINEAL MEASURE.

Inches.	Gunter's Links.	Feet.	Yards	Fathom	Rods Poles, or Perches	Gunter's Chains.	Furlongs.	Mile.
$7\frac{23}{25}$	= 1							
12	$1\frac{17}{33}$	= 1						
36	4	3	= 1					
72	$9\frac{1}{11}$	6	2	= 1				
198	25	$16\frac{1}{2}$	$5\frac{1}{2}$	$2\frac{3}{4}$	= 1			
792	100	66	22		4	= 1		
7920	1000	660	220	110	40	10	= 1	
63360	8000	5280	1760	880	320	80	8	= 1

Besides the above, there are the Barley-corn, which equals ⅓ of an inch; the Palm=3 inches; the Hand=4 inches; the Span=9 inches: the Cubit=1½ foot; the League=3 miles; and 60 Geographical, or 69.114 English Miles, = 1 Degree.

II. SQUARE MEASURE.

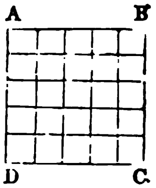
Sq. Inches.	Sq. Links.	Sq. Feet.	Sq. Yards.	Sq. Poles Perches, or Rods	Sq. Chains	Sq Roods	Sq. Acres	Sq. Mile.
$62\frac{454}{625}$	= 1							
144	$2\frac{322}{1089}$	= 1						
1296	$20\frac{80}{121}$	9	= 1					
39204	625	$272\frac{1}{4}$	$30\frac{1}{4}$	= 1				
627264	10000	4356	484	16	= 1			
1568160	25000	10890	1210	40	$2\frac{1}{2}$	= 1		
6272640	100000	43560	4840	160	10	4	=1	
4014489600	64000000	27878400	3097600	102400	6400	2560	640	=1

PROBLEM I.

To find the area of a parallelogram, whether it is a square, a rectangle, or an oblique angled rhomboid.

Rule.—Multiply the length of the base by the breadth, or perpendicular height, and the product will be the area.

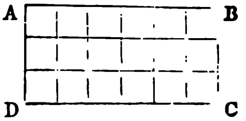
Ex.—To find the area of a square ABCD, whose side AB=5 feet.



Here $AB^2=5^2=25$ square feet, the area of the square ABCD.

Schol. 1.—The side of a square may be found by extracting the square root of its area.

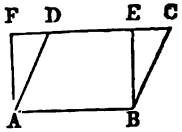
Ex.—To find the area of the rectangle ABCD, whose length AB=6, and breadth AD=3 feet.



Here $AB\times AD=6\times3=18$ square feet, the area of the rectangle.

Schol. 2.—If the area of the rectangle be divided by one of its sides, the quotient will be the adjacent side.

Ex.—To find the area of a rhomboid ABCD, whose length AB=6.20 chains, and perpendicular height=5.25 chains.



Here AB×BE=6.20×5.45=33.79 square chains.

Then divide 33.79 by 10, because 10 chains make an acre; and the decimals are multiplied by 4, and then by 40, because 4 roods make 1 acre, and 40 rods 1 rood.

Thus (33.79÷10)×4×40=3 acres, 1 rood, 20.64 rods, the area of the rhomboid ABCD.

The perpendicular breadth of a rhomboid may be found by dividing the area by the length of the base or longest side; and the length may be found by dividing the area by the breadth.

Examples for Practice.

Ex. 1. To find the area of the square whose side=35.25 chains. Ans. 124 ac. 1 ro. 1 perch.

Ex. 2. What is the area of a square, whose side = 1275 links. Ans. 16 ac. 1 ro. 1 pole.

Ex. 3. The side of a square = 280 yards, how many acres does it contain? Ans. 16 ac. 0 ro. 31.7355 poles.

Ex. 4. How many square yards are there in a floor which is 20 feet square? Ans. 44 yds. 4 feet.

Ex. 5. To find the area of a rectangular board, whose length = 12.5 feet, and breadth = 9 inches. Ans. 9⅜ ft.

Ex. 6. What is the area of the rhombus, the length = 12.5 feet, and the perpendicular height = 6.25 feet.
Ans. 78.125 square feet.

Ex. 7. To find the square yards of painting in the rhomboid, whose length = 37 feet, and the perpendicular breadth =5¼ feet. Ans. 21 [illegible] square yards.

Ex. 8. If a room be painted whose height being 16 ft. 6 in., and the compass of the room 97 ft. 9 in., how many yards of painting are there in that room?
Ans. 179 yds. 1 ft. 10 pt. 6 in.

Ex. 9. How many square yards of paving are there in a court yard, being in the form of a rhomboid, whose length is 64 ft. 6 in. and perpendicular breadth 47 ft. 8 in.
Ans. 341 yds. 5 ft. 6 pt.

Ex. 10. The area of a square = 16 ac. 1 ro. 1 pole, required the length of its side? Ans. 1275 links.

Ex. 11. The area of a rectangle = 10 ac. 1 ro. 26 perches, it is required to find the length, the breadth being 8.5 chains. Ans. 12.25 chains.

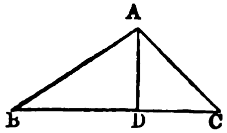
Ex. 12. Required the perpendicular breadth of a rhomboid, whose area = $65\frac{1}{3}$ square feet, the length $12\frac{1}{4}$ feet.
Ans. 5 ft. 4 in.

PROBLEM II.

To find the area of a triangle.

Rule.—Multiply the base by the perpendicular height, and take half the product for the area, (Prop. VIII. B. IV.)

Schol.—If the area of a triangle be divided by half the perpendicular height, the quotient will be the base. If the area be divided by half the base, the quotient will be the perpendicular height.



Required the area of a triangle, whose base BC=10 chains, and perpendicular height AD = 5.5 chains.

Here $\frac{1}{2}(BC \times AD) = \frac{1}{2}(10 \times 5.5) = 55.0 \div 2 = 27.5$ square chains.

Then $(27.5 \div 10) \times 4 \times 40 = 2$ acres, 3 roods, the area of the triangle ABC.

Examples for Practice.

Ex. 1. Required the area of a triangle whose base = 17 ft. 7 in., and perpendicular height = 6 ft. 11 in.
Ans. 60 ft. 9 pt. 8 in. 6‴.

Ex. 2. How many square yards are there in the right-angled triangle, whose base = 40 feet, and perpendicular height = 30 feet. Ans. $66\frac{2}{3}$ square yards.

Ex. 3. How many square yards contains the triangle, whose base is 49 feet, and height $25\frac{1}{4}$ feet?
Ans. $68\frac{53}{72}$ or 68.7361 square yards.

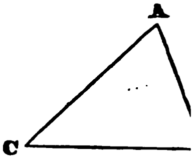
Ex. 4. To find the area of a triangle, whose base is 18 ft. 4 in., and height 11 ft. 10 in. Ans. 108 ft. 5 pt. 8 in.

Ex. 6. The area of a triangle = 46.25 square chains, and its perpendicular height = 5.20 chains, required the length of the base. Ans. 6.25 chains.

PROBLEM III.

To find the area of a triangle by having the three sides given.

Rule.—Add the three sides together, and take half the sum; subtract each side separately from the half sum, t multiply the half sum and the three remainders continu together, and the square root of the last product will the area, (Prop. XL. B. IV.)



Ex. To find the number of square yards in the triangle ABC, whose three sides AB=13, BC=14, and AC = 15 feet.

Here $\frac{1}{2}(AB+BC+AC)=\frac{1}{2}(13+14+15)=42\div2=21$ half the sum of the sides.

$$\left.\begin{array}{l}21-13=8\\21-14=7\\21-15=6\end{array}\right\}\text{the three remainders.}$$

Then $\sqrt{(21\times8\times7\times6)}=\sqrt{7056}=84$ square feet. H $84\div9=9\frac{1}{3}$ square yards, the area of the triangle ABC quired.

Examples for Practice.

Ex. 1. To find the area of a triangle, whose three s equal 20, 30 and 40 chains, respectively.

Ans. 29 acres, 0 roods, 6.568 perche

Ex. 2. How many acres, &c. are there in a trian whose three sides are 380, 420, and 765 yards?

Ans. 9 acres, 0 roods, 37.654 perche

Ex. 3. How many acres, &c. are contained in a trian whose three sides are 49, 50.25, and 25.69 chains?

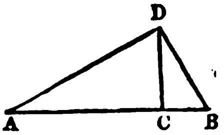
Ans. 61 acres, 1 rood, 39.68 perche

PROBLEM IV.

To find the area of a triangle by having two sides and ratio of one of those sides to a perpendicular from it tremity to the other given side, or side produced.

Rule.—Multiply the two given sides together, and product by the given ratio, and the last product is the (Prop. XXXII. B. IV.)

Ex. What is the area of a triangular field, two of w sides AB and DB are 10 and 8 rods respectively, and ratio of DB to the perpendicular DC = ,734.



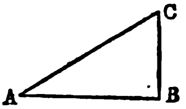
$10 \times 8 \times .734 = 58.72$, the area of the triangle.

PROBLEM V.

Any two sides of a right angled triangle being given, to find the third side.

RULE. I.—When the base and perpendicular are given, to find the hypothenuse. To the square of the base, add the square of the perpendicular; the square root of the sum will give the hypothenuse, or the longest side, (Prop. XXIV. B. IV.)

Ex. Required the hypothenuse AC of a right angled triangle, whose base AB = 1600 links, and perpendicular BC = 1340 links.



Here $\sqrt{(AB^2 + BC^2)} = \sqrt{(1600^2 + 1340^2)} = \sqrt{4355600} = 2087.007426+$ links, the length of the hypothenuse AC.

RULE II.—When the hypothenuse and one side are given, to find the other side. Multiply the sum of the hypothenuse and one side, by their difference; the square root of the product will give the other side.

Otherwise:

From the square of the hypothenuse, subtract the square of the given side, and the square root of the remainder will be the required side.

Ex. What is the perpendicular of a right angled triangle, whose base $AB = 56$, and the hypothenuse $AC = 65$.

Here $\sqrt{(AC + AB) \times (AC - AB)} = \sqrt{(65 + 56) \times (65 - 56)} = \sqrt{(121 \times 9)} = \sqrt{(11^2 \times 3^2)} = 33$, the length of the perpendicular BC required.

Otherwise:

$\sqrt{(AC^2 - AB^2)} = \sqrt{(65^2 - 56^2)} = \sqrt{(4225 - 3136)} = \sqrt{1089} = 33$, the length of the perpendicular AC, as before.

Examples for Practice.

Ex. 1. Required the length of a scaling ladder to rea
the top of a wall whose height = 28 feet, and the brea
of a ditch before it being 45 feet. Ans. 53 feet.

Ex. 2. To find the length of a shore, which, strutti
twelve feet from the upright of a building, may suppor
jamb 20 feet from the ground. Ans. 23.3238 feet.

Ex. 3. A line of 320 feet will reach from the top o
precipice, standing close by the side of a river, to the op
site bank: required the breadth of the river; the height
the precipice being 103 feet. Ans. 302.9703 feet.

Ex. 4. A ladder of 60 feet long being placed in a stre
reached a window 28 feet from the ground on one side; a
by turning the ladder over, without removing the foot out
its place, it touched a moulding 36 feet high on the ot
side: required the breadth of the street.
Ans. 75.1233335 feet.

Ex. 5. Two ships set sail from the same port; one
them sails due West 50 miles; the other due north 84 mil
how many miles are they from each other, supposing t
earth's surface a plane?
Ans. 97.7547953 miles.

Ex. 6. A tree 60 feet high is cut so, that, as it falls,
top reaches the ground, which is supposed to be level, 25 f
from the foot of the tree, while the portion cut off remains
contact with the upper end of the trunk remaining; what
the length of each part?

PROBLEM VI.

*To find the area of a triangle, when neither its sides nor
angles are known.*

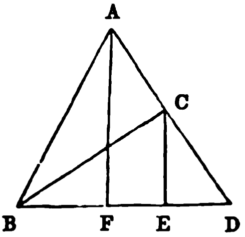
Rule I.—From one of the angles of the triangle draw
base line till it meets the opposite side produced; draw al
perpendiculars from the other angles of this triangle to th
base; multiply this base into half the difference of the pe
pendiculars, and the product is the area.

Ex. From one of the angles B of a triangular field AB
I measured a base line BD, until it met AC produced in
12 chains; from the angles A and C, I then measured lin
AF and CE perpendicular to BD, and found them to be

and 8 chains respectively. Required the area of the triangle.

Here $BD\left(\frac{AF-CE}{2}\right)=12\left(\frac{12-8}{2}\right)=24$ square chains.

Then $(24\div10)\times4\times40=2$ acres, 1 rood, 24 rods, the area of the triangle.



Ex. Having a triangular field ABC destitute of access, except by two ways, viz: by the lines BD terminating in the angle B, and CE in the angle C, perpendicular to each other. I set out to determine its area; and find by measuring, that the base BD is 30 chains, the perpendicular EC 14 chains, BF 14 chains, and by observations, that the side AB is 26 chains, supposing A, C and D to be in the same right line. Required the area of the field.

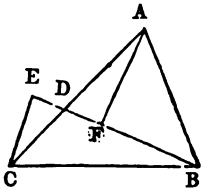
Ans. $60\sqrt{30}-210=113.633536$ chains.

Rule II.—If the base line BD falls within the triangle ABC, multiply that base by half the sum of the two perpendiculars AF and CE, drawn from the other angles to that base produced, and their product will be the area of the triangle, (Prop. XXXI. Schol. B. IV.)

Ex. From one of the angles B of the triangular field ABC, I measured a line BD through said field 25 chains, AF 20 chains, and CE 12 chains, perpendicular to BD and BD produced. Required the area of the field.

Here $\frac{AF+CE}{2}\times DB=\frac{20+12}{2}\times25=16\times25=400$ square chains.

Then $400\div10=40$ acres, the area of the triangle.



Ex. There being three towns, A, B and C; a public highway runs from B in a direct line until it meets a right line uniting A and C, 40 miles; a perpendicular AF dropped from A to the highway AD=30 miles; a line drawn from C perpendicular to a line drawn from D, the end of the highway BD in a direct line with such highway, 10 miles; and the area of the triangle CDE thus formed, 120 miles. Required the area of the triangle formed by highways connecting these towns.

Ans. 1080 square miles.

PROBLEM VII.

To find the area of any quadrilateral figure.

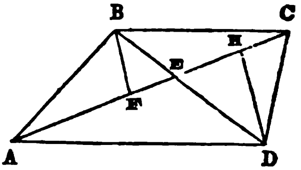
Rule.—Multiply half of one of the diagonals by the sum of the two perpendiculars let fall upon it from the extremities of the other diagonal.

Or,

Divide the quadrilateral into triangles by a diagonal; then find the areas of these triangles, and add them together.

Ex. Required the area of the quadrilateral ABCD, one of whose diagonals AC is 32 feet, and the perpendiculars BF and DH drawn from the extremities of the other diagonal BD to AC, 6 and 13 feet respectively.

Here $\frac{AC}{2} \times (BF + DH) = 16\times(6+13)=16\times19=304=$ the area of the triangle.



Schol. If the quadrilateral is a *trapezium*, or has two of its sides parallel, its area is equal to half the sum of its parallel sides, multiplied by the perpendicular distance between them.

Ex. In a trapezium, whose parallel sides are 20 and 8 ft., respectively, and the perpendicular distance between them 10 ft., what is the area?

$(20+8)\div2\times10=140$ square feet.

Case II.—*When the quadrilateral can be inscribed in a circle.*

Rule.—Add all the four sides together, and take half the sum; next subtract each side separately from the half sum; then multiply the four remainders continually together, and take the square root of the last product for the area of the quadrilateral.

Ex. If a quadrilateral can be inscribed in a circle, and have four sides 25, 26, 28, 30; required its area.

$\frac{1}{2}(24+26+28+30)=108\div2=54$ yards, the half sum of the sides.

Then $\sqrt{[(54-24)\times(54-26)\times(54-28)\times(54-30)]}=\sqrt{(30\times28\times26\times24)}=\sqrt{524160}=24\sqrt{910}=24\times30.1662062=723.9889488$ square yards, the area of the inscribed quadrilateral required.

Examples for Practice.

Ex. 1. How many square yards of paving are there in a quadrilateral, whose diagonal is 65 feet, and the two perpendiculars let fall on it are 28 and 33.5 feet?

Ans. $222\frac{1}{12}$ yards.

Ex. 2. What is the area of a quadrilateral, whose south side is 27.40 chains, east side 35.75 chains, north side 37.55 chains, west side 41.05 chains, and the diagonal from south-west to north-east 48.35 chains?

Ans. 123 ac. 0 ro. 11.862048 perches.

Ex. 3. What is the area of a quadrilateral, whose diagonal is $108\frac{1}{2}$ feet, and the perpendiculars $56\frac{1}{4}$ and $60\frac{3}{4}$ feet?

Ans. $6347\frac{1}{4}$ feet.

Ex. 4. What is the area of a quadrilateral inscribed in a circle, the four sides being 12, 13, 14, 15?

Ans. 108.9972372.

Ex. 5. In a four-sided field, on account of obstructions in two of its sides, and in the perpendiculars, the following measures only could be taken: the other two sides equal to 265 and 220 yards respectively, the diagonal 378 yards, and the two distances of the perpendiculars from the ends of the diagonal 100 yards and 70 yards: required the area in acres, when 4840 square yards make an acre.

Ans. 17 acres, 2 roods, 21 perches.

Ex. 6. In a trapezium, whose parallel sides are 186 ft 6 in. and 15 ft., and the perpendicular distance between those sides 8 ft. 6 in., what is the superficial content?

Ans. 142 ft. $4\frac{1}{2}$ inches.

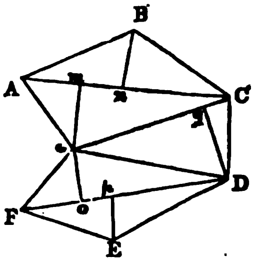
PROBLEM VIII.

To find the area of an irregular polygon.

Rule.—Draw diagonals dividing the figure into quadrilaterals and triangles. Then find the areas of these separately, and add them together for the area of the polygon.

Ex. To find the content of the irregular figure ABCDEFGA in which are given the following diagonals and perpendiculars, namely,

$AC = 5.5$
$FD = 5.2$
$GC = 4.4$
$Gm = 1.3$
$Bn = 1.8$
$Go = 1.2$
$Ep = 0.8$
$Dq = 2.3$



$(Gm+Bn)\times AC=(1.3+1.8)\times 5.5$ . . . . . . $= 17.05$
double the area of the quadrilateral ABCG.
$GC\times Dq=4.4\times 2.3$ . . . . $= 10.12$
double the area of the triangle CGD.
$(Go+Ep)\times FD=(1.2+.8)\times 5.2.$ . . . . . . $= 10.40$
double the area of the quadrilateral GDEF.
The sum $= 37.57$

double the area of the whole, which, divided by 2, equals 18.785, the area of the irregular figure ABCDEFGA required.

Ex. 1. Suppose that in the first figure there were given AC equal 44 chains, FD 41.6, GC 35.2, G*m* 11.4, B*n* 14.4, G*o* 9.6, E*p* 6.4, and D*q* 18.4 chains, what would be the contents in acres? Ans. 122 acres, 1 rood, 27.84 perches.

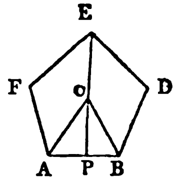
Ex. 2. It is required to draw the plan and find the area of a five-sided field ABCDEA (*by the Rules to Prob. II, III, and V*), whose dimensions are in chains as follow: $AB = 14$, $BC = 7$, $CD = 10$, $DE = 12$, $EA = 5$, and the diagonal $AC = 17$ chains; also the angle at E being a right angle. Ans. 14 acres 1.022701 roods.

PROBLEM IX.

To find the area of a regular polygon.

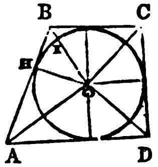
RULE I.—Find the perimeter of the figure, or sum of its sides, and multiply it by the perpendicular falling from its centre on one of its sides, and take half the product for the area.

Ex. Required the area of the regular pentagon ABDEFA, whose side AB is 25 feet, and perpendicular CP 17.204774 feet.



$\frac{1}{2}(5AB\times CP)=\frac{1}{2}[(5\times25)\times17.204774]=2150.59675\div2=1075.298375$ square feet, the area of the pentagon ABDEFA required.

Scholium 1.—If the perimeter of any rectilinear figure in which a circle may be inscribed (Def. B. III.) is multiplied by half the radius of its inscribed circle, the product will be the area.



What is the area of the quadrilateral ABCD, whose perimeter is 21 inches, and the radius of whose inscribed circle is 2 inches.

Ans. $21\times\frac{2}{2}=21$ square inches.

Ex. What is the radius of a circle inscribed in an irregular polygon, whose area is 30 square feet, and whose perimeter is 20 feet, the sides being all tangents to the circle.

Ans. 3 ft.

Scholium 2.—Since the areas of similar polygons are as the squares of their like sides (Prop. XXIV. Cor. F. B. IV.), if we have the side of a regular polygon whose area is known, we can from this find the area of any other similar polygon, having the length of one of its sides.

Thus, if the side of a regular pentagon is 25 feet, and its area 1075.298375 square feet, the area of a similar pentagon, whose side is 5 feet, will be

$25\times25:5\times5::1075.298375:43.011935$ square feet, the area required.

Hence, if a table is prepared, containing regular polygons of any specified number of sides, the length of one of whose sides is unity, the area of any similar polygon may be found by simply multiplying the tabular area by the square of the side of the given polygon.

And since the areas of similar polygons are also as the squares described on the radii of their inscribed or circumscribed circles, or as the squares of lines similarly drawn in each, it follows that if tables are prepared, containing the areas of regular polygons, the radii of whose inscribed or circumscribed circles are 1, those areas may be used as multipliers into the squares of the radii of the circles to any other similar polygons, to discover their areas.

TABLE I.

When the side of the Polygon is 1.

No. of Sides.	Area.	Radius of circumscribing circle.	Radius of inscribed circle.
3	0.4330127	0.5773503	0.2886751
4	1.0000000	0.7071068	0.5000000
5	1.7204774	0.8506508	0.6881910
6	2.5980762	1.0000000	0.8660254
7	3 6339124	1.1523825	1.0382617
8	4.8284271	1.3065630	1.2071068
9	6.1818242	1.4619022	1.3737387
10	7.6942088	1.6186340	1 5388418
11	9.3656404	1.7747329	1.7028437
12	11.1961524	1.9318516	1.8660254

TABLE II.

When the radius of the Circumscribing Circle is 1.

No. of Sides.	Area.	Length of the side.	Radius of inscribed circle.
3	1.2990381	1.7320508	0.5000000
4	2.0000000	1.4142136	0.7071068
5	2.3776412	1.1755705	0.8090170
6	2.5980762	1.0000000	0.8660254
7	2.7364102	0.8677674	0.9009689
8	2.8284271	0.7653668	0.9238795
9	2.8925137	0.6840403	0.9396926
10	2.9389263	0.6180340	0.9510565
11	2.9735250	0.5634651	0.9594931
12	3.0000000	0 5176381	0.9659259

TABLE III.

When the radius of the Inscribed Circle is 1.

No. of Sides.	Area.	Length of the side.	Radius of circumscribed circle.
3	5.1961524	3.4641016	2.0000000
4	4.0000000	2.0000000	1.4142136
5	3.6327128	1.4530851	1.2360680
6	3.4641016	1.1547005	1.1547005
7	3.3710222	0.9631491	1.1099160
8	3.3137084	0.8284271	1.0823919
9	3.2757315	0.7279405	1.0641776
10	3.2491970	0.6498394	1.0514622
11	3.2298913	0.5872521	1.0422172
12	3.2153904	0.5358984	1.0352750

TABLE IV.

When the area of the Polygon is 1.

No. of Sides.	Length of the side.	Radius of circumscribing circle.	Radius of inscribed circle.
3	1.5196716	0.8773827	0.4386912
4	1.0000000	0.7071068	0.5000000
5	0.7623870	0.6485251	0.5246678
6	0.6204033	0.6204033	0.5372849
7	0.5245813	0.6045183	0.5446520
8	0.4550899	0.5946034	0.5493420
9	0.4021996	0.5879764	0.5525172
10	0.3605106	0.5833184	0.5547687
11	0.3267617	0.5799148	0.5564212
12	0.2988585	0.5773503	0.5576775

And because the radii of the circumscribed and inscri circles of similar polygons are directly as the sides of polygons, it follows that tables may be prepared, contain each of those parts, viz: the side of the polygon, the r of the inscribed and circumscribed circles, and also t areas; and from having given a polygon, with either of above quantities, the other parts may be found, as in following:

Ex. 1. When the side of a square is 5; then, by Ta I.

$5^2\times1 = 25$ area of square.
$5\times.7071068=3.535534$, rad. of circumscr. circle.
$5\times.5 = 2.5$ *rad. of inscribed circle.*

Ex. 2. When the radius of the circumscribing cir about an octagon, is 5; then, by Table II.

$5^2\times2.8284271=70.71068$, area of octagon.
$5\times.7653668=3.826834$, length of the side.
$5\times.9238795=4.619397$, *rad. of inscribed circle.*

Ex. 3. When the radius of the inscribed circle abou hexagon is 7; then, by Table III.

$7^2\times3.4641016=49\times3.4641016=169.7409784$, the are the hexagon.

$7\times1.1547005=8.0829035$, the length of the side.

$7\times1.1547005=8.0829035$, the radius of circumscri circle.

Ex. 4. When the area of a dodecagon is 81; then, Table IV.

$9\times.2988585=2.6897265$, length of the side.
$9\times.5773503=5.1961527$, rad. of circumsc. circle.
$9\times.5576775=5.0190975$, *rad. of inscribed circle.*

Ex. 5. Required the side of a hexagon, an octagon decagon, and of a dodecagon, the area of each of wh shall be 144.

$\sqrt{144}=12$; therefore, by Table IV.

$12\times.6204033=7.2448396$, side of hexagon.
$12\times.4550899=5.4610788$, side of octagon.
$12\times.3605106=4.3261272$, side of decagon.
$12\times.2988585=3.5863020$, side of dodecagon.

Note 1. The radii of the circumscribing circles of th polygons, by means of which they may readily be c structed on paper, may be found from the same table.

Note 2. From a careful inspection of the previous tables, several curious relations and properties well known to mathematicians, and some of which have been shown in Book V. of the *Elements*, may be observed.

That the side of a hexagon is equal to the radius of the circumscribing circle.

That when the radius of the circumscribing circle is 1, the area of the square which it circumscribes is 2, that of the octagon which it circumscribes is $2\sqrt{2}$, and the area of the dodecagon which the said circle circumscribes is 3, &c. &c.

Examples for Practice.

Ex. 1. To find the area of the hexagon, whose side is 20. Ans. 1039.23048.

Ex. 2. To find the area of the trigon, or equilateral triangle, whose side is 20. Ans. 173.20508.

Ex. 3. Required the area of an octagon, whose side is 20. Ans. 1931.37084.

Ex. 4. What is the area of a decagon, whose side is 20? Ans. 3077.68352.

PROBLEM X.

Given the diameter of a circle, to find the circumference; or, given the circumference, to find the diameter.

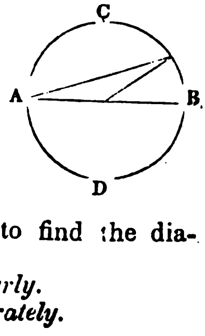
Rule 1.—As 7 is to 22, so is the diameter to the circumference; or as 22 to 7, so is the circumference to the diameter.

Or as 113 is to 355, so is the diameter to the circumference; and inversely as 355 is to 113, so is the circumference to the diameter.

Ex. 1. To find the circumference ACBD of a circle, whose diameter AB=10.

$7:22::10:31\frac{3}{7}$, the circumference *nearly*, or which is more nearly,

$113:355::10:31,415926$, the circumference.



Ex. 2. Given the circumference = 50, to find the diameter.

$22:7::50:15.90$, the diameter, *nearly*.

Or, $355:113::50:15.915409$, *more accurately*.

Rule II. Multiply the diameter by 3.1416: or more exactly, by 3.14159, and the product will give the circumference: or divide the circumference by 3.1416, and the quotient will be the diameter.

Ex. 1. Required the circumference of a circle, whose diameter is 10.

3.1416×10=31.416, the circumference.

Ex. 2. What is the diameter of a circle, whose circumference is 50.

50÷3.146=15.90+, the diameter.

Scholium. If we divide 1 by 3.1416, the quotient may be used as a multiplier into the circumference to find the diameter.

Thus 1÷3,1416=.318309, which multiplied by the circumference gives the diameter.

Ex. Taking the same example as before,

50×.318309=15.90+, as before.

Examples for Practice.

Ex. 1. If the diameter of the earth be 7958 miles, as it is very nearly, what is the circumference, supposing it to be exactly round? Ans. 25000.8528 miles.

Ex. 2. To find the diameter of the globe of the earth, supposing its circumference to be 25000 miles.

Ans. 7957¾ miles, *nearly.*

Ex. 3. The distance between the cogs of a mill-wheel, and the centre of the axle, is 5 feet; how many cogs, at 4 inches pitch, will the wheel admit of? Ans. 94.248.

PROBLEM XI.

To find the length of the arc of a circle.

Case I.—When the number of degrees in the arc, and the radius or diameter of the circle, are given.

Scholium.—All circles are arbitrarily supposed to be divided into 360 equal parts called degrees, marked thus 360°; and each degree consists of 60 parts called minutes, marked thus 60′: and each minute consists of 60 seconds, marked thus 60″, &c. Hence any arc of a circle will contain as many of those degrees, minutes and seconds, compared to the whole number, or 360°, as its ratio to the whole circle: hence, as in the following

Rule.—First find the circumference of the whole circle; then say, as 360, the number of degrees in the circumference, is to the number of degrees in the arc, so is the length of the circumference, to the length of the arc.

Ex. To find the length of an arc of 30 degrees, the diameter being 9 feet.

(By Prob. X. Rule II.) 1 : 31416::9 : 28.2754.
Then 360 : 30::28.2754 : 2.3562 feet=Ans.

CASE II.—When the length of the chord of the arc, and the radius of the circle, are given.

Scholium.—Let there be a table prepared containing the lengths of all the arcs of a circle whose radius is 1, from one degree to 180°, or to the semicircumference, and we can from this find the length of an arc whose chord is given of a circle with any given radius, by the following

RULE.—Divide the chord of the arc by the radius, and seek in the following table, in the column of chords, for a number corresponding to the quotient; and opposite thereto, in the column of arcs at the left hand, will be found the number of degrees in the arc corresponding to that chord for a circle whose radius is 1; then proceed as in Case I. to find the length of the arc.

Let it be required to find the length of the arc of a circle, whose chord is 14.142 inches and radius 10 inches.

Here 14,142÷10=1,4143, which number being found in the column of chords, there will be found opposite thereto, in the column of arcs, 90°. Then, by Case I., after having found the circumference of the circle whose radius is 10 feet =62,832,

360° : 90°::62,832 : 1,5706, the length of the arc required.

Note. — When the quotient of the chord, divided by the radius, cannot be exactly found in the table, proportions may be instituted so as to find the arc between the next higher and the next lower, in the same ratio that this chord varies from the next higher and lower chords.

Thus as the difference of the next higher and next lower number in the column of chords, is to the difference of either of these from the one sought, so is the difference of the next higher and next lower arc, to the corresponding difference, or correction, to be added or subtracted, according as we use the higher or lower number for our correction.

Ex. 1. Thus, if it be required to find the arc answering to the chord ,5.

TABLE OF CHORDS FOR ANY ARC OF A CIRCLE, WHOSE RADIUS IS 1.

Degrees in the Arc.	Length of Chord.	Degrees in the Arc.	Length of Chord.	Degrees in the Arc.	Length of Chord.	Degrees in the Arc.	Length of Chord.
1	.01745	46	.78146	91	1.42650	136	1.85436
2	.03490	47	.79749	92	1.43868	137	1.86083
3	.05235	48	.81347	93	1.45074	138	1.86716
4	.06979	49	.82938	94	1.46270	139	1.87334
5	.08723	50	.84523	95	1.47455	140	1.87938
6	.10467	51	.86102	96	1.48629	141	1.88528
7	.12209	52	.87674	97	1.49791	142	1.89103
8	.13951	53	.89239	98	1.50942	143	1.89661
9	.15691	54	.90798	99	1.52081	144	1.90211
10	.17431	55	.92349	100	1.53208	145	1.90743
11	.19169	56	.93894	101	1.54325	146	1.91261
12	.20905	57	.95431	102	1.55429	147	1.91764
13	.22640	58	.96962	103	1.56521	148	1.92252
14	.24373	59	.98484	104	1.57602	149	1.92726
15	.26105	60	1.00000	105	1.58670	150	1.93185
16	.27831	61	1.01507	106	1.59727	151	1.93629
17	.29561	62	1.03007	107	1.60771	152	1.94059
18	.31286	63	1.04499	108	1.61803	153	1.94474
19	.33009	64	1.05983	109	1.62823	154	1.94874
20	.34729	65	1.07460	110	1.63830	155	1.95359
21	.36447	66	1.08927	111	1.64825	156	1.95629
22	.38161	67	1.10387	112	1.65807	157	1.95985
23	.39873	68	1.11838	113	1.66777	158	1.96325
24	.41582	69	1.13281	114	1.67734	159	1.96651
25	.43288	70	1.14715	115	1.68678	160	1.96961
26	.44990	71	1.16140	116	1.69609	161	1.97257
27	.46689	72	1.17557	117	1.70528	162	1.97537
28	.48381	73	1.18964	118	1.71553	163	1.97803
29	.50076	74	1.20363	119	1.72325	164	1.98053
30	.51763	75	1.21752	120	1.73205	165	1.98289
31	.53447	76	1.23132	121	1.74071	166	1.98409
32	.55127	77	1.24503	122	1.74924	167	1.98714
33	.56803	78	1.25864	123	1.75763	168	1.98904
34	.58474	79	1.27215	124	1.76589	169	1.99079
35	.60141	80	1.28557	125	1.77402	170	1.99239
36	.61803	81	1.24889	126	1.78201	171	1.99383
37	.63461	82	1.31211	127	1.78986	172	1.99512
38	.65113	83	1.32524	128	1.79758	173	1.99627
39	.66761	84	1.33826	129	1.80517	174	1.99726
40	.68404	85	1.35118	130	1.81261	175	1.99809
41	.70041	86	1.36399	131	1.81992	176	1.99878
42	.71673	87	1.37671	132	1.82709	177	1.99931
43	.73300	88	1.38931	133	1.83412	178	1.99969
44	.74921	89	1.40181	134	1.84101	179	1.99992
45	.76536	90	1.41421	135	1.84776	180	2.00000

Here the next higher chord will be found = ,50076
And the next lower, = ,48384

The difference, = ,01692

The difference between the required chord, viz: ,5, and the next lower one, viz: ,48384, is ,01616.

The arc answering to the next higher chord, is = 29°.

That answering to the next lower one is=28°.

The difference is = 1°.

Then .01692 : 01616: : 1° : .955°, the correction to be added to 28°, the arc answering to the next lower chord. Hence 28°+.955=28.955° is the arc corresponding to the chord .5.

Ex. 2. What is the length of the arc of a circle, whose chord is 16 inches, the radius of the circle being 12 inches.

16÷12=1.33333;

The next higher number in the column of chords is = 1.33826, and the corresponding arc is=84°
The next lower= 1.32524, the corresponding arc is=83°
The difference = .01302, the difference of the arcs, 1°
The difference of the chord sought and the next lower one is = .00807.

Then .01302 : 00807: : 1 : .0619, the correction to be added to 83°; whence we have 83.0619° for the arc. The circumference of a circle whose radius is 12 inches is 75.3981 inches. *By Rule I.*, 360° : 83.0619°: :75.8981 : 17.3964 inches, length of the arc required.

PROBLEM XII.

To find the length of the chord of any given arc.

RULE.—Seek for the number of degrees in the arc, and take out the chord corresponding thereto; and if the arc cannot be exactly found in the column of arcs, a proportion may be instituted, as before, for finding the arc in Prob. XI. Case II. Multiply the tabular chord thus found, by the radius of the circle, which will give its length.

Ex. 1. Let it be required to find the length of the chord of 20°, in a circle whose radius is 10 inches.

Opposite 20° in the table is found .34729=the chord of 20° in a circle whose radius is 1; this multiplied by 10 =3.4729, the length of the chord required.

Ex. 2 What is the length of the chord of an arc of 80°, in a circle whose radius is 12 inches.

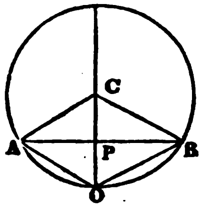
Ans. 15.42684 inches.

Note.—The chord of 60° is = the radius, (Prop. V. B. V.)
the chord of 90° is $=\sqrt{2r^2}$ (Prop. VI. Sch. B. V.)
the chord of 120° is $=2\sqrt{\frac{3}{4}r^2}$
or $\sqrt{3}:1::$the chord of 120° $:r$ or radius, (Prop. VII. Sch. B. V.)

PROBLEM XIII.

To find the length of a perpendicular drawn from the centre of a circle to a chord, or the altitude of a triangle formed by a chord and two radii.

Rule.—From the square of the radius subtract the square of half the chord; the square root of this remainder will be the length of the perpendicular or the altitude.



Let CAB be a triangle formed by the chord AB, and the two radii CA, CB; then will CP be the altitude of the triangle, which is evidently $=\sqrt{(AC^2-AP^2.)}$

Ex. 1. Let it be required to find the altitude of the triangle CAB, the chord AB being 12 feet, and the radius CA 8 feet.

$$CA^2=64$$
$$AP^2=36$$
$$CA^2-AP^2=28$$
$$\sqrt{(CA^2-AP^2)}=\sqrt{28}=5.2915 \text{ ft., the altitude CP.}$$

Scholium 1. If the altitude CP of the triangle be subtracted from the radius CO, the remainder OP will be the altitude of the segment ABO.

Hence, Ex. 2. To find the altitude of a segment of a circle whose arc is 80°, the radius of the circle being 12 ft.

By Prob. XII., the chord of $80°=15.4268$ ft.
$(15.4268\div2)^2=59.496539.$
$12\times12=144,$
$144-59.496539=84,503461=OP^2.$

Whence $OP=9.192$;
And $12-9.192=2.808=CP$, the height of the segment.

Scholium 2. Let r=the radius CA, c=the chord AB, p=the perpendicular CP, and h=the height OP of the segment.

Then will $p=r-h$,

And $r^2-2rh+h^2+\frac{1}{4}c^2=r^2$,

or $h^2+\frac{1}{4}c^2=2rh$;

hence $r=\frac{h^2+\frac{1}{4}c^2}{2h}$ - - - - - - - - - - - - (1)

and $c=\sqrt{(8rh-4h^2)}$ - - - - - - - - - - - (2)

and $h=r-\sqrt{(r^2-\frac{1}{4}c^2)}$ - - - - - - - - - - (3)

Again $p=r-h=\frac{r^2+\frac{1}{4}c^2}{2h}-\sqrt{(2rh-\frac{1}{4}c^2)}$ - (4)

Again $\frac{1}{4}c^2+p^2=r^2$,

hence we have $r=\sqrt{(\frac{1}{4}c^2+p^2,)}$ - - - - - - - (5)

and $c=\sqrt{(4r^2-4p^2)}$ - - - - - - (6)

and $p=\sqrt{(r^2-\frac{1}{4}c^2)}$ - - - - - - - (7)

Hence, by having two of these parts given, the others may be found.

Examples for Practice.

1. If the height of a segment is 2 inches, and the chord 8 inches, what is the radius of the circle?
Ans. 5 inches.

2. If the height of a segment be 3 feet, and the radius of the circle be 10 feet, what is the length of the chord?
Ans. 10.198039 feet.

3. If the chord of a segment be 12 feet, and the radius of the circle 10 feet, what is the height of the segment?
Ans. 2 feet.

4. If the chord of a segment be 12 feet, and the radius of the circle be 10 feet, what is the altitude of the triangle formed on the chord by the radii from its extremities?
Ans. 8 feet.

5. If the length of a chord is 8 inches, and the altitude of a triangle formed by the chord and two radii is 3 inches, what is the radius of the circle? Ans. 5 inches.

6. The altitude of a triangle formed by a chord and two radii is 6 feet, and the radius of the circle is 10 feet, what is the length of the base of the triangle? Ans.

7. The radius and chord of a circle are 10 feet and 4 feet respectively, what is the altitude of the triangle formed on the chord, and whose vertex is the centre of the circle?

Ans.

PROBLEM XIV.

To find the area of a circle.

RULE I.—*When the diameter and circumference are given.* Multiply half the circumference by half the diameter for the area;

Or, multiply the whole circumference by $\frac{1}{4}$ of the diameter, (Prop. XVI. B. V.)

Ex. Required the area of a circle, whose diameter is 10, and circumference 31.4159.

Here $(31.4159 \div 2) \times (10 \div 2) = 15.7079 \times 5 = 78.5398$, the area of the circle required.

Or thus:

$$31.4159 \times \frac{10}{4} = 78.5398 \text{ as before.}$$

RULE II.—*When the diameter only is given.* Multiply the square of the diameter by .7854, and the product will be the area of the circle.

For the ratio of the square described on the diameter of a circle, is to the circle's area, approximately, as 1 to .7854.

Ex. Taking the same example as in Rule I.,

We have the diameter $= 10$,

and $10^2 \times .7854 = 100 \times .7854 = 78.54$, the area required.

Examples for Practice.

Ex. 1. Required the area of a circle, whose diameter is 22.6, and the circumference 71. Ans 401.15.

Ex. 2. What is the area of a circle, whose diameter is 7? Ans. 38.4846.

Ex. 3. Find the area of a semicircle, the diameter of the whole circle being 66 feet.

Ans. 1710.6012 square feet.

Ex. 4. The diameter of a circle is .318309, and the circumference being 1, required the area of the quadrant.

Ans. .019894.

Ex. 5. The area of a circular plantation is known to occupy an acre and a half of ground, what must be the length of a line, which will strike the circle?

Ans. 48.072 yards.

Ex. 6. In the midst of a meadow well stored with feed,
I took just an acre to tether my steed;
How long must the chord be, that, feeding all round,
He mayn't graze less or more than his acre of ground?

Ans. 39.25075 yards.

Ex. 7. Required the diameter and circumference of a circle, whose area is 5760 square feet.

$$\text{Ans.}\begin{cases}\text{Diameter} = 85.637 \text{ feet.}\\ \text{Circumference} = 269.039 \text{ feet.}\end{cases}$$

PROBLEM XV.

To find the area of a Circular Ring, or the space included between two Concentric Circles.

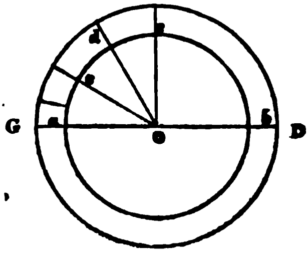
RULE.—*When the two circumferences and radii are given.* Multiply half the sum of the circumferences by the difference of their radii, and the product will be the answer, (Prop. XVI. B. V.)

Or thus:

Take the difference of the areas of the two circles, for the area of the ring.

Ex. The radii O*a* and OG of two concentric circles *ab* and GD being 3 and 5, and their circumferences 18.8496 and 31.416 respectively, required the area of the ring or space contained between the two circumferences.

½(31,416+18,8496) × (5—3) = 50,2656, the area of the ring.



Scholium.—When the two diameters are given. Multiply the sum of the diameters of the two circles by their difference, and that product by .7854, the area of a circle whoes diameter is 1, and the last product will be the area of the ring.

Taking the same example as before:

$$(ab+GD)\times(ab-GD)\times.7854=(10+6)\times(10-6)\times .7854=50.2656,\text{ the area required.}$$

Examples for Practice.

Ex. 1. The diameters of two concentric circles being 20 and 10 respectively, required the area of the ring between their circumferences. Ans. 235.62.

Ex. 2. The circumferences of two concentric circles are 1000 and 800 respectively; required the area of the ring included between their circumferences. Ans. 28648.8.

Ex. 3. Supposing a circular fish-pond is to be dug, whose diameter being 60 yards, a gravel walk of 6 feet wide, and 3 feet from the edge of the water, is to be made round it; required the area of the gravel walk.

Ans. 402.1248 sq. yards.

Ex. 6. Required the area of the front of a circular arch, whose dimensions are as follows: the length of the upper bounding arch=36.5 feet, the length of the lower arch= 34.25 feet, and the breadth of the arch=2.5 feet.

Ans. 88.375 sq. feet.

PROBLEM XVI.

To find the area of a sector of a circle.

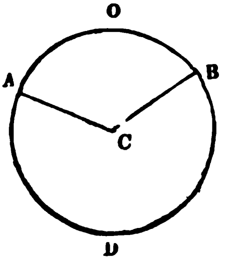
Rule.—Multiply the radius into half the length of the arc:

Or,

As 360, to the number of degrees in the arc, so is the area of the circle, to the area of the sector.

Note.—It is evident that the area of the sector has the same ratio to the area of the circle, which the length of the arc has to the length of the circumference, or which the number of degrees in the arc has to the number of degrees in the circumference.

Ex. If the arc AOB be 120°, and the diameter of the circle 226, what is the area of the sector AOBC?



Here (by the Rule) the area of the whole circle being 40115, we have 360° : 120°::40114 : 13371⅓, the area of sector.

Examples for Practice.

Ex. 1. What is the area of a quadrant, whose radius is 62. Ans. 1509.4675.

Ex. 2. What is the area of a semicircle, whose diameter is 32.

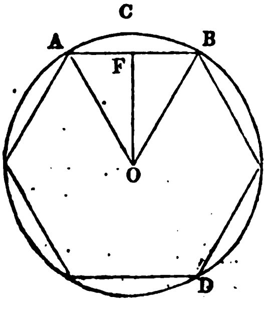
Ex. 3. If the arc ADB be 240°, and the radius of the circle 113, what is the area of the sector ADBC.

PROBLEM XVII.

To find the area of the segment of a circle.

Rule I.—Find the area of the sector which has the same arc, and also the area of the triangle formed by the chord of the segment and the radii of the sector.

Then if the segment is less than a semicircle, subtract the area of the triangle from the area of the sector. But if it is greater, add the area of the triangle to the area of the sector.



If the triangle ABO be taken from the sector AOBC, it is evident the difference will be the segment ABC less than a semicircle. And if the same triangle be added to the sector ADBO, the sum will be the segment ADBA greater than a semicircle.

The area of the triangle is equal to the product of h the chord AB into OF; OF is the difference between t radius and the height of the segment.

Ex. 1 What is the area of the segment ABC, the arc the segment being 60° and radius 12?

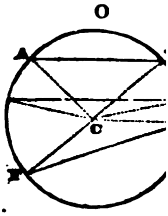
By Prop. V. B. V., the chord of the segment equals t radius. Hence the line $OF=\sqrt{(12^2-6^2)}=10.3923048$.

Then (by the Rule) $10.3923048\times6=62.3538$, the area the triangle AOB.

Also $.7854\times24^2\div6=75.3984$.

Hence $75.3984-62.3538=13.0446$, the area of the se ment.

Ex. 2. To find the area of the segment ABO, its chord AB being 12 and its radius CA 10.



By Prob. XI. Case 2, the arc of the segment will be fou $=73.74°$.

And by Prob. XVI., the area of sector CAOB=64,3504.

The altitude of the triangle CAB, by Prob. XIII.,=8.

Hence $12\times8\div2=18$=the area of the triangle.

Therefore $64.3504-48=16.3504$=the area of the segme ABO.

Ex. 3. What is the area of the segment of a quadrant a circle, whose radius is 25 feet. Ans. 178.37.

Ex. 4. What is the area of the segment contained by arc of 120°, if the radius is 2 feet.

Ex. 5. What is the area of the segment, whose arc 270°, when the radius is 22 feet.

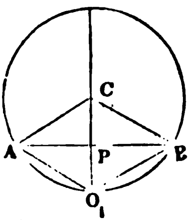
RULE II.—Divide the height of the segment by the di meter, and find the quotient in the column of heights in t following table. Take out the corresponding area in t next column on the right hand, and multiply it by the squa of the circle's diameter, for the area of the segment.

Note.—When the quotient is not found exactly in the t ble, proportions may be made between the next less a greater area, in the same manner as is done in the table chords and arcs, (Prob. XI. Case 2.)

TABLE OF THE AREAS OF CIRCULAR SEGMENTS.

Height.	Area of the segment.	Height.	Area of the segment.	Height.	Area of the segment.	Height.	Area of the segment.	Height.	Area of the segment.
.01	00133	.11	.04701	.21	.11990	.31	.20738	.41	.30319
.02	.00375	.12	05339	22	.12811	.32	.21667	42	31304
.03	.00687	.13	.06000	.23	.13646	33	.22603	43	.32293
.04	.01054	.14	.06683	24	14494	34	.23517	44	33284
.05	.01468	.15	.07387	.25	.15354	.35	.24498	.45	.34278
.06	.01924	.16	.08111	.26	.16226	.36	.25455	.46	.35274
.07	.02417	.17	.08853	.27	.17109	.37	.26418	47	.36272
.08	.02944	.18	.09613	.28	.18002	.38	.27386	48	.37270
.09	.03502	.19	.10390	.29	.18905	.39	28359	49	.38270
.10	.04088	.20	.11182	.30	.19817	.40	.29337	.50	.39270

Examples for Practice.



Ex. 1. What is the area of the segment ABO, its chord AB being 12, and the radius CB or CO 10?

Here $\sqrt{}(BC^2—BP^2)=\sqrt{}(100—36)=\sqrt{}64=8=CP$,

then OC—CP=OP, the height of the segment, = 10—8=2.

Hence OP÷2OC=2÷20=.1, the height of a similar segment in a cirle whose diameter is 1. This being found in the first column of the table, the corresponding tabular area is .04088. Then $.04088\times20^2=.04088\times400=16.352$, the area of the segment ABO.

Ex. 2. What is the area of the segment whose height is 18, and diameter of the circle 50? Ans. 636.375.

Ex. 3. Required the area of the segment, whose chord is 16, the diameter being 20? Ans. 44.728.

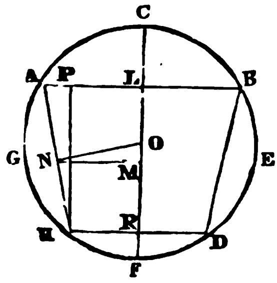
Ex. 4. Find the area of a segment, whose chord is 60 feet and whose altitude is 5 feet. Ans. 201.1063.

PROBLEM XVII.

To find the area of a circular zone, or the space ABEDHGA, included between two parallel chords AB, HD, and the two arcs AGH, BED.

Rule.—From the area of the whole circle subtract the two segments on the sides of the zone.

If from the whole circle there be taken the two segments ABC and DHF, there will remain the zone ABDH. Or the area of the zone may be found, by subtracting the segment ABC from the segment HDC: or by adding two small segments GAH and EBL to the trapezoid ABDH.



Ex. What is the area of the zone ABDH, if AB is 7.75, DH 6.93, and the diameter of the circle 8?

The area of the whole circle is		50.26
of the segment ABC	17.32	
of the segment DFH	9.82	
of the zone ABDH		23.12

Scholium.—If the diameter of the circle is not given, it may be found from the sides and breadth of the zone.

Let the centre of the circle be at O. Draw ON perpendicular to AH, NM perpendicular to LR, and HP perpendicular to AL. Then

$AN=\frac{1}{2}AH$, (Prop. VI. B. III.) $MN=\frac{1}{2}(LA+RH.)$
$LM=\frac{1}{2}LR$, (Prop. XIV. B. IV.) $PA=LA-RH.$

The triangles APH and OMN are similar, because the sides of one arc are perpendicular to those of the other, each to each. Therefore

$$PH : PA :: MN : MO.$$

MO being found, we have $ML-MO=OL$,
And the *radius* $CO=\surd(OL^2+CL^2)$.

Ex. If the breadth of a zone ACDH be 6.4, and the sides 6.8 and 6; what is the radius of the circle?

$$PA=3.4-3=0.4.\quad\text{And } MN=\tfrac{1}{2}(3.4+3)=3.2.$$

Then $6.4 : 0.4 :: 3.2 : 0.2=MO$,
And $3.2-0.2=3=OL$,
And the radius $CO=\surd[3^2+(3.4)^2]=4.534$.

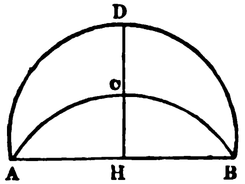
Note.—For a full development of the principles relative to circular segments and zones, the student is referred to a subsequent volume of this work, where, from principles there demonstrated, the area of the segment or zone of a circle may be calculated with as little labor as the area of a triangle.

PROBLEM XVIII.

To find the area of a lune or crescent.

RULE.—Find the difference of the two segments which are between the arcs of the crescent and its chords.

If the segment ABC be taken from the segment ABD, there will remain the lune or crescent ACBD.



Ex. If the chord AB be 88, the height CH 20, and the height DH 40; what is the area of the crescent ACBD?

The area of the segment ABD is - - - - - 2698
of the segment ABC - - - - - - - 1220

of the crescent ACBD - - - - - - - 1478

A COLLECTION OF QUESTIONS FOR EXERCISE IN GEOMETRY AND MENSURATION.

1. In a right angled triangle, given the base, the sum or difference of the other two sides, and the area, to determine the triangle.

2. Given the base, the ratio of the other two sides, and the area, to determine the triangle.

3. Let it be demonstrated that if from the angles of any right angled triangle three lines be drawn perpendicular to the opposite sides, they will all cut each other in the same point;

Or, that if lines be drawn bisecting these angles, these lines will all pass through the same point.

Also, that if lines be drawn from the three angles, bisecting the opposite sides, those lines shall all pass through the same point.

4. A trapezium being given, to divide it into two parts having any given ratio, by a line passing through any given point, either in one of the given sides, or within the trapezium or without it.

5. In a given circle, to inscribe an isosceles triangle of a given magnitude.

6. In a given circle, to inscribe the greatest triangle possible.

7. To circumscribe a given circle by the least triangle possible.

8. To circumscribe about a given circle an isosceles triangle of a given magnitude.

9. To divide a trapezium into four equal parts, by lines intersecting each other at right angles.

10. Given the area of a right angled triangle, and the sum of the three sides, to determine the triangle.

11. Given the base of a triangle, the sum of the other two sides, and the line drawn from the vertex to the middle of the base, to determine the triangle.

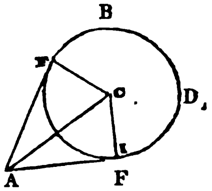
12. Given the three lines drawn from the angles of a triangle to the middle of each of the opposite sides, to determine the triangle.

13. Given the base of a triangle, and the sum and difference of the squares of the sides, to determine the triangle.

This problem is susceptible of a very simple and elegant construction, for the vertex of this triangle is in the circumference of a certain circle, and is also in a certain right line

14. What is the area of the surface AIBDFA enclosed by the circular arc IBDF, and the tangents FA and IA, whose sum is 100 feet, the radius CI being 10 feet.

Ans. 500 square feet.



15. The radius of the earth at the equator being found to be 3963.2 miles, what is the circumference of the equator?

Ans. 24901.578 miles.

16. The polar diameter of the earth being found to be 7900.8 miles, what is the circumference of a meridian, on the supposition that it is a circle. Ans. 24821.153.

17. What is the area of the circle, of which the equator is the circumference. Ans. 29314812.4 sq. miles.

18. What is the area of a sector, whose arc is 120°, and whose radius is 100 feet. Ans. 10472 sq. feet.

ELEMENTS

OF

SOLID GEOMETRY

AND

MENSURATION,

BEING THE

SECOND PART OF A SERIES

ON

ELEMENTARY AND HIGHER

GEOMETRY, TRIGONOMETRY, AND MENSURATION;

CONTAINING MANY VALUABLE DISCOVERIES AND IMPROVEMENTS IN MATHEMATICAL
SCIENCE, ESPECIALLY IN RELATION TO THE QUADRATURE OF THE CIRCLE,
AND SOME OTHER CURVES, AS WELL AS THE CUBATURE OF CERTAIN
CURVILINEAR SOLIDS; DESIGNED AS A TEXT-BOOK FOR COLLEGIATE
AND ACADEMIC INSTRUCTION, AND AS A PRACTICAL
COMPENDIUM ON MENSURATION.

BY NATHAN SCHOLFIELD.

LONDON:
WILEY AND PUTNAM, 6, WATERLOO PLACE.
1845.

PREFACE.

THIS division of the subject is based on LEGENDRE'S ELEMENTS, and contains only the elementary part of solid geometry, with its application to the mensuration of such solids as are therein treated of. The higher branches of solid geometry, together with the subject of mensuration thereto pertaining, is reserved for the fourth division of the series. In this division will be found some new propositions in relation to surfaces and solidities. In the second book, the subject of small pyramidal portions, or elementary pyramids, is discussed, showing that all polyedrons and solids, bounded by plane surfaces, may be conceived to be formed from these as their elements. In the third book, the reasoning in relation to elementary pyramidals is extended to solids of revolution, showing that the cylender, cone, and sphere may be divided into regular pyramidal portions. The discussions in relation to consected cylinders, cones, and conic frustra, and also on a new class of solids, termed polyedroids, it is presumed will be found both novel and interesting, by means of which the demonstrations in relation to the surface and solidity of the sphere are rendered more rigorous than by the methods usually pursued. Book IV. consists of the isoperimetry of solids. The mensuration of elementary solids follows, to which is appended a table of squares and cubes, square roots and cube roots, which will be found highly useful in tedious calculations where powers and roots are concerned.

CONTENTS.

BOOK FIRST.

PLANES AND SOLID ANGLES.

BOOK SECOND.

POLYEDRONS—THEIR PROPERTIES—THEIR SURFACES AND SOLIDITIES, WHETHER PRISMS, PYRAMIDS, PRISMOIDS, OR WEDGES.

SECTION 2D—REGULAR POLYEDRONS.

BOOK THIRD.

SOLIDS OF REVOLUTION—THE CONE, CYLINDER, POLYEDROID, AND SPHERE—THEIR SURFACES AND SOLIDITIES.

BOOK FOURTH.

ISOPERIMETRY OF SOLIDS.

NOTES.

DISCUSSIONS RESPECTING ELEMENTARY PRISMS, PYRAMIDS, &c.—ON SOME NEW GEOMETRICAL FIGURES INTRODUCED INTO THE ELEMENTS OF GEOMETRY.

MENSURATION OF SOLIDS.

AN APPLICATION OF THE ELEMENTARY PRINCIPLES OF GEOMETRY TO THE MENSURATION OF SOLIDS—TABLE OF SQUARES AND CUBES, SQUARE ROOTS AND CUBE ROOTS OF NUMBERS, FROM 1 TO 1000.

ELEMENTS OF SOLID GEOMETRY.

BOOK I.

PLANES AND SOLID ANGLES.

DEFINITIONS.

1. The *common section* of two planes, is the line in which they meet to cut each other.

2. A *line* is perpendicular to a plane when it is perpendicular to every line in that plane which meets it.

3. One *plane* is *perpendicular* to another, when every line of the one which is perpendicular to the line of their common section is perpendicular to the other.

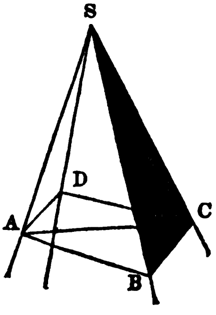
4. A line is *parallel* to a *plane*, when it is equidistant from it in all its parts.

5. Parallel *planes* are such as are everywhere at the same perpendicular distance, or such as being produced ever so far in either direction, will never meet.

6. The angle or mutual inclination of two planes, is the angle contained between two lines, one in each plane, and both perpendicular to the common intersection at the same point. This angle may be acute, obtuse, or a right angle. If it is a right angle, the two planes are perpendicular to each other.

7. A *solid angle* is that which is made by three or more plane angles meeting each other in the same point.

Thus the solid angle S is formed by the union of the planes ASB, BSC, CSD, DSA.



PROPOSITION I. THEOREM.

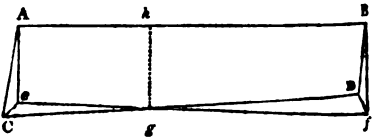
A straight line cannot lie partly in a plane and partly out of it, when both are produced to the same distance.

For, by the definition of a plane, where a straight line has two points common with a plane, it lies wholly in that plane.

Scholium. To discover whether a surface is plane, it is necessary to apply a straight line in different directions to that surface, and ascertain if it touches the surface throughout its whole extent.

PROPOSITION II. THEOREM.

If one straight line coincides with a plane, and another straight line is cut by the same plane, those lines cannot be parallel.

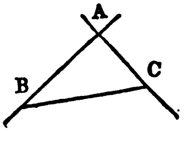


Let AB be a straight line, coinciding in the plane A*ef*B in its whole length; and let CD be another straight line, cut by the same plane in *g*; then the lines AB and CD cannot be parallel.

For make the line *ef*, which coincides in the same plane as the line AB, and which cuts the line CD, parallel to AB, and at the distance *hg*. Then (Prop. XX, B. II. *El. Geom.*) will A*e* and B*f* be each equal to *hg*, the two lines being equidistant in all their parts by definition. Now, as the plane cuts the line CD in *g*, so as to leave C*g* below the plane and D*g* above it, let C*e* and D*f* be drawn perpendicular to the plane, and these lines will measure the distance of the extremes of the line CD from the plane, join CA and DB then we shall have the two right angled triangles A*e*C and B*f*D, with the right angl respectively at *e* and *f*. And because the hypothenuse of right angled triangle is greater than either of the legs, the lines CA and DB are each greater than *e*A or *f*B; but th distance *gh* is common to the two lines *ef* and CD, and i equal to *e*A and also to *f*B. Therefore, the line CD is n equidistant from AB in all its parts, and hence is not parallel AB. But suppose it were made to be equidistant in each the points C, *g* and D, by making the lines AC, *hg*, and B equal; this would require that the extremities of the line C should be brought nearer to the line AB, while the point *g* r mains fixed, and hence the line would become curved. Ther fore, the conditions of the proposition are not answere Hence right lines that cannot coincide in the same plane a not parallel.

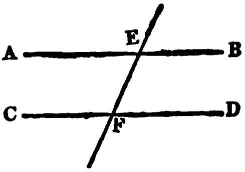
PROPOSITION III. THEOREM.

raight lines, which intersect each other, lie in the same plane, and determine its position.



AB, AC be two straight lines which t each other in A; a plane may be ed in which the straight line AB is if this plane be turned round AB, pass through the point C, then the , which has two of its points, A and is plane, lies wholly in it; hence the position of the determined by the single condition of containing the ight lines AB, AC.

A triangle ABC, or three points, A, B, C, not in a line, determine the position of the plane.



2. Hence, also, two parallels) determine the position of a or, drawing the secant EF, the the two straight lines AE, EF, f the parallels AB, CD.

PROPOSITION IV. THEOREM.

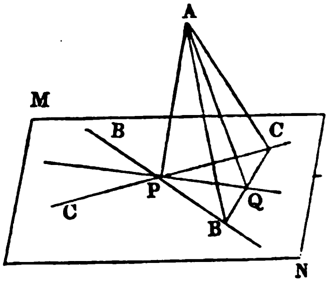
lanes cut each other, their common intersection will be a straight line.

f among the points common to the two planes, there which are not in the same straight line, then the passing each through these three points, must form and the same plane; which contradicts the hypo-

PROPOSITION V. THEOREM.

traight lines intersect each other, and a third line is dicular to both of them at their point of intersection, also be perpendicular to all lines drawn through its nd in the same plane of the two first lines, and will, ore, be perpendicular to the plane of those lines.

P be perpendicular to PB, PC, at the point P, and plane of the lines CP, BP; then will AP be perpendicular to any line of the plane passing through P, and consequently to the plane itself, (Def. 2.)



Through any point Q in PQ, draw (Prob. V. B IV. *El. Geom.*) the straight line BC in the angle BPC, so that BQ=QC; join AB, AQ, AC.

The base BC being divided into two parts at the point Q, the triangle BPC (Prop. XXVII. B. IV. *El. Geom.*) will give

$$PC^2+PB^2=2PQ^2+2QC^2.$$

The triangle BAC will in like manner give

$$AC^2+AB^2=2AQ^2-2QC^2.$$

Taking the first equation from the second, and observing that the triangles APC, APB, which are both right-angled at P, give $AC^2-PC^2=AP^2$, and $AB^2-PB^2=AP^2$; we shall have

$$AP^2+AP^2=2AQ^2-2PQ^2.$$

Therefore, by taking the halves of both, we have $AP^2=AQ^2-PQ^2$, or $AQ^2=AP^2+PQ^2$; hence the triangle APQ is right angled at P; hence AP is perpendicular to PQ.

Scholium. Thus it is evident, not only that a straight line may be perpendicular to all the straight lines which pass through its foot in a plane, but that it always must be so, whenever it is perpendicular to two straight lines drawn in the plane; which proves our second Definition to be accurate.

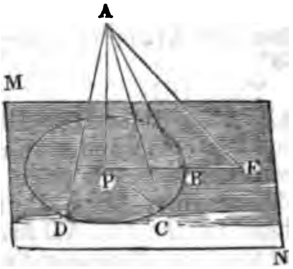
Cor. 1. The perpendicular AP is shorter than any oblique line AQ; therefore it measures the true distance from the point A to the plane MN.

Cor. 2. At a given point P on a plane, it is impossible to erect more than one perpendicular to that plane; for if there could be two perpendiculars at the same point P, draw through these two perpendiculars a plane, whose intersection with the plane MN is PQ; then these two perpendiculars would be perpendicular to the line PQ, at the same point, and in the same plane, which is impossible, (Prop. I. B. II. *El. Geom.*)

It is also impossible to let fall from a given point out of a plane two perpendiculars to that plane; for let AP. AQ, be these two perpendiculars, then the triangle APQ would have two right angles APQ, AQP, which is impossible.

PROPOSITION VI. THEOREM.

lique lines drawn from the same point without a plane, to points equidistant from the perpendicular to the plane, drawn from such point are equal, and of two oblique lines drawn to points unequally distant from the perpendicular, the more distant is the longer.



Let AP be perpendicular to the lane MN ; AB, AC, AD, oblique nes equally distant from the per- endicular, and AE a line more re- iote: then will AB=AC=AD ; nd AE will be greater than AD.

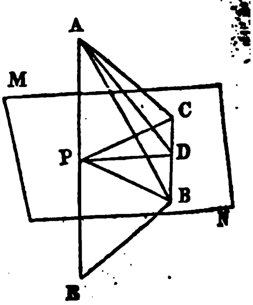
For, the angles APB, APC, APD, eing right angles, if we suppose ie distances PB, PC, PD, to be qual to each other, the triangles PB, APC, APD, will have in each an equal angle contained two equal sides ; therefore they will be equal ; hence the pothenuses, or the oblique lines AB, AC, AD, will be equal to ch other. In like manner, if the distance PE is greater than D or its equal PB, the oblique line AE will evidently be eater than AB, or its equal AD.

Cor. All the equal oblique lines, AB, AC, AD, &c., termi- te in the circumference BCD, described from P the foot of e perpendicular as a centre ; therefore a point A being given t of a plane, the point P at which the perpendicular let fall om A would meet that plane, may be found by marking upon at plane three points B, C, D, equally distant from the point A, d then finding the centre of the circle which passes through ese points ; this centre will be P, the point sought.

Scholium. The angle ABP is called the *inclination of the lique line* AB *to the plane* MN ; which inclination is evidently ual with respect to all such lines AB, AC, AD, as are equally stant from the perpendicular ; for all the triangles ABP, ACP, DP, &c. are equal to each other.

PROPOSITION VII. THEOREM.

If from a point without a plane, a perpendicular be let fall on the plane, and from the foot of the perpendicular a perpendicular be drawn to any line of the plane, and from the point of intersection a line be drawn to the first point, this latter line will be perpendicular to the line of the plane.



Let AP be perpendicular to the plane NM, and PD perpendicular to BC; then will AD be also perpendicular to BC.

Take $DB=DC$, and draw PB, PC AB, AC. Since $DB=DC$, the oblique line $PB=PC$: and with regard to the perpendicular AP, since $PB=PC$, the oblique line $AB=AC$ (Prop. VI.); therefore the line AD has two of its points A and D equally distant from the extremities B and C; therefore AD is a perpendicular to BC, at its middle point D (Prop. XVIII. B. II. *El. Geom.*)

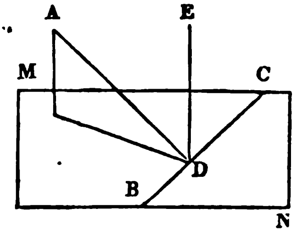
Cor. It is evident likewise, that BC is perpendicular to the plane APD, since BC is at once perpendicular to the two straight lines AD, PD.

Scholium. The two lines AE, BC, afford an instance of two lines which do not meet, because they are not situated in the same plane. The shortest distance between these lines is the straight line PD, which is at once perpendicular to the line AP and to the line BC. The distance PD is the shortest distance between them, because if we join any other two points, such as A and B, we shall have $AB>AD$, $AD>PD$; therefore $AB>PD$.

The two lines AE, CB, though not situated in the same plane, are conceived as forming a right angle with each other, because AE and the line drawn through one of its points parellel to BC would make with each other a right angle. In the same manner, the line AB and the line PD, which represent any two straight lines not situated in the same plane, are supposed to form with each other the same angle, which would be formed by AB and a straight line parallel to PD drawn through one of the points of AB.

PROPOSITION VIII. THEOREM.

f two parallel lines be perdendicular to a plane, the other will also be perpendicular to the same plane.



he lines ED, AP, be parallel ;
s perpendicular to the plane
en will ED be also perpen-
to it.
ugh the parallels AP, DE,
plane ; its intersection with
e MN will be PD ; in the
N draw BC perpendicular
and draw AD.
e Corollary of the preceding Theorem, BC is perpen-
to the plane APDE ; therefore the angle BDE is a right
but the angle EDP is also a right angle, since AP is
licular to PD, and DE parallel to AP (Prop. XXIII.
B. II. *El. Geom.*) ; therefore the line DE is perpendicu-
he two straight lines DP, DB ; hence it is perpendicu-
eir plane MN (Prop. V.).

1. Conversely, if the straight lines AP, DE, are per-
lar to the same plane MN, they will be parallel ; for if
not so, draw through the point D, a line parallel to
s parallel will be perpendicular to the plane MN ; there-
ough the same point D, more than one perpendicular
e erected to the same plane, which is impossible (Prop.
. 2.).

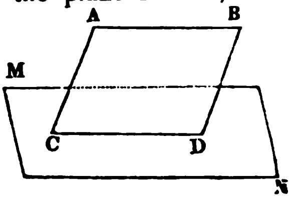
2. Two lines A and B, parallel to a third C, are par-
each other ; for, conceive a plane perpendicular to the
the lines A and B, being parallel to C, will be perpen-
to the same plane ; therefore, by the preceding Corol-
ey will be parallel to each other.
three lines are supposed not to be in the same plane ;
se the proposition would be already known (Prop. XXV.
El. Geom.).

PROPOSITION IX. THEOREM.

aight line is parallel to a straight line drawn in a plane, it will be parallel to that plane.

AB be parallel to CD of the plane NM ; then will it be
l to the plane MN.

For, if the line AB, which lies in the plane ABDC, could meet the plane MN, this could only be in some point of the line CD, the common intersection of the two planes: but AB cannot meet CD, since they are parallel; hence it will not meet the plane MN; hence it is parallel to the plane (Def. 4.).

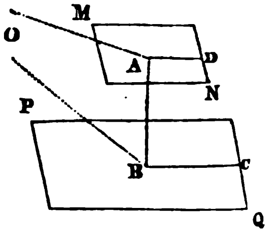


PROPOSITION X. THEOREM.

Two planes which are perpendicular to the same straight line, are parallel to each other.

Let the planes NM, QP, be perpendicular to the line AB, then will they be parallel.

For, if they can meet any where, let O be one of their common points, and draw OA, OB; the line AB which is perpendicular to the plane MN, is perpendicular to the straight line OA drawn through its foot in that plane; for the same reason AB is perpendicular to BO; therefore OA and OB are two perpendiculars let fall from the same point O, upon the same straight line; which is impossible; therefore, the planes MN, PQ, cannot meet each other; consequently they are parallel.

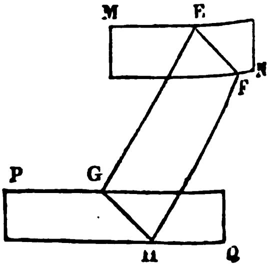


PROPOSITION XI. THEOREM.

If a plane cut two parallel planes, the lines of intersections will be parallel.

Let the parallel planes NM, PQ, be intersected by the plane EH; then will the lines of intersection EF, GH, be parallel.

For, if the lines EF, GH, lying in the same plane, were not parallel, they would meet each other when produced; therefore, the planes MN, PQ, in which those lines lie, would also meet; and hence the planes would not be parallel.



PROPOSITION XII. THEOREM.

planes are parallel, a straight line which is perpendicular to one, is also perpendicular to the other.

MN, PQ, (see diagram to Prop. X.) be two parallel , and let AB be perpendicular to NM ; then will it also pendicular to QP.

ing drawn any line BC in the plane PQ, through the B and BC, draw a plane ABC, intersecting the plane AD : the intersection AD will be parallel to BC (Prop. but the line AB, being perpendicular to the plane MN, endicular to the straight line AD ; therefore, also, to its l BC (Prop. XXIII. Cor. 1. B. II. *El. Geom.*) : hence e AB being perpendicular to any line BC. drawn through in the plane PQ, is consequently perpendicular to that Def. 2.)

PROPOSITION XIII. THEOREM.

arallels comprehended between two parallel planes are equal.

MN, PQ, (see diagram to Prop. XI.) be two parallel , and FH, GE, two parallel lines : then will EG=FH.

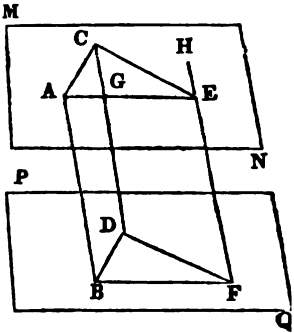
through the parallels EG, FH, draw the plane EGHF, cting the parallel planes in EF and GH. The intersec- F, GH, are parallel to each other (Prop. XI.) ; so like- re EG, FH ; therefore the figure EGHF is a parallelo- consequently, EG=FH.

. Hence it follows, that *two parallel planes are every equidistant :* for, suppose EG were perpendicular to the PQ ; the parallel FH would also be perpendicular to it VIII.) and the two parallels would likewise be perpen- r to the plane MN (Prop. XII.) ; and being parallel, they equal, as shown by the Proposition.

PROPOSITION XIV. THEOREM.

If two angles, not situated in the same plane, have their sid parallel and lying in the same direction, those angles will equal, and their planes will be parallel.

Let the angles be CAE and DBF.



Make AC=BD. AE=BF; and draw CE, DF, AB, CD, EF. Since AC is equal and parallel to BD, the figure ABDC is a parallelogram; therefore CD is equal and parallel to AB. For a similar reason, EF is equal and parallel to AB; hence also CD is equal and parallel to EF; hence the figure CEFD is a parallelogram, and the side CE is equal and parallel to DF; therefore the triangles CAE, DBF, have their corresponding sides equal; therefore the angle CAE= DBF.

Again, the plane ACE is parallel to the plane BDF. F suppose the plane drawn through the point A, parallel to BD were to meet the lines CD, EF, in points different from C an E, for instance in G and H; then, the three lines AB, GD, F would be equal (Prop. XIII.): but the lines AB, CD, EF, ar already known to be equal; hence CD=GD, and FH=E which is absurd; hence the plane ACE is parallel to BDF.

Cor. If two parallel planes MN, PQ, are met by two othe planes CABD, EABF, the angles CAE, DBF, formed by th intersections of the parallel planes will be equal; for the intersection AC is parallel to BD, and AE to BF (Prop. XI.); therefore the angle CAE=DBF.

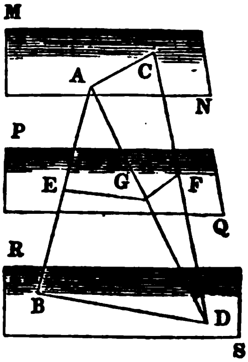
PROPOSITION XV. THEOREM.

If three straight lines, not situated in the same plane, are equal and parallel, the opposite triangles formed by joining the extremities of these lines will be equal, and their planes will b parallel.

Let AB, CD, EF, (see diagram to Prop. XIV.) be the lines. Since AB is equal and parallel to CD, the figure ABDC is parallelogram; hence the side AC is equal and parallel to BD For a like reason the sides AE, BF, are equal and parallel, also CE, DF; therefore the two triangles ACE, BDF, are equal; hence, by the last Proposition, their planes are parallel.

PROPOSITION XVI. THEOREM.

If two straight lines be cut by three parallel planes, they will be divided proportionally.



Suppose the line AB to meet the parallel planes MN, PQ, RS, at the points A, E, B; and the lines CD to meet the same planes at the points C, F, D: we are now to show that

AE : EB : CF : FD.

Draw AD meeting the plane PQ in G, and draw AC, EG, GF, BD; the intersections EG, BD, of the parallel planes PQ, RS, by the plane ABD, are parallel (Prop. XI.); therefore

AE : EB : : AG : GD;

in like manner, the intersections AC, GF, being parallel,

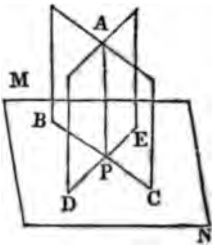
AG : GD : : CF : FD;

the ratio AG : GD is the same in both; hence

AE : EB : : CF : FD.

PROPOSITION XVII. THEOREM.

If a line is perpendicular to a plane, every plane passed through the perpendicular, will also be perpendicular to the plane.



Let AP be perpendicular to the plane NM; then will every plane passing through AP be perpendicular to NM.

Let BC be the intersection of the planes AB, MN; in the plane MN, draw DE perpendicular to BP: then the line AP, being perpendicular to the plane MN, will be perpendicular to each of the two straight lines BC, DE; but the angle APD, formed by the two perpendiculars PA, PD, to the common intersection BP, measures the angles of the two planes AB, MN (Def. 6.); therefore, since that angle is a right angle, the two planes are perpendicular to each other.

Scholium. When three straight lines, such as AP, BP, DP, are perpendicular to each other, each of those lines is perpendicular to the plane of the other two, and the three planes are perpendicular to each other.

BROPOSITION XVIII. THEOREM.

If two planes are perpendicular to each other, a line drawn in one of them perpendicular to their common intersection, will be perpendicular to the other plane.

Let the plane AB (see diagram to Prop. XVII.) be perpendicular to NM; then if the line AP be perpendicular to the intersection BC, it will also be perpendicular to the plane NM.

For, in the plane MN draw PD perpendicular to PB; then, because the planes are perpendicular, the angle APD is a right angle; therefore, the line AP is perpendicular to the two straight lines PB, PD; therefore it is perpendicular to their plane MN (Prop. V.).

Cor. If the plane AB is perpendicular to the plane MN, and if at a point P of the common intersection we erect a perpendicular to the plane MN, that perpendicular will be in the plane AB: for, if not, then, in the plane AB we might draw AP perpendicular to PB the common intersection, and this AP, at the same time, would be perpendicular to the plane MN; therefore at the same point P there would be two perpendiculars to the plane MN, which is impossible (Prop. V. Cor. 2.)

PROPOSITION XIX. THEOREM.

If two planes are perpendicular to a third plane, their common intersection will also be perpendicular to the third plane.

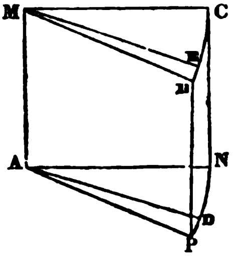
Let the planes AB, AD, (see diagram to Prop. XVII.) be perpendicular to NM; then will their intersection AP be perpendicular to NM.

For, at the point P, erect a perpendicular to the plane MN; that perpendicular must be at once in the plane AB and in the plane AD (Prop. XVIII. Cor.); therefore it is their common intersection AP.

PROPOSITION XX. THEOREM.

The angle included between two planes may be measured, agreeably to our Definition, by the angle which is formed by two lines, one being drawn in each of those planes, and both perpendicular to the common intersection at the same point.

Let the line AN of the plane MAN, and AP of the plane MAP be perpendicular to the common intersection AM at the point A; then will the angle PAN measure the angle included between the planes.



To show the correctness of this measurement, we must, in the first place, prove that it is constant, or that it would be the same at whatever point of the common intersection the perpendiculars were drawn.

Take any other point M; and draw MC in the plane MN, MB in the plane MP, perpendicular to the common intersection AM. Since MB and AP are perpendicular to the same line AM, they are parallel to each other. For the same reason, MC is parallel to AN; therefore (Prop. XIV.) the angle BMC=PAN; therefore it is indifferent whether the perpendiculars be drawn at the point M or at the point A; the included angle will be always the same.

In the second place, we must prove that, if the angle of the two planes increases or diminishes in a certain ratio, the angle PAN will increase or diminish in the same ratio.

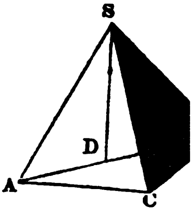
In the plane PAN, from the centre A and with any radius, describe the arc NDP; from the centre M and with an equal radius describe the arc CEB; draw AD to any point D of the arc PD: the two planes PAN, BMC, being perpendicular to the same straight line MA, will (Prop. X.) be parallel; therefore the intersections AD, ME, of these two planes with a third AMD, will be parallel; therefore (Prop. XIV.) the angle BME will be equal to PAD.

Let us for a moment call the angle, which is formed by the two planes MP, MN, a *wedge;* that granted, if the angle DAP were equal to DAN, it is evident that the wedge DAMP would be equal to the wedge DAMN; for the base PAD if placed upon its equal DAN would exactly coincide with it, the height AM would be always the same; therefore the two corners would coincide with each other. In like manner it may be shown, that if the angle DAP, were contained a certain number of times exactly in the angle PAN; the wedge DAMP would be contained just as many times in the wedge P But from the ratio in whole numbers, the conclusion wi gard to any ratio is legitimate, and is demonstrated in a altogether similar (Prop. XVIII. B. III. *El. Geom.*); the whatever be the ratio of the angle DAP to the angle PA wedge DAMP will be in that same ratio with the PAMN; therefore the angle NAP may be taken as the sure of the wedge PAMN, or of the angle which is form the two planes MAP, MAN.

Scholium. The same relation subsists between the which are formed by two planes, as between those whic formed by two straight lines. Thus when two planes ally cross each other, the opposite or vertical angles are and the adjacent angles are together equal to two right a therefore if one plane be perpendicular to another, the is also perpendicular to the former. In like manner, whe parallel planes are met by a third plane, the same equ and the same properties appear, as when two parallel lin met by a third line.

PROPOSITION XXI. THEOREM.

If a solid angle is formed by three plane angles, the sum two of these angles will be greater than the third.



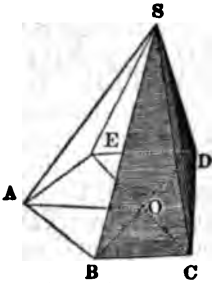
The proposition requires demonstration only when the plane angle, which is compared to the sum of the other two, is greater than either of them. Therefore suppose the solid angle S to be formed by three plane angles ASB, ASC, BSC, whereof the angle ASB is the greatest; we are to show that ASB<ASC+BSC.

In the plane ASB make the angle BSD=BSC, dra straight line ADB at pleasure; and having taken SC= draw AC, BC.

The two sides BS, SD, are equal to the two BS, SC angle BSD=BSC; therefore the triangles BSD, BSC equal; therefore BD=BC. But AB<AC+BC; takin from the one side, and from the other its equal BC, ther mains AD<AC. The two sides AS, SD, are equal to th AS, SC; the third side AD is less than the third side therefore the angle ASD<ASC (Prop. XII. Sch. B. I *Geom.*) Adding BSD=BSC, we shall have ASD+BS ASB<ASC+BSC.

PROPOSITION XXII. THEOREM.

The sum of the plane angles which form a solid angle is always less than four right angles.



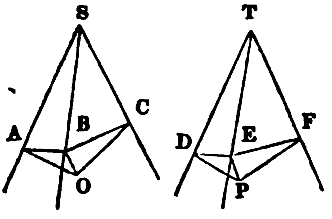
Cut the solid angle S by any plane ABCDE; from O, a point in that plane, draw to the several angles the straight lines AO, OB. OC, OD, OE.

The sum of the angles of the triangles ASB, BSC, &c. formed about the vertex S, is equal to the sum of the angles of an equal number of triangles AOB, BOC, &c. formed about the point O. But at the point B the sum of the angles ABO, OBC, equal to ABC, is less than the sum of the angles ABS, SBC, (Prop. XXI.); in the same manner at the point C we have $BCO+OCD<BCS+SCD$; and so with all the angles of the polygon ABCDE: whence it follows, that the sum of all the angles at the bases of the triangles whose vertex is in O, is less than the sum of the angles at the bases of the triangles whose vertex is in S; hence to make up the deficiency, the sum of the angles formed about the point O, is greater than the sum of the angles formed about the point S. But the sum of the angles about the point O is equal to four right angles (Prop. VI. B. II. Sch. *El. Geom.*); therefore the sum of the plane angles, which form the solid angle S, is less than four right angles.

Scholium. This demonstration is founded on the supposition that the solid angle is convex, or that the plane of no one surface produced can ever meet the solid angle; if it were otherwise, the sum of the plane angles would no longer be limited, and might be of any magnitude.

PROPOSITION XXIII. THEOREM.

If two solid angles are contained by three plane angles which are equal to each other, each to each, the planes of the equal angles will be equally inclined to each other.



Let the angle $ASC=DTF$, the angle $ASB=DTE$, and the angle $BSC=ETF$; then will the inclination of the planes ASC, ASB, be equal to that of the planes DTF, DTE.

Having taken SB at pleasure, draw BO perpendicular to the plane ASC; from the point O, at which the perpendicular meets the plane, draw OA, OC, perpendicular to SA, SC; draw AB, BC; next take TE=SB; draw EP perpendicular to the plane DTF; from the point P draw PD, PF, perpendicular respectively to TD, TF; lastly, draw DE, EF.

The triangle SAB is right angled at A, and the triangle TDE at D (Prop. VII.); and since the angle ASB=DTE we have SBA=TED. Likewise SB=TE; therefore the triangle SAB is equal to the triangle TDE; therefore SA=TD, and AB=DE. In like manner, it may be shown, that SC=TF, and BC=EF. That granted, the quadrilateral SAOC is equal to the quadrilateral TDPF; for, place the angle ASC upon its equal DTF; because SA=TD, and SC=TF, the point A will fall on D, and the point C on F; and at the same time, AO, which is perpendicular to SA, will fall on PD which is perpendicular to TD, and in like manner OC on PF; wherefore the point O will fall on the point P, and AO will be equal to DP. But the triangles AOB, DPE, are right angled at O and P; the hypothenuse AB=DE, and the side AO=DP; hence those triangles are equal (Prop. XIX. B. II. *El. Geom.*); and consequently, the angle OAB=PDE. The angle OAB is the inclination of the two planes ASB, ASC; and the angle PDE is that of the two planes DTE, DTF; hence those two inclinations are equal to each other.

It must, however, be observed, that the angle A of the right angled triangle AOB is properly the inclination of the two planes ASB, ASC, only when the perpendicular BO falls on the same side of SA, with SC; for if it fell on the other side, the angle of the two planes would be obtuse, and the obtuse angle together with the angle A of the triangle OAB would make two right angles. But in the same case, the angle of the two planes TDE, TDF, would also be obtuse, and the obtuse angle together with the angle D of the triangle DPE, would make two right angles; and the angle A being thus always equal to the angle at D, it would follow in the same manner that the inclination of the two planes ASB, ASC, must be equal to that of the two planes TDE, TDF.

Scholium. If two solid angles are contained by three plane angles, respectively equal to each other, and if at the same time the equal or homologous angles are *disposed in the same manner* in the two solid angles, these angles will be equal, and they will coincide when applied the one to the other. We have already seen that the quadrilateral SAOC may be placed upon its equal TDPF; thus placing SA upon TD, SC falls upon TF, and the point O upon the point P. But because the triangles

AOB, DPE, are equal, OB, perpendicular to the plane ASC, is equal to PE, perpendicular to the plane TDF; besides, those perpendiculars lie in the same direction; therefore, the point B will fall upon the point E, the line SB upon TE, and the two solid angles will wholly coincide.

This coincidence, however, takes place only when we suppose that the equal plane angles are *arranged in the same manner* in the two solid angles; for if they were *arranged in an inverse order*, or, what is the same, if the perpendiculars OB, PE, instead of lying in the same direction with regard to the planes ASC, DTF, lay in opposite directions, then it would be impossible to make these solid angles coincide with one another. It would not, however, on this account, be less true, as our Theorem states, that the planes containing the equal angles must still be equally inclined to each other; so that the two solid angles would be equal in all their constituent parts, without, however, admitting of superposition. This sort of equality, which is not absolute, or such as admits of superposition, deserves to be distinguished by a particular name: we shall call it *equality by symmetry*.

Thus those two solid angles, which are formed by three plane angles respectively equal to each other, but disposed in an inverse order, will be called *angles equal by symmetry*, or simply *symmetrical angles*.

The same remark is applicable to solid angles, which are formed by more than three plane angles: thus a solid angle, formed by the plane angles A, B, C, D, E, and another solid angle, formed by the same angles in an inverse order A, E, D, C, B, may be such that the planes which contain the equal angles are equally inclined to each other. Those two solid angles, are likewise equal, without being capable of superposition, and are called *solid angles equal by symmetry*, or *symmetrical solid angles*.

Among plane figures, equality by symmetry does not properly exist, all figures which might take this name being absolutely equal, or equal by superposition; the reason of which is, that a plane figure may be inverted, and the upper part taken indiscriminately for the under. This is not the case with solids; in which the third dimension may be taken in two different directions.

BOOK II.

SECTION I.—POLYEDRONS.

DEFINITIONS.

1. The name *solid polyedron*, or simple *polyedron*, is given to every solid terminated by planes or plane faces; which planes, it is evident, will themselves be terminated by straight lines.

Solids which have a certain number of faces, receive particular names: the solid which has four faces is named a *tetraedron;* that which has six, a *hexaedron;* that which has eight, an *octaedron;* that which has twelve, a *dodecaedron;* that which has twenty, an *icosaedron;* and so on.

The tetraedron is the simplest of all polyedrons; because at the least, three planes are required to form a solid angle, and these three planes leave a void, which cannot be closed without at least one other plane.

2. The common intersection of two adjacent faces of a polyedron is called the *side*, or *edge* of the polyedron.

3. A *regular polyedron* is one whose faces are all equal regular polygons, and whose solid angles are all equal to each other.

4. The *prism* is a solid bounded by several parallelograms, which are terminated at both ends by equal and parallel polygons. The solidity of a prism then, fills all the space between its equal and parallel bases.

To construct this solid, let ABCDE (see diagram to Prop. III.) be any polygon; then if in a plane parallel to ABCDE, the lines FG, GH, HI, &c. be drawn equal and parallel to the sides AB, BC, CD, &c. thus forming the polygon FGHIK equal to ABCDE; if in the next place, the vertices of the angles in the one plane be joined with the homologous vertices in the other, by straight lines, AF, BG, CH, &c., the faces ABGF, BCHG, &c. will be parallelograms, and ABCDEFGHIK, the solid so formed, will be a prism.

5. The equal and parallel polygons ABCDE, FGHIK (see diagram to Prop. III.) are called the *bases of the prism;* the parallelograms taken together constitute the *lateral* or *convex surface of the prism;* the equal straight lines AF, BG, CH, &c. are called the *sides of the prism.*

6. The *altitude of a prism* is the distance between its two bases, or the perpendicular drawn from a point in the upper base to the plane of the lower base.

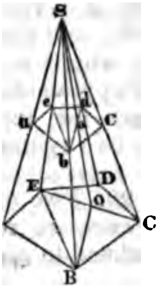
7. A *prism* is *right,* when the sides AF, BG, CH, &c. (see diagram to Prop. III.) are perpendicular to the planes of the bases; and then each of them is equal to the altitude of the the prism. In any other case the prism is *oblique,* and the altitude less than the side.

8. A *prism* is *triangular, quadrangular, pentagonal, hexagonal,* &c. when the base is a triangle, a quadrilateral, a pentagon, a hexagon, &c.

9. A prism whose base is a parallelogram, and which has all its faces parallelogram, is named a *parallelopipedon.* (See diagram to Prop. IV.)

The *parallelopipedon* is *rectangular* when its faces are rectangles.

10. Among rectangular parallelopipedons, we distinguish the *cube,* or regular hexaedron, bounded by six equal squares,



11. A *pyramid* is a solid contained by several triangular planes proceeding from the same point S, and terminating in the different sides of the same polygonal plane ABCDE.

The polygon ABCDE is called the *base* of the pyramid, the point S its *vertex;* and the triangles ASB, BSC, &c., form its *convex* or *lateral surface.*

The solidity of a pyramid, then, is equal to the space included between its base and a point called its vertex.

12. If a pyramid SABCDE, be cut by a plane *abcde* parallel to its base, that portion cut off by this plane and lying next to its base, is called a *frustum,* or *trunk of a pyramid.*

The solidity of a frustum of a pyramid, then, is equal to the space included between two similar but unequal parallel bases.

13. A pyramid is *triangular, quadrangular,* &c., according as its base is a triangle, a quadrilateral, &c.

14. A pyramid is *regular,* when its base is a regular polygon, and when, at the same time, the perpendicular let fall from the vertex to the plane of the base, passes through the

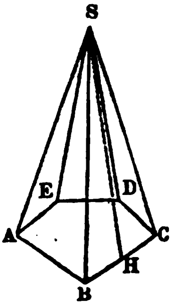
centre of this base. That perpendicular is then called the *axis* of the pyramid.

15. The *altitude* or *height* of a pyramid, is the perpendicular let fall from the vertex upon the plane of the base; as SO in the preceding figure. In a regular pyramid, it is the length of the axis.

16. The *slant-height* of a regular pyramid, is the distance from the vertex to the middle of one of the sides of the base; as SH.

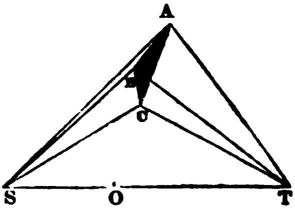
17. The *diagonal* of a polyedron is a straight line joining the vertices of two solid angles which are not adjacent to each other.

18. *Similar polyedrons* are such as are contained by the same number of similar planes, similarly situated, and having like inclinations to each other.



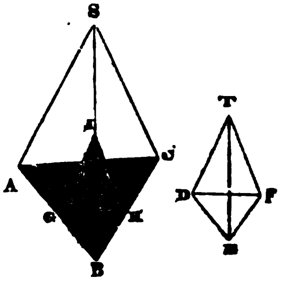
19. We shall give the name *symmetrical polyedrons*, to any two polyedrons which, having a common base, are constructed similarly, the one above this base, the other beneath it, and so that the vertices of their homologous solid angles are situated at equal distances from the plane of the base; and in the same straight line perpendicular to that plane.

If the straight line ST, for example, is perpendicular to the plane ABC, and also bisected at the point O, where it meets this plane, the two pyramids SABC, TABC, which have the common base ABC, will be two symmetrical polyedrons.



20. Two *triangular pyramids* are *similar*, when two faces in each are respectively similar, similarly placed, and equally inclined to each other.

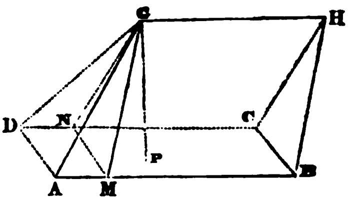
Thus supposing the angles ABC =DEF, BAC=EDF, ABS=DET, BAS=EDT, if the inclination of the planes ABS, ABC is likewise equal to that of their homologous planes DTE, DEF, the pyramids SABC, TDEF, will be similar.



21. If a triangle is formed by joining the vertices of three angles taken upon the same face, or on the base of a polye-

dron, then the vertices of the different solid angles of the polyedron, which are situated without this base, may be conceived as being the vertices of so many triangular pyramids having the triangle just described for a common base; and each of those pyramids will determine the position of a solid angle of the polyedron with reference to the base. Now, two *polyedrons* are *similar*, when having similar bases, the vertices of their corresponding solid angles lying without those bases, are determined by triangular pyramids which are similar each to each.

22. A *Wedge* is a solid of five sides, viz., a rectangular base, two rhomboidal sides meeting in an edge, and two triangular ends; as ABHG. The base is ABCD, the sides are ABHG and DCHG, meeting in the edge GH, and the ends are BCH and ADG. The *height* of the wedge is a perpendicular drawn from any point in the edge, to the plane of the base, as GP.



23. A *Prismoid* is a solid whose ends or bases are parallel, but not similar, and whose sides are quadrilateral. It differs from a prism or a frustum of a pyramid, in having its ends dissimilar. It is a *rectangular* prismoid when its ends are right parallelograms.

24. A *linear side* or *edge* of a solid, is the line of intersection of two of the planes which form the surface.

Note 1. By the *vertices* of a polyedron, we mean the points situated at the vertices of its different solid angles.

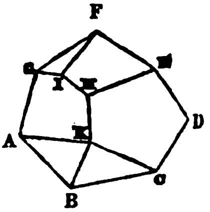
2. The only polyedrons we intend at present to treat of, are polyedrons with salient angles, or *convex* polyedrons. They are such that their surfaces cannot be intersected by a straight line in more than two points. In polyedrons of this kind, the plane of any face, when produced, can in no case cut the solid; the polyedron therefore cannot be in part above the plane of any face, and in part below it; it must lie wholly on the same side of this plane.

POPOSITION I. THEOREM.

Two polyedrons having the same vertices in number and position, will coincide.

For, suppose one polyedron to be already constructed; if a second is to be formed, having the same vertices, and in the same number, the planes of the latter must either not all pass through the same points with the planes of the former, or the two polyedrons will not differ from each other. But if those planes of the latter do not all pass through the same points with the planes of the former, some of them must cut the first polyedron; one or more of whose vertices must therefore lie above these planes, one or more below; which cannot be the case with a convex polyedron; hence, if two polyedrons have the same vertices, and in the same number, they must necessarily coincide with each other.

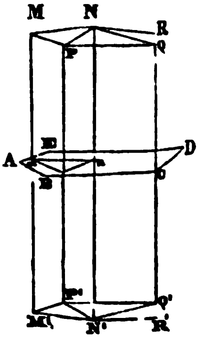
The points A, B, C, K, &c. which are to be the vertices of a polyedron, being given, it is easy to describe the polyedron.



First choose three adjacent points, D, E, H, such that the plane DEH shall pass, if need be, through the new points K, C, but leaving all the rest on the same side, all above the plane or all below it; the plane DEH or DEHKC, thus determined, will be one face of the solid. Through EH one of its sides, pass a plane, which turn round upon EH as an axis till it embraces a new vertex F, or several at once, as F, I; it will give a second face FEH or FEHI. Continue the same process, making planes to pass through the sides successively determined till the solid is bounded on all quarters: this solid will be the polyedron required, since there cannot be two which have the same vertices.

PROPOSITION II. THEOREM.

In two symmetrical polyedrons, the homologous faces are respectively equal, and the inclination of the two adjacent faces in one of those solids, is equal to the inclination of the two homologous faces in the other.



Let ABCDE be the common base of the two polyedrons; M and N the vertices of any two solid angles in the one, M′ and N′ the homologous vertices of the other: then (Def. 19) the straight lines MM′, NN′ must be perpendicular to the plane ABC, and be divided into two equal parts at the points *m* and *n*, where they meet it. Now we are to show that MN is equal to M′N′.

For, if the trapezoid *m*M′N′*n* be made to revolve about *mn* till the plane of it falls pon the plane *m*MN*n*; by reason of the right angles at *m* ıd *n*, the side *m*M′ will fall on its equal *m*M, and *n*N′ upon N; hence the trapezoids will coincide, and we shall have $N=M'N'$.

Let P be a third vertex of the upper polyedron, and P′ its omologous vertex in the other; we shall, as before, have $P=M'P'$, and $NP=N'P'$; hence *the triangle* MNP, *which ins any three vertices of the upper polyedron, is equal to the iangle* M′N′P′ *which joins the three corresponding vertices of e other polyedron.*

If among those triangles, we confine our attention to such s are formed at the surface of the polyedrons, we may aleady conclude that the surfaces of the two polyedrons are ach composed of the same number of triangles respectively qual in both.

It is now to be shown, that if any of those triangles lie on he same plane in the upper surface, and form one and the ame polygonal face, the corresponding triangles will lie on he same plane in the under surface, and there form an equal olygonal face.

To prove this, let MPN, NPQ, be two adjacent triangles upposed to lie on the same plane; and let M′P′N′, N′P′Q′, e their corresponding triangles. The angle $MNP=M'N'P'$, he angle $PNQ=P'N'Q'$; and if MQ and M′Q′ were joined, he triangle MNQ would be equal to M′N′Q′; hence we hould have the angle $MNQ=M'N'Q'$. But since MPNQ is ne single plane, the angle $MNQ=MNP+PNQ$; hence we hall likewise have $M'N'Q'=M'N'P'+P'N'Q'$. Now, if the hree planes M′N′P′, P′N′Q′, M′N′Q′ were not all in one lane, those three planes would form a solid angle, and (Prop. XI. B. I.) we should have the angle $M'N'Q'<M'N'P'+ N'Q'$; which conclusion not being true, the two triangles N′P′, P′N′Q′ are in one and the same plane.

Hence each face, whether triangular or polygonal, in the ne polyedron, corresponds to an equal face in the other polydron, and thus the two polyedrons are each included under he same number of planes respectively equal in both.

We have still to show, that the inclination of any two adjaent faces in the one polyedron is equal to the inclination of he two corresponding faces in the other.

Let MPN, NPQ be two triangles formed on the common dge NP, in the planes of two adjacent faces; let M′P′N′, ′P′Q′ correspond to them: conceive a solid angle to be formd at N, by the three plane angles MNQ, MNP, PNQ; and nother at N′, by the three M′N′Q′, M′N′P′, P′N′Q′. Now has been shown already, that those plane angles are respectively equal; hence the inclination of the two planes MNP PNQ is equal (Prop. XXIII. B. I.) to that of their corresponding planes M'N'P', P'N'Q'.

Therefore, in symmetrical polyedrons, the faces are equ each to each; and the planes of any two adjacent faces, in th one solid, have the same inclination as the planes of the tw corresponding faces in the other solid.

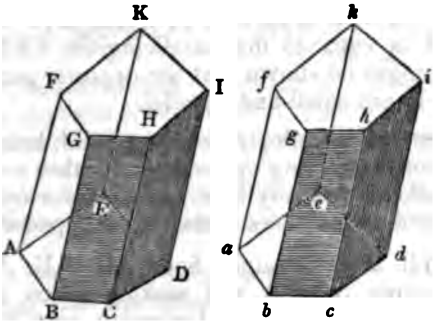
Scholium. It may be observed, that the *solid angles of th one polyedron are symmetrical with the solid angles of the ot er;* for as the solid angle of N is formed by the planes MN PNQ, QNR, &c., so its corresponding angle N' is formed b the planes M'N'P', P'N'Q', Q'N'R', &c. The latter appea to be arranged in the same order as the former; but since th two solid angles are in an inverse position with regard t each other, the real arrangement of the planes which form th solid angle N' must be the reverse of the arrangement whic occurs in the corresponding angle N. Farther, in both solid the inclination of the consecutive planes are respectively th same; hence the solid angles are symmetrical each with th other. (Prop. XXIII. *Sch.* B. I.)

The observation we have just made, proves, moreover, th *no polyedron can have more than one polyedron symmetric with it.* For if upon a second base, a new polyedron we constructed symmetrical with the given one, the solid angl of this new polyedron would still be symmetrical with thos of the given polyedron; hence they would be equal to thos of the symmetrical polyedron constructed on the first bas Besides, the homologous faces would still be equal: henc those two symmetrical polyedrons constructed on the first an on the second base, would have their faces equal and thei solid angles equal; hence they would coincide if applied t each other; hence they would form one and the same polyedron.

PROPOSITION III. THEOREM.

Two prisms are equal when the three planes containing a solid angle of the one, are respectively equal to the three planes containing a solid angle of the other, and are similarly situated.

Let the base ABCDE be equal to the base *abcde*, the parallelogram ABGF equal to the parallelogram *abgf*, and the parallelogram BCHG equal to *bchg;* then will the prism ABCI be equal to the prism *abci.*

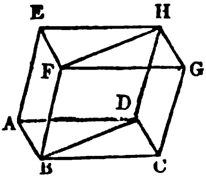


For, lay the base ABCDE upon its equal *abcde;* these two bases will coincide. But the three plane angles which form the solid angle B, are respectively equal to the three plane angles, which form the solid angle *b*, namely, ABC=*abc*, ABG=*abg*, and GBC=*gbc;* they are also similarly situated: hence the solid angles B and *b* are equal, and therefore the side BG will fall on its equal *bg*. It is likewise evident, that by reason of the equal parallelograms ABGF, *abgf*, the side GF will fall on its equal *gf*, and in the same manner GH on *gh;* hence the upper base FGHIK will exactly coincide with its equal *fghik*, and the two solids will be identical (Prop. 1) since their vertices are the same.

Cor. Two right prisms, which have equal bases and equal altitudes, are equal. For, since the side AB is equal to *ab*, and the altitude BG to *bg*, the rectangle ABGF will be equal to *abgf;* so also will the rectangle BGHC be equal to *bghc;* and thus the three planes, which form the solid angle B, will be equal to the three which form the solid angle *b*. Hence the two prisms are equal.

PROPOSITION IV. THEOREM.

In every parallelopipedon, the opposite planes are equal and parallel.



By the definition of this solid, the bases ABCD, EFGH are equal parallelograms, and their sides are parallel: it remains only to show that the same is true of any two opposite lateral faces, such as AEHD, BFGC. Now AD is equal and parallel to BC, because the figure ABCD is a parallelogram; for a like reason, AE is parallel to BF: hence (Prop. XIV. B. I.) the angle DAE is equal to the angle CBF, and the planes DAE, CBF are parallel; hence also the parallelogram DAEH is equal to the parallelogram CBFG. In th same way, it might be shown that the opposite parallelogram ABFE, DCGH are equal and parallel.

Cor. Since the parallopipedon is a solid, bounded by si planes, whereof those lying opposite each other are equal an parallel, it follows that any face and the one opposite to i may be assumed as the bases of the parallopipedon.

Scholium. If three straight lines AB, AE, AD, passin through the same point A, and making given angles wit each other, are known, a parallelopipedon may be formed o those lines. For this purpose, a plane must be passed throug the extremity of each line, and parallel to the plane of th other two; that is, through the point B a plane parallel t DAE, through D a plane parallel to BAE, and through E plane parallel to BAD. The mutual intersections of thos planes will form the parallelopipedon required.

PROPOSITION V. THEOREM.

In every parallelopipedon, the opposite solid angles are symmetrical; and the diagonals drawn through the vertices o those angles bisect each other.

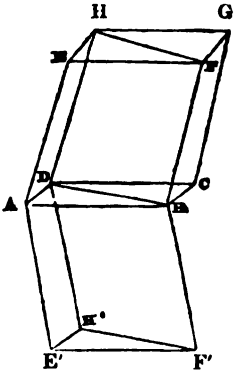
First. Compare the solid angle A (see last diagram) wit its opposite one G. The angle EAB, equal to EFB, is als equal to HGC; the angle DAE=DHE=CGF; and the angle DAB=DCB=HGF; therefore the three plain angles which form the solid angle A, are equal to the three which form the solid angle G, each to each. It is easy, moreover, to see that their arrangement in the one is different from their arrangement in the other: hence (Prop. XXIII. Sch. B. I.) the two solid angles A and G are symmetrical.

Secondly. Imagine two diagonals EC, AG to be drawn both through opposite vertices: since AE is equal and parallel to CG, the figure AEGC is a parallelogram; hence the diagonals EC, AG will mutually bisect each other. In the same manner, we could show that the diagonal EC and another DF bisect each other; hence the four diagonals will mutually bisect each other, in a point which may be regarded as the centre of the parallelopipedon.

PROPOSITION VI. THEOREM.

If a plane be passed through the two edges of a parallelopipedon diagonally opposite to each other, it will divide the parallelopipedon into two symmetrical triangular prisms.

Let the plane BDHF be passed through the edges DH, BF of the parallelopipedon AG; the prisms ABDE, DBCG are symmetrical.



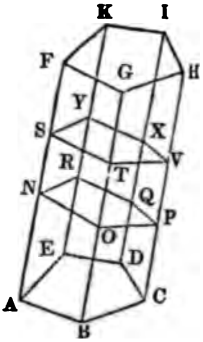
In the first place, those solids are evidently prisms; for the triangles ABD, EFH, having their sides equal and parallel are equal; also the lateral faces ABFE, ADHE, BDHF are parallelograms; hence the solid ABDHEF is a prism; so likewise is GHFBCD. We are now to show that those prisms are symmetrical.

On the base ABD, construct the prism ABDE'F'H' such that it be symmetrical with the prism ABDEFH. According to what has been already proved (Prop. II.). the plane ABF'E' must be equal to ABFE, and the plane ADH'E' to ADHE: but comparing the prism GHFBCD with the prism ABDH'E'F', we find the base GHF=ABD; the parallelogram GHDC, which is equal to ABFE, also equal to ABF'E'; and the parallelogram GFBC, which is equal to ADHE, also equal ADH'E'; therefore the three planes which form the solid angle G in the prism GHFBCD are equal to the three planes which form the solid angle A in the prism ABDHE'F', each to each; and they are similarly arranged: hence those two prisms are equal, and might be made to coincide. But one of them ABDH'E'F' is symmetrical with the prism ABDHEF; hence the other GHFBCD is also symmetrical with that prism.

PROPOSITION VII. THEOREM.

In every prism, the sections formed by parallel planes, are equa polygons.

Let the prism AH be intersected by the parallel planes NP. SV; then are the polygons NOPQR, STVXY equal.

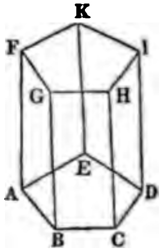


For the sides ST, NO, are parallel, being the instructions of two parallel planes with a third plane ABGF; those same sides, ST, NO, are included between the parallel NS, OT, which are sides of the prism: hence NO is equal to ST. For like reasons, the sides OP, PQ, QR, &c. of the section NOPQR, are respectively equal to the sides TV, VX, XY, &c. of the section STVXY. And since the equal sides are at the same time parallel, it follows that the angles NOP, OPQ, &c. of the first section are respectively equal to the angles STV, TVX of the second (Prop. XIV. B. I.) Hence the two sections NOPQR, STVXY are equal polygons.

Cor. Every section is a prism, if drawn parallel to the base, is also equal to that base.

PROPOSITION VIII. THEOREM.

The convex surface of a right prism is equal to the perimeter of its base multiplied by its altitude.



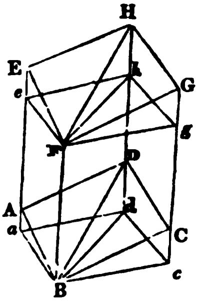
Let ABCDE-K be a right prism: then will its convex surface be equal to (AB+BC+CD+DE+EA)×AF.

For, the convex surface is equal to the sum of all the rectangles AG, BH, CI, DK, EF, which compos it. Now, the altitudes AF, BG, CH, &c. of the rectangles, are equal to the altitude of the prism. Hence, the sum of these rectangles, or the convex surface of the prism, is equal to (AB+BC+CD+DE+EA)×AF; that is, to the perimeter of the base of the prism multiplied by its altitude.

Cor. If two right prisms have the same altitude, their convex surfaces will be to each other as the perimeters of their bases.

PROPOSITION IX. THEOREM.

ymmetrical triangular prisms into which a parallelo- n is divided by a plane passing through the opposite al edges, are equivalent.



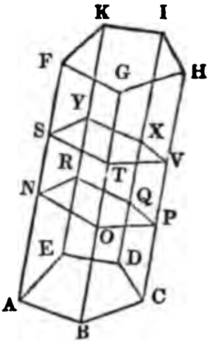
gh the vertices B and F, draw the *dc*, F*ehg* at right angles to the the former meeting AE, DH, CG other sides of the parallelopipe- e points *a*, *d*, *c* the latter in *e*, *h*, ections B*adc*, F*ehg* will be equal grams. They are equal (Prop. ause they are formed by planes ular to the same straight line and ntly parallel; they are parallelo- ecause *a*B, *dc*, two opposite sides me section, are formed by the meeting of one plane parallel planes ABFE, DCGH.

like reason, the figure B*ae*F is a parallelogram; so BF*gc*, *cdhg*, *adhe*, the other lateral faces of the solid *g*; hence that solid is a prism (Def. 4.); and that right, because the side BF is perpendicular to its

eing proved, if the right prism B*h* is divided, by the FHD, into two right triangular prisms *a*B*de*F*h*, ; we are now to show that the oblique triangular BDEFH will be equivalent to the right triangular B*de*F*h*. And since those two prisms have a part in common, it will only be requisite to prove that aining parts, namely, the solids B*a*AD*d*, F*e*EH*h* are nt.

by reason of the parallelograms ABFE, *a*BF*e*, the *ae*, being equal to their parallel BF, are equal to er; and taking away the common part A*e* there re- *a*=E*e*. In the same manner we could prove D*d*=H*h*. to bring about the superposition of the two solids F*e*EH*h*, let us place the base F*eh* on its equal B*ad*; t *e* falling on *a*, and the point *h* on *d*, the sides *e*E, *h*H on their equals *a*A, *d*D, because they are perpendicu- e same plane B*ad*. Hence the two solids in question cide exactly with each other; hence the oblique prism H is equivalent to the right one B*ad*F*eh*.

same manner might the oblique prism BDCFHG be quivalent to the right prism B*dc*F*hg*. But (Prop. III. two right prisms B*ad*F*eh*, B*dc*F*hg* are equal, since

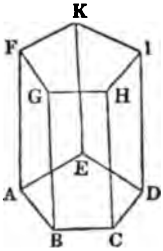


For the sides ST, NO, are parallel, being the instructions of two parallel planes with a third plane ABGF; those same sides, ST, NO, are included between the parallel NS, OT, which are sides of the prism: hence NO is equal to ST. For like reasons, the sides OP, PQ, QR, &c. of the section NOPQR, are respectively equal to the sides TV, VX, XY, &c. of the section STVXY. And since the equal sides are at the same time parallel, it follows that the angles NOP, OPQ, &c. of the first section are respectively equal to the angles STV, TVX of the second (Prop. XIV. B. I.) Hence the two sections NOPQR, STVXY are equal polygons.

Cor. Every section is a prism, if drawn parallel to the base, is also equal to that base.

PROPOSITION VIII. THEOREM.

The convex surface of a right prism is equal to the perimeter of its base multiplied by its altitude.



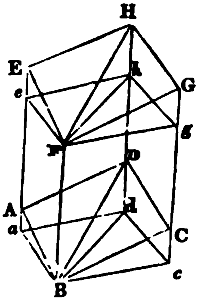
Let ABCDE-K be a right prism: then will its convex surface be equal to (AB+BC+CD+DE+EA)×AF.

For, the convex surface is equal to the sum of all the rectangles AG, BH, CI, DK, EF, which compos it. Now, the altitudes AF, BG, CH, &c. of the rectangles, are equal to the altitude of the prism. Hence, the sum of these rectangles, or the convex surface of the prism, is equal to (AB+BC+CD+DE+EA)×AF; that is, to the perimeter of the base of the prism multiplied by its altitude.

Cor. If two right prisms have the same altitude, their convex surfaces will be to each other as the perimeters of their bases.

PROPOSITION IX. THEOREM.

The two symmetrical triangular prisms into which a parallelopipedon is divided by a plane passing through the opposite diagonal edges, are equivalent.



Through the vertices B and F, draw the planes B*adc*, F*ehg* at right angles to the side BF, the former meeting AE, DH, CG the three other sides of the parallelopipedon, in the points *a*, *d*, *c* the latter in *e*, *h*, *g*; the sections B*adc*, F*ehg* will be equal parallelograms. They are equal (Prop. VII.) because they are formed by planes perpendicular to the same straight line and consequently parallel; they are parallelograms, because *a*B, *dc*, two opposite sides of the same section, are formed by the meeting of one plane with two parallel planes ABFE, DCGH.

For a like reason, the figure B*ae*F is a parallelogram; so also are BF*gc*, *cdhg*, *adhe*, the other lateral faces of the solid B*adc*F*ehg*; hence that solid is a prism (Def. 4.); and that prism is right, because the side BF is perpendicular to its base.

This being proved, if the right prism B*h* is divided, by the plane BFHD, into two right triangular prisms *a*B*de*F*h*, B*dc*F*hg*; we are now to show that the oblique triangular prism ABDEFH will be equivalent to the right triangular prism *a*B*de*F*h*. And since those two prisms have a part ABD*he*F in common, it will only be requisite to prove that the remaining parts, namely, the solids B*a*AD*d*, F*e*EH*h* are equivalent.

Now, by reason of the parallelograms ABFE, *a*BF*e*, the sides AE, *ae*, being equal to their parallel BF, are equal to each other; and taking away the common part A*e* there remains A*a*=E*e*. In the same manner we could prove D*d*=H*h*.

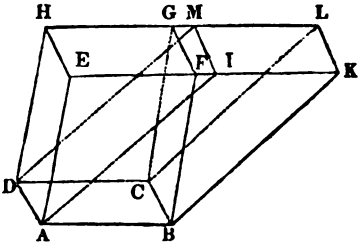
Next, to bring about the superposition of the two solids B*a*AD*d*, F*e*EH*h*, let us place the base F*eh* on its equal B*ad*; the point *e* falling on *a*, and the point *h* on *d*, the sides *e*E, *h*H will fall on their equals *a*A, *d*D, because they are perpendicular to the same plane B*ad*. Hence the two solids in question will coincide exactly with each other; hence the oblique prism BADFEH is equivalent to the right one B*ad*F*eh*.

In the same manner might the oblique prism BDCFHG be proved equivalent to the right prism B*dc*F*hg*. But (Prop. III. Cor.) the two right prisms B*ad*F*eh*, B*dc*F*hg* are equal, since they have the same altitude BF, and since their bases B*ad*, B*dc*, are halves of the same parallelogram. Hence the two triangular prisms BADFEH, BDCFHG, being equivalent to the equal right prisms, are equivalent to each other.

Cor. Every triangular prism ABDHEF is half of the parallelopipedon AG described with the same solid angle A, and the same edges AB, AD, AE.

PROPOSITION X. THEOREM.

If two parallelopipedons have a common base, and their upper bases in the same plane and between the same parallels, they will be equivalent.



Let the parallelopipedons AG, AL, have the common base AC, and their upper bases EG, MK in the same plane, and between the same parallels HL, EK; then will they be equivalent.

There may be three cases, according as EI is greater, less than, or equal to, EF; but the demonstration is the same for all. In the first place, then we shall show that the triangular prism AEIDHM is equal to the triangular prism BFKCGL.

Since AE is parallel to BF, and HE to GF, the angle AEI =BFK, HEI=GFK, and HEA=GFB. Of these six plane angles the first three form the solid angle E, the last three the solid angle F; therefore, the plane angles being respectively equal, and similarly arranged, the solid angles F and E must be equal. Now, if the prism AEM is laid on the prism BFL, the base AEI being placed on the base BFK will coincide with it because they are equal; and since the solid angle E is equal to the solid angle F, the side EH will fall on its equal FG: and nothing more is required to prove the coincidence of the two prisms throughout their whole extent, for (Prop. III.) the base AEI and the edge EH determine the prism AEM, as the base BFK and the edge FG determine the prism BFL; hence these prisms are equal.

But if the prism AEM is taken away from the solid AL, there will remain the parallelopipedon AIL; and if the prism BFL is taken away from the same solid, there will remain the parallelopipedon AEG; hence those two parallelopepidons AIL, AEG, are equivalent.

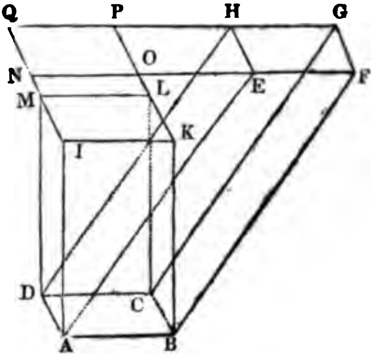
PROPOSITION XI. THEOREM.

*llelopipedons having the same base and the same alti-
tude, are equivalent.*

CD (see diagram to Prop. XII) be the common base
o parallelopipedons AG, AL; since they have the
ıde, their upper bases EFGH, IKLM will be in the
e. Also the sides EF and AB will be equal and
, well as IK and AB; hence EF is equal and par-
; for a like reason, GF is equal and parallel to LK.
es EF, HG be produced, and likewise LK, IM, till
ntersections they form the parallelogram NOPQ;
elogram will evidently be equal to either of the
ЗH, IKLM. Now if a third parallelopipedon be
having for its lower base the same ABCD, and
its upper, the third parallelopipedon will (Prop. X.)
:o the parallelopipedon AG, since with the same
, their upper base lie in the same plane and be-
same parallels, GQ, FN. For the same reason, this
lelopipedon will also be equivalent to the parallelo-
L; hence the two parallelopipedons AG, AL, which
ıme base and the same altitude, are equivalent.

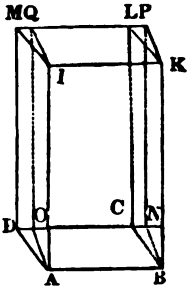
PROPOSITION XII. THEOREM.

*lopipedon may be changed into an equivalent rectan-
ırallelopipedon having the same altitude and an
nt base.*



be the parallelo-
oposed. From the
3, C, D, draw AI,
ıM, perpendicular
e of the base; you
ırm the parallelo-
.L equivalent to
having its lateral
BL, &c. rectan-
nce if the base
a rectangle, AL
:ectangular paral-
ı equivalent to AG,
uently the parallelopipedon required. But if ABCD

is not a rectangle, draw AO and BN perpendicular to CD, and OP and NP perpendicular to the base; you will then have the solid ABNOIKPQ, which will be a rectangular parallelopipedon: for by construction, the bases ABNO, and IKPQ are rectangles; so also are the lateral faces, the edges AI, OQ, &c. being perpendicular to the plane of the base; hence the solid AP is a rectangular parallelopipedon. But the two parallelopipedons AP, AL may be conceived as having the same base ABKI and the same altitude AO; hence the parallelopipedon AG, which was at first changed into an equivalent parallelopidedon AL, is again changed into an equivalent rectangular parallelopipedon AP, having the same altitude AI, and a base ABNO equivalent to the base ABCD.

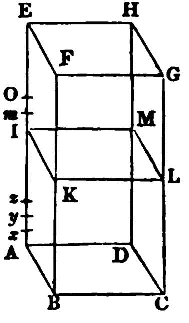


PROPOSITION XIII. THEOREM.

Two rectangular parallelopipedons, which have the same base, are to each other as their altitudes.

Let the parallelopipedons AG, AL have the same base BD; then will they be to each other as their altitudes AE, AI.

First, suppose the altitudes AE, AI, to be to each other, as two whole numbers, as 15 is to 8, for example. Divide AE into 15 equal parts; whereof AI will contain 8; and through x, y, z, &c. the points of division, draw planes parallel to the base. These planes will cut the solid AG into 15 partial parallelopipedons, all equal to each other, because they have equal bases and equal altitudes—equal bases, since (Prop. VII.) every section MIKL, made parallel to the base ABCD of a prism, is equal to that base—equal altitudes, because these altitudes are equal divisions Ax, xy, yz, &c. But of those 15 equal parallelopipedons, 8 are contained in AL; hence the solid AG is to the solid AL as 15 is to 8, or generally, as the altitude AE is to the altitude AI.

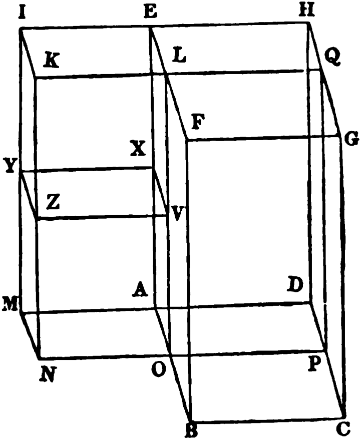


Again, if the ratio of AE to AI cannot be expressed in numbers, it is to be shown, that notwithstanding, we shall have *solid.* AG : *solid.* AL : : AE : AI. For, if this proportion is not correct, suppose we have *sol.* AG : *sol.* AL : : AE : AO greater than AI. Divide AE into equal parts, such that each shall be less than OI; there will be at least one point of division *m*, between O and I. Let P be the parallelopipedon, whose base is ABCD, and altitude A*m*; since the altitudes AE, A*m* are to each other as the two whole numbers, we shall have *sol.* AG : P : : AE : A*m*. But by hypothesis, we have *sol.* AG : *sol.* AL : : AE : AO; therefore *sol.* AL : P : : AO : A*m*. But AO is greater than A*m*; hence if the proportion is correct, the solid AL must be greater than P. On the contrary. however, it is less; hence the fourth term of this proportion *sol.* AG : *sol.* AL : : AE : *x*, cannot possibly be a line greater than AI. By the same mode of reasoning, it might be shown that the fourth term cannot be less than AI; therefore it is equal to AI; hence rectangular parallelopipedon having the same base are to each other as altitudes.

PROPOSITION XIV. THEOREM.

Two rectangular parallelopipedons, having the same altitude, are to each other as their base.

Let the parallelopipedons AG, AK have the same altitude AE; then will they be to each other as their bases AC, AN.



Having placed the two solids by the side of each other, as the figure represents, produce the plane ONKL till it meets the plane DCGH in PQ; you will thus have a third parallelopipedon AQ, which may be compared with each of the parallelopipedons AG, AK. The two solids AG, AQ, having the same base AEHD are to each other as their altitudes AB, AO; in like manner, the two solids AQ, AK, having the same base AOLE, are to each other as their altitudes AD, AM. Hence we have the two proportions.

$$sol.\ AG : sol.\ AQ : : AB : AO,$$
$$sol.\ AQ : sol.\ AK : : AD : AM.$$

Multiplying together the corresponding terms of those proportions, and omitting in the result the common multiplier *sol.* AQ; we shall have

$$sol.\ AG : sol.\ AK : : AB \times AD : AO \times AM$$

But AB×AD represents the base ABCD; and AO×AM represents the base AMNO; hence two rectangular parallelopidons of the same altitude are to each other as their bases.

PROPOSITION XV. THEOREM.

Any two rectangular parallelopipedons are to each other as the products of their bases by their altitudes, that is to say, as the products of their three dimensions.

For, having placed the two solids AG, AZ, (see diagram to Prop. XIV.), so that their surfaces have the common angle BAE, produce the planes necessary for completing the third parallelopipedon AK having the same altitude with the parallelopipedon AG. By the last proposition, we shall have

sol. AG : *sol.* AK : : ABCD : AMNO.

But the two parallelopipedons AK, AZ, having the same base AMNO, are to each other as their altitudes AE, AX; hence we have

sol. AK : *sol.* AZ : : AE : AX.

Multiplying together the corresponding terms of these proportions, and omitting in the result the common multiplier *sol.* AK; we shall have

sol. AG : *sol.* AZ : : ABCD×AE : AMNO×AX.

Instead of the bases ABCD and AMNO, put AB×AD and AO×AM it will give

sol. AG : *sol.* AZ : : AB×AD×AE : AO×AM×AX.

Hence any two rectangular parallelopipidons are to each other, &c.

Scholium. We are consequently authorized to assume, as the measure of a rectangular parallelopipedon, the product of its base by its altitude, in other words, the product of its three dimensions.

In order to comprehend the nature of this measurement, it is necessary to reflect, that the number of linear units in one dimension of the base multiplied by the number of linear units in the other dimension of the base, will give the number of superficial units in the base of the parallelopipedon (Prop. B. IV. Sch. *El. Geom.*) For each unit in height there are evidently as many solid units as there are superficial units in the base. Therefore, the number of superficial units in the base multiplied by the number of linear units in the altitude, gives the number of solid units in the parallelopipedon.

If the three dimensions of another parallelopipedon are valued according to the same linear unit, and multiplied together in the same manner, the two products will be to each other as the solids, and will serve to express their relative magnitude.

The magnitude of a solid, its volume or extent, forms what is called its *solidity ;* and this word is exclusively employed to designate the measure of a solid: thus we say the solidity of a rectangular parallelopipedon is equal to the product of its base by its altitude, or to the product of its three dimensions.

As the cube has all its three dimensions equal, if the side is 1, the solidity will be $1\times1\times1=1$: if the side is 2, the silidity will be $2\times2\times2=8$; if the side is 3, the solidity will be $3\times3\times3=27$; and so on: hence, if the sides of a series of cubes are to each other as the numbers 1, 2, 3, &c., the cubes themselves or their solidities will be as the numbers 1, 8, 27, &c. Hence it is, that in arithmetic, the *cube* of a number is the name given to a product which results from three factors, each equal to this number.

If it were proposed to find a cube double of a given cube, the side of the required cube would have to be to that of the given one, as the cube-root of 2 is to unity. Now, by a geometrical construction, it is easy to find the square root of 2; but the cube-root of it cannot be so found, at least not by the simple operations of elementary geometry, which consists in employing nothing but straight lines, two points of which are known, and circles whose centres and radii are determined.

Owing to this difficulty the problem of the *duplication of the cube* became celebrated among the ancient geometers, as well as that of the *trisection of an angle,* which is nearly of the same species. The solutions of which such problems are susceptible, have however long since been discovered; and though less simple than the constructions of elementary geometry, they are not, on that account, less rigorous or less satisfactory.

PROPOSITION XVI. THEOREM.

The solidity of a parallelopipedon, and generally of any prism, is equal to the product of its base by its altitude.

For, in the first place, any parallelopipedon is equivalent to a rectangular parallelopipedon, having the same altitude and an equivalent base (Prop. XII.) Now the solidity of the latter is equal to its base multiplied by its height; hence the solidity of the former is, in like manner, equal to the product of its base by its altitude.

In the second place, any triangular prism is half of the parallelopipedon so constructed as to have the same altitude and a double base (Prop. IX.) But the solidity of the latter is equal to its base multiplied by its altitude; hence that of a triangular prism is also equal to the product of its base, which is half that of the parallelopipedon, multiplied into its altitude.

In the third place, any prism may be divided into as many triangular prisms of the same altitude, as there are triangles capable of being formed in the polygon which constitutes its base. But the solidity of each triangular prism is equal to its base multiplied by its altitude; and since the altitude is the same for all, it follows that the sum of all the partial prisms must be equal to the sum of all the partial triangles, which constitute their bases, multiplied by the common altitude.

Hence the solidity of any polygonal prism, is equal to the product of its base by its altitude.

Cor. 1. Comparing two prisms, which have the same altitude, the products of their bases by their altitudes will be as the bases simply; hence *two prisms of the same altitude are to each other as their bases.* For a like reason, *two prisms of the same base are to each other as their altitudes.* And when neither bases nor their altitudes are equal, their solidities will be to each other as the products of their bases and altitudes.

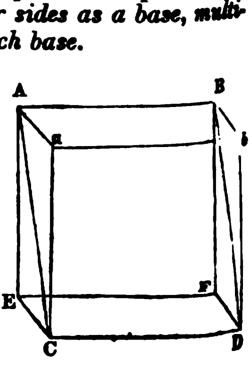
Cor. 2. As either side of a rectangular parallelopipedon may be taken as a base, it follows, that its solidity is equal to the surface of either side as its base multiplied by the vertical height on that base.

PROPOSITION XVII. THEOREM.

The solidity of a triangular prism is equal to the product of the surface of either of its rectangular sides as a base, multiplied by half its vertical height on such base.

Let ACEBFD, be a triangular prism, and let ECDF be its base, and EA its vertical height; then will the product of the surface ECDF into half EA=the solidity of the prism ACEBDF.

For, draw the lines Aa, aC, and Bb, bD, completing the parallelograms AaCE and BbFD, join ab, and we have the quadrangular prism A$a$ECB$b$FD,



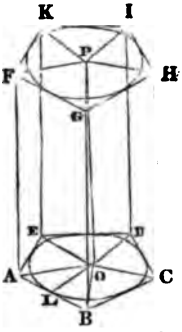
solidity (Prop. XVI. *Cor.* 2) is equal to the base ECDF ied by its vertical height EA, or its solidity is equal to e A*a*CE multiplied by the vertical height AB. But the lar base AEC of the triangular prism, is equal to half e A*a*EC of the parallelopipedon, (Prop. IV. B. IV., *El.*

Hence. if these solids are taken on equal bases, viz., base ECDF, which is common to both, the triangular eing equal to half the quadrangular prism, is equal to se multiplied by half the vertical height EA.

lium. It may be observed, that when one of the rec- r sides of a triangular prism is taken as a base, the generally called a wedge: the wedge, then, is nothing t from a triangular prism, except as its ends may be arallel or inclined toward each other.

PROPOSITION XVIII. THEOREM.

prism whose base is a regular polygon, may be divided as many equal symmetrical prisms as the polygon of the has sides, each of which is equal to the rectangle of one e vertical planes of the prism, with half of the radius of ircle inscribed in the base.



FC be a pentagonal prism, then may id be divided into as many symmetri- ngular prisms AOEFKP, EODKPI, ual, each to each, as the polygon E of the base has sides, each equal to the vertical faces AEFK, multiplied the radius OL of the inscribed circle. conceive vertical planes, to be passed h the angles of the prism, reaching to tres of the circles inscribed in the two these planes will evidently divide the the ratio of their bases, (Prop. XIV.,) base will be divided into triangles, equal each to each, XI. B. V. *El. Geom.*) and each of these triangular is symmetrical with those with which it is in contact; two contiguous prisms, as AOBFPG, and AOEFPK, e referred to the plane FAOP as their base, and triangles forming the ends of the several prisms are and similar, (Prop. XI. B. V. *El. Geom.*) their angles e base AO or FP are equal and similar, each to each, e angle BAO=the angle EAO; the angle GFP=the KFP, hence the angles which the planes ABGF and

AEKF make with the plane AOPF are equal; hence, also, the other planes KPEO and GPBO form angles with the base AOFP which are similar, each to each; therefore, each of the contiguous triangular prisms are symmetrical, (see Definition 19.) And because the altitude of the triangle forming the end of each of these prisms is equal to the radius of the circle inscribed in the base of the polygon, (Proposition XI. B. V. *El. Geom.*) their solidities are equal, each, to one of the vertical planes of the polygon multiplied by half its radius. (Prop. XVII.)

Cor. 1. Hence the solidity of any prism, whose base is a regular polygon, is equal to the convex surface multiplied by half the radius of the circle inscribed in the base.

Scholium 1. As all the triangular prisms composing any polygonal prism, are equal, each to each, they may all be regarded as symmetrical with each other. And as two similar triangular prisms are said to be symmetrical when a homologous side of each is placed in contact, (Def. 19.) I shall call a series of these associated equal triangular prisms, symmetrical by association.

Cor. 2. From the foregoing demonstration, and by reference to Prop. XI., *Sch.*, B. V. *El. Geom.*, it will appear that every polygonal prism whatever, whether regular or irregular, may be divided into triangular prisms, the sum of whose solidities shall be equal to the solidity of the polygonal prism.

Scholium 2. As a triangular prism is the most simple of all prisms into which a polygonal prism can be divided, it may, therefore, be regarded as the element of all polygonal prisms; for, from triangular prisms all others may be formed, and triangular prisms may be divided indefinitely, by dividing their rectangular bases by lines parallel to one of its sides, and passing planes perpendicular to the bases through such divisions, and through the vertical plane angles; and these prisms, so divided, will be the elementary prisms of the prism so divided.

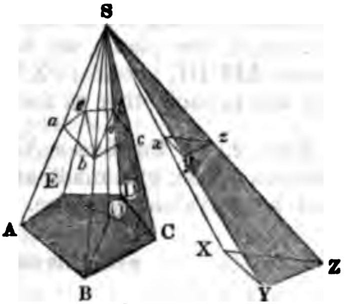
PROPOSITION XIX. THEOREM.

If a pyramid is cut by a plane parallel to its base,

First, *The edges and the altitudes will be divided proportionally.*

Secondly, *The section will be a polygon similar to the base.*

And thirdly, *These two surfaces will be to each other as the square of their distances from their vertex.*



Let the pyramid SABCDE, of which SO is the altitude, be cut by the plane *abcde;* then will $Sa : SA :: So : SO$, and the same for the other edges; and the polygon *abcde*, will be similar to the base ABCDE.

First. Since the planes ABC *abc* are parallel, their intersections AB, *ab*, by a third plane SAB will also be parallel (Prop. XI. B. I.); hence the triangles SAB, S*ab* are similar, and we have $SA : Sa :: SB : Sb$; we might also have $SB : Sb :: SC : Sc$; and so on. Hence all the edges SA, SB, SC, &c. are cut proportionally in *a*, *b*, *c*, &c. The altitude SO is likewise cut in the same proportion at the point *o*; for BO and *bo* are parallel, therefore we have $SO : So :: SB : Sb$.

Secondly. Since *ab* is parallel to AB, *bc* to BC, *cd* to CD, &c. the angle *abc* is equal to ABC, the angle *bcd* to BCD, and so on. Also, by reason of the similar triangles SAB, S*ab*, we have $AB : ab :: SB : Sb$; and by reason of the similar triangles SBC, S*bc*, we have $SB : Sb :: BC : Bc$; hence $AB : ab :: BC : bc$; we might likewise have $BC : bc :: CD : cd$, and so on. Hence the polygons ABCDE, *abcde*, have their angles respectively equal and their homologous sides proportional; hence they are similar.

Thirdly. Similar planes being to each other as the squares of their like sides (Prop.XXIV. Cor. C. B. IV.*El. Ge.*) the plane $ac : AC :: ab^2 : AB^2$ or $:: Sa^2\ SA^2$, since we have shown that $ab : AB$ and $Sa : SA$ have the same ratio. Also, the two triangles SOB, S*ob*, having the angles O and *o* right angles (Prop. V. B. I.) and the angle S common are eqiangular, and have therefore their like sides proportional, viz. $SB : Sb :: SO : So$, or $SB^2 : Sb^2 :: SO^2 : So^2$. Consequently the two planes AC, *ac*, which are as the former squares AB^2, ab^2, will be also as the latter squares SO^2, So^2, that is, *area* AC : *area* $ac :: SO^2, : So^2$.

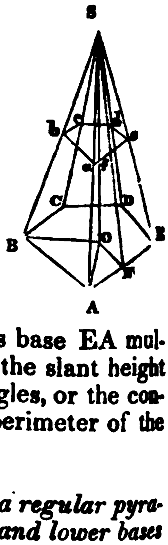
Cor. 1. Let SABCDE, SXYZ be two pyramids, having a common vertex and the same altitude, or having their bases situated in the same plane; if those pyramids are cut by a plane parallel to the plane of their bases, and the sections *abcde*, *xyz* result from it, *then will the sections abcde, xyz be to each other as the bases* ABCDE, XYZ.

For, the polygons ABCDE, *abcde* being similar, their surfaces are as the squares of the homologous sides AB, *ab*; but AB : *ab* : : SA : S*a*; hence ABCDE : *abcde* : : $SA^2 : Sa^2$. For the same reason, XYZ : *xyz* : : $SX^2 : Sx^2$. But since *abc* and *xyz* are in one plane, we have likewise SA : S*a* : : SX : S*x*; hence ABCDE : *abcde* : : XYZ : *xyz*; hence the sections *abcde*, *xyz* are to each other as the bases ABCDE, XYZ.

Cor. 2. If the bases ABCDE, XYZ are equivalent, any sections *abcde*, *xyz*, made at equal distances from those bases, will be eqivalent likewise.

PROPOSITION XX. THEOREM.

The convex surface of a regular pyramid is equal to the perimeter of its base multiplied by half its slant height.



For, since the pyramid is regular, the point O, in which axis meets the base, is the centre of the polygon ABCDE (Def. 14.); hence the lines OA, OB, OC, &c. drawn to the vertices of the base, are equal.

In the right angled triangles SAO, SBO, the bases and perpendiculars are equal: since the hypothenuses are equal: and it may be proved in the same way that all the sides of the right pyramid are equal. The triangles, therefore, which form the convex surface of the prism are all equal to each other. But the area of either of these triangles, as ESA, is equal to its base EA multiplied by half the perpendicular SF, which is the slant height of the pyramid: hence the area of all the triangles, or the convex surface of the pyramid, is equal to the perimeter of the base multiplied by half the slant height.

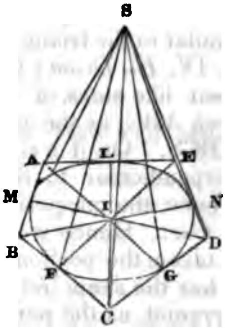
Cor. *The convex surface of the frustum of a regular pyramid is equal to half the perimeters of its upper and lower bases multiplied by its slant height.*

For, since the section *abcde* is similar to the base (Prop. XIX.) and since the base ABCDE is a regular polygon (Def. 14.), it follows that the sides *ea*, *ab*, *bc*, *cd* and *de* are all equal to each other. Hence the convex surface of the fustrum ABCDE-*d* is formed by the equal trapezoids EA*ae*, AB*ba*, &c. and the perpendicular distance between the parallel sides of either of these trapezoids is equal to F*f*, the slant height of the frustum. But the area of either of the trapezoids, as AE*ea*, is

$(EA+ea)\times Ff$ (Prop. IX. B. IV. *El. Geom.*): hence of all of them, or the convex surface of the frustrum, o half the perimeters of the upper and lower bases l by the slant height.

PROPOSITION XXI. THEOREM.

x surface of a regular pyramid exceeds the area of in the same ratio that the slant height of its side ex- e radius of the circle inscribed in its base.

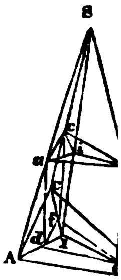


IS be a regular pyramid, and ygon ABCDE of the base de- ircle with the radius IF or IG centre I; draw IA, IB, IC, ID, o the angles of the polygon, ase will be divided into trian- se bases are the several sides lygon, and whose several alti- IF, IG, &c., each equal to the the circle, and the convex sur- ie pyramid consists in like man- angles ABS, BCS, &c., whose ases are the sides of the polygon of the base of the and whose several altitudes are the common slant S, GS, &c., of the sides of the pyramid. Now, since gles of the base and of the sides of the pyramid have bases, each to each, that is, as the triangle BCI, and gle BCS, have the common base BC, and the triangle the triangle CDS have the common base CD their e areas in each corresponding triangle are as their viz. as FI to FS, or GI to GS, and as this is true of of the pyramid in relation to its corresponding por- ie base, it follows that the convex surface exceeds the ts base in the same ratio that the slant height exceeds is of the circle inscribed in the base.

PROPOSITION XXII. THEOREM.

ılar pyramid is cut by a plane perpendicular to its base, ıvex surface of the pyramid and its base will be divided same ratio.

irst suppose as many planes *abde*, *acdf*, and *cbfe* to d through the pyramid, as its base has sides and which planes is parallel to one of the sides of the base,

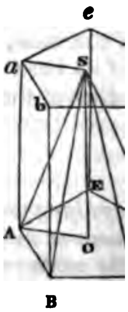


cutting the pyramid so as to form a similar pyramid the plane *abc* be taken as a base, then will the base *abc* be equal and similar to the section *def*, perpendicularly below. Now, the base or section *abc* is to the base ABC as the square of *i*S is to the square of IS, (Prop. XIX.) or as the square of *a*S to the square of AS, and the several sides of the new pyramid *abc*S are each triangles, similar to those forming the convex surface of the former pyramid, viz. the triangle *ab*S is similar to the triangle ABS. The triangle *cbs* is similar to the triangle CBS, &c. Hence (Prop. XXIV. C B. IV. *El. Geom.*) they are also proportional to the squa their like sides, or as aS^2 to AS^2, hence the area *def* is area ABC, as the convex surface *abc*S to the convex s ABCS. Or, if a single plane is passed through the py perpendicular to the base, the same argument will Hence the proposition as enunciated is true.

Cor. Hence we may infer that if any portion of th is taken, the portion of the convex surface perpendicular it has the same relation to the whole convex surface c pyramid, as the portion of the base so taken, has to the base.

PROPOSITION XXIII. THEOREM.

The convex surface of a regular pyramid exceeds half that circumscribed prism, in the same ratio that the vertical or slant height of one of its sides exceeds the vertical hei the pyramid.

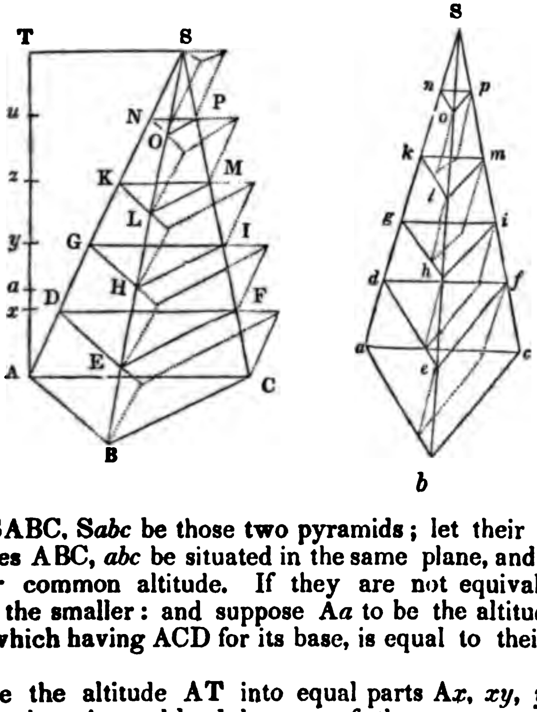


Let ABCDE be a regular polygon on which the pyramid ABCDE-S is described as its base; and on the same polygon describe the prism at BCDE*abcde;* then let AS=the vertical length of one of its sides; and OS the vertical height of the pyramid; and also that of its circumscribing prism. Then will the surface of the pyramid exceed half the surface of the prism, in the same ratio that the length of the line AS=the vertical length of the side of the pyramid exceeds the vertical height of the pyramid.

For, each side of the prism is a parallelogram, and eac of the pyramid is a triangle, on the same base. Now, a gle on the same base, and of the same vertical height, is=half the rectangle, or parallelogram: moreover, triangles on the same base are as their vertical heights. But the vertical or slant height of a triangle forming one of the sides of the pyramid, is to the vertical height of the pyramid, or of the prism, as the line AS to the line OS, or Aa, Bb, &c. Therefore, the surface of one side of the pyramid exceeds half the surface of the corresponding side of the prism, in the same ratio that its vertical length exceeds the vertical height of the pyramid or prism. Hence the sum of the surfaces of all the sides of the pyramids, exceeds the sum of the surfaces of all the sides of the prisms in the same ratio.

PROPOSITION XXIV. THEOREM.

Two triangular pyramids, having equivalent bases and equal altitudes, are equivalent, or equal in solidity.



Let SABC, Sabc be those two pyramids; let their equivalent bases ABC, abc be situated in the same plane, and let AT be their common altitude. If they are not equivalent, let Sabc be the smaller: and suppose Aa to be the altitude of a prism, which having ACD for its base, is equal to their difference.

Divide the altitude AT into equal parts Ax, xy, yz, &c. each less than Aa, and let k be one of those parts; through the points of division pass planes parallel to the plane of the bases; the corresponding sections formed by these planes in the two pyramids will be respectively equivalent (Prop. XIX. Cor. 2.) namely DEF to *def* GHI to *ghi*, &c.

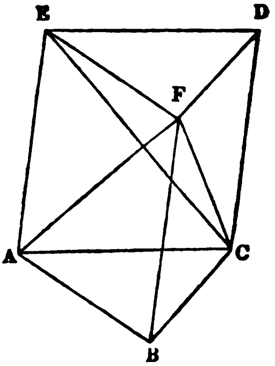
This being granted, upon the triangles ABC, DEF, GHI, &c. taken as bases, construct exterior prisms having for edges the parts AD, DG, DK, &c. of the edge SA; in like manner, on the bases *def*, *ghi*, *klm*, &c. in the second pyramid, construct interior prisms, having for edges the corresponding parts of *sa*. It is plain that the sum of all the exterior prisms of the pyramid SABC will be greater than this pyramid; and also the sum of all the interior prisms of the pyramid *sabc* will be less than this. Hence the difference, between the sum of all all the exterior prisms and the sum of all the interior ones, must be greater than the difference between the two pyramids themselves.

Now, beginning with the bases ABC, *abc*, the second exterior prism DEFG is equivalent to the first interior prism *defg*, because they have the same altitude *k*, and their bases DEF, *def*, are equivalent; for like reasons, the third exterior prism GHIK, and the second interior prism *ghik* are equivalent; the fourth exterior and the third interior; and so on, to the last in each series. Hence all the exterior prisms of the pyramid SABC, excepting the first prism DABC, have equivalent corresponding ones in the interior prisms of the pyramid *sabc*: hence the prism DACB is the difference between the sum of all the exterior prisms of the pyramid SABC, and the sum of all the interior prisms of the pyramid S*abc*. But the difference between those two sets of prisms has already been proved to be greater than that of the two pyramids; which latter difference we suppose to be equal to the prism *a*ABC: hence the prism DABC must be greater than the prism *a*ABC. But in reality it is less; for they have the same base ABC, and the altitude Ax of the first is less than A*a* the altitude of the second. Hence the supposed inequality between the two pyramids cannot exist; hence the two pyramids SABC, *sabc*, having equal altitudes and equivalent bases, are themselves equivalent.

PROPOSITION XXV. THEOREM.

Every triangular pyramid is a third part of the triangular prism having the same base and the same altitude.

Let FABC be a triangular pyramid, ABCDEF a triangular prism of the same base and the same alitude; the pyramid will be equal to a third of the prism.



Cut off the pyramid FABC from the prism, by the plane FAC; there will remain the solid FACDE, which may be considered as a quadrangular pyramid, whose vertex is F, and whose base is the parallelogram ACDE. Draw the diagonal CE; and pass the plane FCE, which will cut the quadrangular pyramid into two triangular ones FACE, FCDE. These two triangular pyramids have for their common altitude the perpendicular let fall from F on the plane ACDE; they have equal bases, the triangles ACE, CDE being halves of the same parallelogram; hence (Prop. XXIV.) the two pyramids FACE, FCDE are equivalent. But the pyramid FCDE and the pyramid FABC have equal bases ABC, DEF; they have also the same altitude, namely, the distance of the parallel planes ABC, DEF; hence the two pyramids are equivalent. Now the pyramid FCDE has already been proved equivalent to FACE; hence the three pyramids FABC, FCDE, FACE, which compose the prism ABCD are all equivalent. Hence the pyramid FABC, is the third part of the prism ABCD which has the same base and the same altitude.

Cor. 1. The solidity of a triangular pyramid is equal to a third part of the product of its base by its altitude.

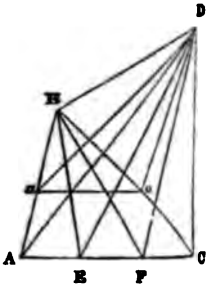
2. The solidities of triangular pyramids of equal bases are to each other as their altitudes.

3. The solidities of triangular pyramids of equal altitudes are to each other as their bases.

Scholium 1. As all prisms (Prop. XVIII. Cor. and Sch.) may be divided into triangular prisms; so all triangular prisms may be divided into triangular pyramids; and for the same reasons (Prop. XVIII.) every polygonal pyramid may also be divided into triangular pyramids. And hence, as pyramids are the most simple solids to which a prism or polygonal pyramid can be referred, triangular pyramids may be considered as the elements of all other pyramids as well as prisms, as also of any polyedral body.

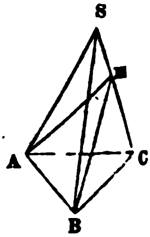
2. And as triangular prisms, the elements of other prisms, may be divided indefinitely by planes perpendicular to their

rectangular bases (Prop. XVIII. Sch. 2.) so triangular pyramids, the elements of other polyedral bodies may be divided indefinitely by planes passing through their bases and vertices, and these new triangular pyramids will be the elementary pyramids of the pyramid so divided.



Thus, if the pyramid ABCD be divided by the planes BED, or BFD, passing through the base and the vertice D, then each of the pyramids ABE-D, BEC-D, &c. and also the pyramid B*ac*-D cut off by the plane *ac*D will be an elementary pyramid of the pyramid ABC-D.

For, the pyramid ABC-D may be conceived to be made up of some multiple of each of the elementary pyramids into which it is divided, without any change of the character of the base, except in its proportions.



But if the triangular pyramid ABC S be divided by the plane ABE not passing through the vertice S, the triangular pyramid ABC-E cut off by this plane is not an elementary pyramid referred to the pyramid ABCS, but if one triangular pyramid is an element of another triangular pyramid, they must both be referred to the same vertice, or their vertical heights must be equal.

Hence, then, and in reference to (Prop. XVIII. Sch. 2.) a polyedron may be considered as elementary to another polyedron when it is one of the natural or proportional divisions of the other.

A triangular pyramid may be considered as elementary to a triangular prism, and hence to all other prisms.

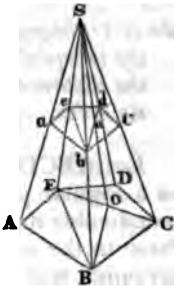
One triangular pyramid is elementary to another triangular pyramid where their vertices are referred to the same point in relation to their bases, or where they are of the same vertical height.

PROPOSITION XXVI. THEOREM.

The solidity of every pyramid is equal to the product of the base by a third of its altitude.

For, pass the planes SEB, SEC through the diagonals EB, EC; the pylygonal pyramid SABCDE will be divided into several triangular pyramids all having the same altitude SO.

rop. XXV. Cor.) each of these pyramids is measured ltiplying its base ABE, BCE, or CDE, third part of its altitude SO; hence m of these triangular pyramids, or the nal pyramid SABCDE will be mea- by the sum of the triangles ABE, BCE, or the polygon ABCDE, multiplied by ird of SO; hence every pyramid is red by a third part of the product of e by its altitude.



. 1. Every pyramid is the third part of a prism which e same base and the same altitude.

. 2. Two pyramids having the same altitude are to ther as their bases, and conversely; two pyramids hav- uivalent bases are as their altitudes.

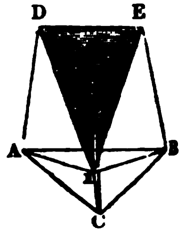
. 3. Pyramids are to each other as the product of their by their altitudes.

. 4. As in a triangular pyramid either side may be as a base and its opposite solid angle its vertex, it fol- hat the solidity of a triangular pyramid is equal to either aces multiplied by a third of the distance of its opposite from such base.

lium. The solidity of any polyedral body may be com- by dividing the body into pyramids; and this division e accomplished in various ways. One of the simplest ake all the planes of division pass through the vertex of lid angle; in that case, there will be formed as many pyramids as the polyedron has faces, *minus* those faces form the solid angle whence the planes of division prois that of the frustum, and whose bases are the lower base of the frustum, the upper base, and a mean proportonal between the two bases.

PROPOSITION XXIX. THEOREM.

The frustum of a pyramid is equal to the sum of the several pyramids formed by each vertical side, and also the top, as bases, whose several vertices are referred to the centre of the base of the frustum, or to any point within that base.



Let AE be a frustum of a pyramid; this may be divided into as many pyramids as it has vertical faces, together with a pyramid erected on one of the bases, which suppose, in the present case, to be the smaller base, DEF. Draw the lines DI, EI and FI, from the several angles of the smaller base to the centre of the larger, then will there be enclosed, by planes passing through these lines as angles, the pyramid DEFI. Now let the frustum be further divided by planes passing through the angles AD, CF, and BC, to the centre of the base I; then will the whole of the frustum be divided into pyramids whose bases are all the several plane faces of the frustum, except the lower base; and they all terminate with points as vertices in the centre, I, of the larger base. Hence, the whole frustum is equal to the pyramids formed on each of the vertical sides, as well as on the smaller base of the frustum as bases, and whose vertices all terminate in the centre of the larger base, since the whole is equal to the sum of all its parts; and if, instead of the vertices of the several pyramids all terminating in the centre of the base at I, they all terminate in any other point in that base, the whole of the frustum would still be divided into triangular pyramids, whose sum would be equal to the frustum, by joining the angles about each of the bases, with the point designated for the vertices.

Scholium 1. We might, instead of erecting a pyramid on the smaller base of the frustum, erect it on the larger, in which case the vertices would all terminate in the smaller; or, we might erect a pyramid in each base: and if the sum of their vertical heights are equal to the vertical height of the frustum, the vertices would all centre in the same point within the frustum.

Cor. 1. Hence the solidity of a frustum of a pyramid is equal to the smaller base multiplied by one-third of its vertical height, plus all of the vertical surfaces multiplied by one-third of theirs from the centre of the larger base, perpendicular to their surfaces respectively.

Cor. 2. Hence, if the bases of the frustum of the pyramid are regular polygons, its solidity is equal to the surface of the upper base multiplied by one-third of its vertical height, plus the sum of its vertical surfaces multiplied by one-third of a line drawn from the centre of its lower base perpendicular to one of its sides.

Scholium 2. If the smaller end of the frustrum of the pyramid becomes enlarged, till it is made equal to the other end, the frustum will become a prism ; and if its base is a regular polygon, the vertical height of the several pyramids erected on their vertical surfaces as bases, will be equal to the radius of a circle inscribed in the base.

Scholium 3. And if from a prism whose bases are regular polygons, a pyramid be withdrawn whose base equals the base of the prism, and altitude the altitude of the prism, the remainder will be equal to the vertical surface multiplied by one-third of the radius of a circle inscribed in the base.

PROPOSITION XXX. THEOREM.

If a triangular prism be cut by a plane inclined to the plane of its base, the solid included between this plane and the plane of the base is equivalent to three pyramids, having the base of the prism for a common base, and the three vertices terminating the inclined plane for their vertices.

Let ABCDEF (see diagram to Prop. XXV.) be a solid formed by intersecting a triangular prism, by a plane DEF inclined to the plane of the base ABC; then will this solid be equivalent to the three pyramids having ABC for a common base, and the points F, D, E, for vertices.

Through the three points F, A, C, pass the plane FAC; it will cut off, from the truncated prism ABCDEF, the triangular pyramid FABC: this pyramid has ABC for its base, and the point F for its vertex.

This pyramid being cut off, there will remain the quadrangular pyramid FACDE, whose vertex is F, and whose base is ACDE. Through the three points F, E, C, draw another plane FEC; it will divide the quadrangular pyramid into two triangular pyramids FACE, FCDE.

The pyramid FACE, which has for its base the triangle

AEC and for its vertex the point F, is equivalent to a pyramid BAEC having AEC for its base, and for its vertex the point B. For, these two pyramids have the same base; they have also the same altitude, because the line BF, parallel to each of the lines AE, CD, is parallel to their plane ACE: hence the pyramid FAEC is equivalent to the pyramid BACE: which latter may be considered as having ABC for its base, and the point E for its vertex.

Again, the third pyramid FCDE may in the first place be changed into AFCD; for these two pyramids have the same base FCD; they have also the same altitude, AE being parallel to the plane FCD; hence the pyramid EFCD is equivalent to the pyramid AFCD. In the second place, the pyramids AFCD may be changed into BACD: for these two pyramids have the common base ACD; they have the same altitude, since the vertices F and B lie in a line parallel to the plane of the base. Hence the pyramid EFCD, equivalent to AFCD, is also equivalent to BACD: which latter may be regarded as having ABC for its base, and the point D for its vertex.

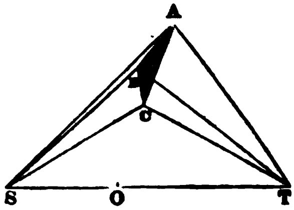
Hence, finally, the truncated prism ABCDEF is equal to the sum of three pyramids whose common base is ABC, and whose vertices are respectively the points D, E, F.

Cor. If the edges AE, BF, CD are perpendicular to the plane of the base, they will at the same time be the altitudes of the three pyramids which compose the truncated prism: so that the solidity of the truncated prism will be expressed by $\frac{1}{3}ABC \times AE + \frac{1}{3}ABC \times BF + \frac{1}{3}ABC \times CD$, or, which is the same thing, by $\frac{1}{3}$ $ABC \times (AE+BF+CD.)$

PROPOSITION XXXI. THEOREM.

Two symmetrical polyedrons are equivalent, or equal in solidity.

For, in the first place, two symmetrical triangular pyramids, such as SABC, TABC, are both measured by the product of their base ABC and the third part of their altitude SO or TO; hence these pyramids are equivalent.



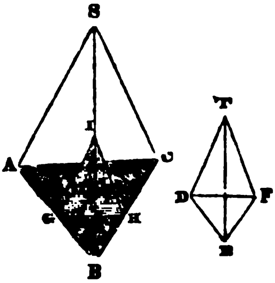
In the second place, if one of the two symmetrical polyedrons is divided in any way into triangular pyramids, the other may always be divided in the same way into as many symmetrical triangular pyramids; and these symmetrical pyramids

are equivalent each to each; hence the whole polyedrons are equivalent, or equal in solidity.

Scholium. This Theorem might seem to result immediately from Proposition II., where it was shown that, in two symmetrical polyedrons, all the constituent parts of the one solid are respectively equal to the constituent parts of the other: but it appeared necessary, notwithstanding, to demonstrate this truth in a rigorous manner.

PROPOSITION XXXII. THEOREM.

Two similar triangular pyramids have their homologous faces similar, and their solid angles equal.



By the definition, (20.) the two triangular pyramids SABC, TDEF are similar when the two triangles SAB, ABC are similar to the two TDE, DEF, and similarly placed, that is to say, when the angles ABS=DET BAS=EDT, ABC=DEF, BAC=EDF, and when also the inclination of the planes SAB, ABC is equal to that of the planes TDE, DEF. This being granted, we are to show that these pyramids have all their faces similar each to each, and their homelogous solid angles equal.

Take BG=ED, BH=EF, BI=ET; and join GH, GI, IH. The pyramid TDEF is equal to the pyramid IGBH; for, the sides GB, BH being made equal to the sides DE, EF, and the angle GBH being equal to the angle DEF by hypothesis, the triangle GBH must be equal to DEF; hence to bring about the superposition of the two pyramids, the base DEF may in the first place be laid on its equal GBH; then, because the plane DTE is inclined to DEF as much as the plane SAB to ABC, it is evident that the plane DET will fall indefinitely on the plane ABS. But, by hypothesis, the angle DET=GBI: therefore ET will fall on its equal BI; and since the four points D, E, F, T, coincide with the four G, B, H, I, it follows (Prop. I.) that the pyramid TDEF coincides with the pyramid IGBH.

Now by reason of the equal triangles DEF, GBH, we have the angle BGH=EDF=BAC; hence GH is parallel to AC. For a like reason, GI is parallel to AS; hence (Prop. XIV, B. I.) the plane IGH is parallel to SAC. It follows, therefore, AEC and for its vertex the point F, is equivalent to a pyramid BAEC having AEC for its base, and for its vertex the point B. For, these two pyramids have the same base; they have also the same altitude, because the line BF, parallel to each of the lines AE, CD, is parallel to their plane ACE: hence the pyramid FAEC is equivalent to the pyramid BACE: which latter may be considered as having ABC for its base, and the point E for its vertex.

Again, the third pyramid FCDE may in the first place be changed into AFCD; for these two pyramids have the same base FCD; they have also the same altitude, AE being parallel to the plane FCD; hence the pyramid EFCD is equivalent to the pyramid AFCD. In the second place, the pyramids AFCD may be changed into BACD: for these two pyramids have the common base ACD; they have the same altitude, since the vertices F and B lie in a line parallel to the plane of the base. Hence the pyramid EFCD, equivalent to AFCD, is also equivalent to BACD: which latter may be regarded as having ABC for its base, and the point D for its vertex.

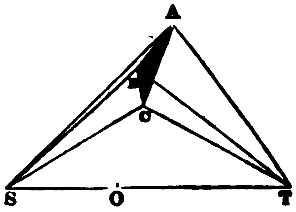
Hence, finally, the truncated prism ABCDEF is equal to the sum of three pyramids whose common base is ABC, and whose vertices are respectively the points D, E, F.

Cor. If the edges AE, BF, CD are perpendicular to the plane of the base, they will at the same time be the altitudes of the three pyramids which compose the truncated prism: so that the solidity of the truncated prism will be expressed by $\frac{1}{3}\text{ABC}\times\text{AE}+\frac{1}{3}\text{ABC}\times\text{BF}+\frac{1}{3}\text{ABC}\times\text{CD}$, or, which is the same thing, by $\frac{1}{3}\ \text{ABC}\times\ (\text{AE}+\text{BF}+\text{CD}.)$

PROPOSITION XXXI. THEOREM.

Two symmetrical polyedrons are equivalent, or equal in solidity.

For, in the first place, two symmetrical triangular pyramids, such as SABC, TABC, are both measured by the product of their base ABC and the third part of their altitude SO or TO; hence these pyramids are equivalent.



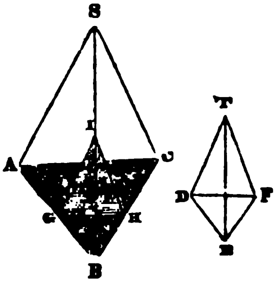
In the second place, if one of the two symmetrical polyedrons is divided in any way into triangular pyramids, the other may always be divided in the same way into as many symmetrical triangular pyramids; and these symmetrical pyramids

are equivalent each to each; hence the whole polyedrons are equivalent, or equal in solidity.

Scholium. This Theorem might seem to result immediately from Proposition II., where it was shown that, in two symmetrical polyedrons, all the constituent parts of the one solid are respectively equal to the constituent parts of the other: but it appeared necessary, notwithstanding, to demonstrate this truth in a rigorous manner.

PROPOSITION XXXII. THEOREM.

Two similar triangular pyramids have their homologous faces similar, and their solid angles equal.



By the definition, (20.) the two triangular pyramids SABC, TDEF are similar when the two triangles SAB, ABC are similar to the two TDE, DEF, and similarly placed, that is to say, when the angles ABS=DET BAS=EDT, ABC=DEF, BAC=EDF, and when also the inclination of the planes SAB, ABC is equal to that of the planes TDE, DEF. This being granted, we are to show that these pyramids have all their faces similar each to each, and their homelogous solid angles equal.

Take BG=ED, BH=EF, BI=ET; and join GH, GI, IH. The pyramid TDEF is equal to the pyramid IGBH; for, the sides GB, BH being made equal to the sides DE, EF, and the angle GBH being equal to the angle DEF by hypothesis, the triangle GBH must be equal to DEF; hence to bring about the superposition of the two pyramids, the base DEF may in the first place be laid on its equal GBH; then, because the plane DTE is inclined to DEF as much as the plane SAB to ABC, it is evident that the plane DET will fall indefinitely on the plane ABS. But, by hypothesis, the angle DET=GBI: therefore ET will fall on its equal BI; and since the four points D, E, F, T, coincide with the four G, B, H, I, it follows (Prop. I.) that the pyramid TDEF coincides with the pyramid IGBH.

Now by reason of the equal triangles DEF, GBH, we have the angle BGH=EDF=BAC; hence GH is parallel to AC. For a like reason, GI is parallel to AS; hence (Prop. XIV, B. I.) the plane IGH is parallel to SAC. It follows, therefore, (Prop. XVI. B. 1.) that the triangle IGH, or its equal TDF, is similar to SAC, and the triangle IBH, or its equal TEF, to SBC; hence the two similar triangular pyramids SABC, TDEF, have their four faces similar each to each. We are now to show that the homologous solid angles are equal.

The solid angle E has already been applied to its homologous one B; and the same thing might be done with any other two homologous solid angles; but it is clear of itself that any two homologous solid angles: T and S for example, are equal, because they are formed by three plane angles, which are respectively equal and similarly placed.

Hence two similar triangular pyramids have their homologous faces similar, and their homologous solid angles equal.

Cor. 1. The similar triangles in the two pyramids furnish the proportions AB : DE : : BC : EF : : AC : DF : : AS : DT : : SB : TE : : SC : TF; hence, *in similar triangular pyramids, the homologous edges are proportional.*

Cor. 2. And since the homologous solid angles are equal, it follows that *the inclination of any two faces of a pyramid, is equal to the inclination of the two corresponding faces in a similar pyramid.* (Prop. XXIII. B. 1.)

Cor. 3. If the triangular pyramid SABC is cut by a plane GIH parallel to SAC one of its faces, the partial pyramid IGBH will be similar to the whole pyramid SABC: for, the triangles BGI, BGH are similar to the triangles BAS, BAC, each to each, and similarly placed; the inclination of their planes is the same in both: hence the two pyramids are similar.

Cor. 4. In general (see diagram to Prop. XIX.), *if any pyramid* SABCDE *is cut by a plane* abcde *parallel to the base, the partial pyramid* Sabcde *will be similar to the whole pyramid* SABCDE. For the bases ABCDE, *abcde*, are similar; and joining AC, *ac*, we have just proved that the triangular pyramid SABC must be similar to the pyramid *Sabc;* hence (Def. 21.) the point S in reference to the base ABC, is determined exactly as it is in reference to the base *abc;* hence the two pyramids SABCDE, *Sabcde* are similar.

Cor. 5. And since every polyedron may be divided into triangular pyramids (Prop. XXV. *Sch.* 1.), it is evident that two similar polyedrons may be divided into a similar number of triangular pyramids, which are respectively similar in each, and similarly placed: hence we may generally infer that any two similar polyedrons have their homologous faces similar, and their homologous solid angles equal.

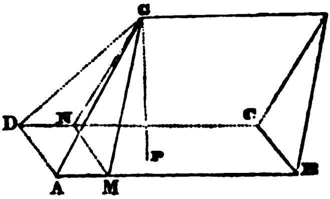
Scholium. Instead of the five data required by the Definition to determine the similarity of two triangular pyramids, five other data might be substituted according to various combinations, and there would result as many theorems as combinations. Among these, the following deserves to be specially noticed : *Two triangular pyramids are similar when they have their homologous edges proportional.*

For (see diagram of the Proposition), if we have the proportions AB : DE : : BC : EF : : AC : DF : : AS : DT : : SB : TE : : SC : TF, which includes five conditions, the triangles ABS, ABC will be respectively similar to the triangles DET, DEF, and similarly placed. The triangle SBC will be similar to TEF ; hence the three plane angles which form the solid angle B will be equal to the plane angles which form the solid angle E, each to each ; hence the inclination of the planes SAB, ABC is equal to that of their homologous planes TDE, DEF ; hence the two pyramids are similar.

PROPOSITION XXXIII. THEOREM.

The solidity of a wedge is equal to the product of the sum of twice the length of the base and the length of the edge, multiplied by one-sixth of the product of the height of the wedge and the breadth of the base.

Let L=AB the length of the base.
l=GH the length of the edge.
b=BC the breadth of the base.
h=PG the height of the wedge.
Then, L—l=AB—GH=AM.



If the length of the base and the edge be *equal*, as BM and GH, the wedge MBHG is half a parallelopipedon of the same base and height. And the solidity (Prop. XVII.) is equal to half the product of the height, into the length and breadth of the base ; that is to $\frac{1}{2}bhl$.

If the length of the base be *greater* than that of the edge, as ABGH ; let a section be made by the plane GMN, parallel to HBC. This will divide the whole wedge into two parts

MBHG and AMG. The latter is a pyramid, whose solidity (Prop. XXVI.) is $\frac{1}{3}bh\times(\mathrm{L}-l.)$

The solidity of the parts together, is, therefore,

$$\tfrac{1}{2}bhl-\tfrac{1}{3}bh\times(\mathrm{L}-l)=\tfrac{1}{6}bh3l+\tfrac{1}{6}bh2\mathrm{L}-\tfrac{1}{6}bh2l=\tfrac{1}{6}bh\times(2\mathrm{L}+l)$$

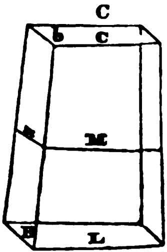
If the length of the base be *less* than that of the edge, it is evident that the pyramid is to be *subtracted* from half the parallelopipedon, which is equal in height and breadth to the wedge, and equal in length to the edge.

The solidity of the wedge is, therefore,

$$\tfrac{1}{2}bhl-\tfrac{1}{3}bh\times(l-\mathrm{L})=\tfrac{1}{6}bh3l-\tfrac{1}{6}bh2l+\tfrac{1}{6}bh2\mathrm{L}=\tfrac{1}{6}bh\times(2\mathrm{L}+l)$$

PROPOSITION XXXIV. THEOREM.

The solidity of a rectangular prismoid, is equal to the product of the sum of the areas of the two ends, and four times the area of a parallel section equally distant between the two ends, multiplied by $\frac{1}{6}$ of the height.



Let L and B be the length and breadth of one end,
C and b the length and breadth of the other end,
M and m the length and breadth of the section in the middle.
and h the height of the prismoid.

The solid may be divided into two wedges, whose bases are the ends of the prismoid, and whose edges are L and C. The solidity of the whole (Prop. XXXIII.) is

$$\tfrac{1}{6}\mathrm{B}h\times(2\mathrm{L}+\mathrm{C})+\tfrac{1}{6}bh\times(2\mathrm{C}+\mathrm{L})=\tfrac{1}{6}h(2\mathrm{BL}+\mathrm{BC}+2b\mathrm{C}+b\mathrm{L})$$

As M is equally distant from L and C,

$$2\mathrm{M}=\mathrm{L}+\mathrm{C}, 2m=\mathrm{B}+b, \text{ and } 4\mathrm{M}m=(\mathrm{L}+\mathrm{C})(\mathrm{B}+b)=\mathrm{BL}+\mathrm{BC}+b\mathrm{L}+\mathrm{C}b$$

Substituting 4Mm for its value, in the preceding expression for the solidity, we have

$$\tfrac{1}{6}h(\mathrm{BL}+b\mathrm{C}+4\mathrm{M}m)$$

That is, the solidity of the prismoid is equal to $\frac{1}{6}$ of the height multiplied into the area of the two ends, and 4 times the area of the section in the middle.

Cor. This demonstration may be applied to prismoids of other forms. For, whatever be the figure of the two ends, there may be drawn in each, such a number of small rectangles, that the sum of them shall differ less, than by any given

are equivalent each to each; hence the whole polyedrons are equivalent, or equal in solidity.

Scholium. This Theorem might seem to result immediately from Proposition II., where it was shown that, in two symmetrical polyedrons, all the constituent parts of the one solid are respectively equal to the constituent parts of the other: but it appeared necessary, notwithstanding, to demonstrate this truth in a rigorous manner.

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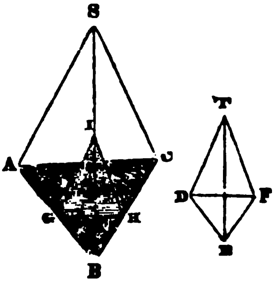
By the definition. (20.) the two triangular pyramids SABC, TDEF are similar when the two triangles SAB, ABC are similar to the two TDE, DEF, and similarly placed, that is to say, when the angles ABS=DET BAS=EDT, ABC=DEF, BAC=EDF, and when also the inclination of the planes SAB, ABC is equal to that of the planes TDE, DEF. This being granted, we are to show that these pyramids have all their faces similar each to each, and their homelogous solid angles equal.



Take BG=ED, BH=EF, BI=ET; and join GH, GI, IH. The pyramid TDEF is equal to the pyramid IGBH; for, the sides GB, BH being made equal to the sides DE, EF, and the angle GBH being equal to the angle DEF by hypothesis, the triangle GBH must be equal to DEF; hence to bring about the superposition of the two pyramids, the base DEF may in the first place be laid on its equal GBH; then, because the plane DTE is inclined to DEF as much as the plane SAB to ABC, it is evident that the plane DET will fall indefinitely on the plane ABS. But, by hypothesis, the angle DET=*GBI*: therefore ET will fall on its equal BI; and since the *four points* D, E, F, T, coincide with the four G, B, H, I, it *follows* (*Prop.* I.) that the pyramid TDEF coincides with th[illegible] *IGBH.*

Now by reason of the equal triangles D[illegible] *we have* the angle IGH=E[illegible] hence G[illegible] *to AC.* For [illegible] XIV. B[illegible] *therefore,* (Prop. XVI. B. 1.) that the triangle IGH, or its equal TDF, is similar to SAC, and the triangle IBH, or its equal TEF, to SBC; hence the two similar triangular pyramids SABC, TDEF, have their four faces similar each to each. We are now to show that the homologous solid angles are equal.

The solid angle E has already been applied to its homologous one B; and the same thing might be done with any other two homologous solid angles; but it is clear of itself that any two homologous solid angles: T and S for example, are equal, because they are formed by three plane angles, which are respectively equal and similarly placed.

Hence two similar triangular pyramids have their homologous faces similar, and their homologous solid angles equal.

Cor. 1. The similar triangles in the two pyramids furnish the proportions AB : DE : : BC : EF : : AC : DF : : AS : DT : : SB : TE : : SC : TF; hence, *in similar triangular pyramids, the homologous edges are proportional.*

Cor. 2. And since the homologous solid angles are equal, it follows that *the inclination of any two faces of a pyramid, is equal to the inclination of the two corresponding faces in a similar pyramid.* (Prop. XXIII. B. 1.)

Cor. 3. If the triangular pyramid SABC is cut by a plane GIH parallel to SAC one of its faces, the partial pyramid IGBH will be similar to the whole pyramid SABC: for, the triangles BGI, BGH are similar to the triangles BAS, BAC, each to each, and similarly placed; the inclination of their planes is the same in both: hence the two pyramids are similar.

Cor. 4. In general (see diagram to Prop. XIX.), *if any pyramid* SABCDE *is cut by a plane* abcde *parallel to the base, the partial pyramid* Sabcde *will be similar to the whole pyramid* SABCDE. For the bases ABCDE, *abcde*, are similar; and joining AC, *ac*, we have just proved that the triangular pyramid SABC must be similar to the pyramid *Sabc;* hence (Def. 21.) the point S in reference to the base ABC, is determined exactly as it is in reference to the base *abc;* hence the two pyramids SABCDE, *Sabcde* are similar.

Cor. 5. And since every polyedron may be divided into triangular pyramids (Prop. XXV. *Sch.* 1.), it is evident that two similar polyedrons may be divided into a similar number of triangular pyramids, which are respectively similar in each, and similarly placed: hence we may generally infer that any two similar polyedrons have their homologous faces similar, and their homologous solid angles equal.

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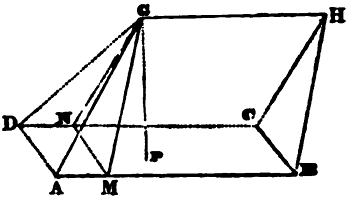
For (see diagram of the Proposition), if we have the proportions AB : DE : : BC : EF : : AC : DF : : AS : DT : : SB : TE : : SC : TF, which includes five conditions, the triangles ABS, ABC will be respectively similar to the triangles DET, DEF, and similarly placed. The triangle SBC will be similar to TEF; hence the three plane angles which form the solid angle B will be equal to the plane angles which form the solid angle E, each to each; hence the inclination of the planes SAB, ABC is equal to that of their homologous planes TDE, DEF; hence the two pyramids are similar.

PROPOSITION XXXIII. THEOREM.

The solidity of a wedge is equal to the product of the sum of twice the length of the base and the length of the edge, multiplied by one-sixth of the product of the height of the wedge and the breadth of the base.

Let $L=AB$ the length of the base.
$l=GH$ the length of the edge.
$b=BC$ the breadth of the base.
$h=PG$ the height of the wedge.
Then, $L-l=AB-GH=AM$.



If the length of the base and the edge be *equal*, as BM and GH, the wedge MBHG is half a parallelopipedon of the same base and height. And the solidity (Prop. XVII.) is equal to half the product of the height, into the length and breadth of the base; that is to $\frac{1}{2}bhl$.

If the length of the base be *greater* than that of the edge, as ABGH; let a section be made by the plane GMN, parallel to HBC. This will divide the whole wedge into two parts

MBHG and AMG. The latter is a pyramid, whose solidity (Prop. XXVI.) is $\frac{1}{3}bh\times(L-l.)$

The solidity of the parts together, is, therefore,

$$\tfrac{1}{2}bhl-\tfrac{1}{3}bh\times(L-l)=\tfrac{1}{6}bh3l+\tfrac{1}{6}bh2L-\tfrac{1}{6}bh2l=\tfrac{1}{6}bh\times(2L+l)$$

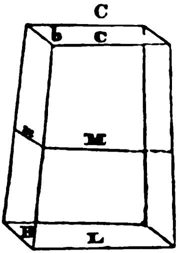
If the length of the base be *less* than that of the edge, it is evident that the pyramid is to be *subtracted* from half the parallelopipedon, which is equal in height and breadth to the wedge, and equal in length to the edge.

The solidity of the wedge is, therefore,

$$\tfrac{1}{2}bhl-\tfrac{1}{3}bh\times(l-L)=\tfrac{1}{6}bh3l-\tfrac{1}{6}bh2l+\tfrac{1}{6}bh2L=\tfrac{1}{6}bh\times(2L+l)$$

PROPOSITION XXXIV. THEOREM.

The solidity of a rectangular prismoid, is equal to the product of the sum of the areas of the two ends, and four times the area of a parallel section equally distant between the two ends, multiplied by $\frac{1}{6}$ *of the height.*



Let L and B be the length and breadth of one end,
C and b the length and breadth of the other end,
M and m the length and breadth of the section in the middle.
and h the height of the prismoid.

The solid may be divided into two wedges, whose bases are the ends of the prismoid, and whose edges are L and C. The solidity of the whole (Prop. XXXIII.) is

$$\tfrac{1}{6}Bh\times(2L+C)+\tfrac{1}{6}bh\times(2C+L)=\tfrac{1}{6}h(2BL+BC+2bC+bL)$$

As M is equally distant from L and C,

$2M=L+C, 2m=B+b$, and $4Mm=(L+C)(B+b)=BL+BC+$
[$bL+Cb$

Substituting $4Mm$ for its value, in the preceding expression for the solidity, we have

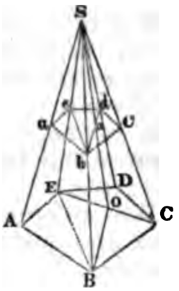
$$\tfrac{1}{6}h(BL+bC+4Mm)$$

That is, the solidity of the prismoid is equal to $\frac{1}{6}$ of the height multiplied into the area of the two ends, and 4 times the area of the section in the middle.

Cor. This demonstration may be applied to prismoids of other forms. For, whatever be the figure of the two ends, there may be drawn in each, such a number ef small rectangles, that the sum of them shall differ less, than by any given antity, from the figure in which they are contained. And ː solids between these rectangles will be rectangular pris-)ids.

PROPOSITION XXXV. THEOREM.

ro similar pyramids are to each other as the cubes of their homologous sides.



For two pyramids being similar, the small- may be placed within the greater, so that solid angle S shall be common to both. that position, the bases ABCDE, *abcde* parallel; because, since (Prop. XXX.) the nologous faces are similar, the angle S*ab* equal to SAB, and S*bc* to SBC; hence op. XIV. B. I.) the plane ABC is parallel he plane *abc*. This granted, let SO be the pendicular drawn from the vertex S to the ne ABC, and *o* the point where this perpen- ılar meets the plane *abc*: from what has ady been shown (Prop. XVI.), we shall have SO : S*o* : : : S*a* : : AB : *ab*; and consequently,

$$\tfrac{1}{3}SO : \tfrac{1}{3}So :: AB : ab.$$

the bases ABCDE, *abcde* being similar figures, we have CDE : *abcde* : : AB^2 ; ab^2. (Prop. XVII. *Sch. El. Geom.* ltiply the corresponding terms of these two proportions; e results the proportion,

$$ABCD \times \tfrac{1}{3}SO : abcde \times \tfrac{1}{3}So :: AB^3 : ab^3.$$

w ABCDE×$\tfrac{1}{3}$SO is the solidity of the pyramid SABCDE *abcde*×$\tfrac{1}{3}$S*o* is that of the pyramid S*abcde* (Prop. XXV. .); hence two similar pyramids are to each other as the es of their homologous sides.

ınd since two similar polyedrons may be conceived to be le up of the same number of triangular pyramids respec- ly similar in each and similarly placed, we may easily in- that, any two similar polyedrons one to each other as the ȝes of their homologous sides.

SECTION II.—REGULAR POLYEDRONS.

DEFINITIONS.

There can only be five regular polyedrons.

For, *regular polyedrons* were defined as having equal regular polygons for their faces, and all their solid angles equal. These conditions cannot be fulfilled except in a small number of cases.

1. If the faces are equilateral triangles, polyedrons may be formed of them, having solid angles contained by three of those triangles, by four, or by five: hence arises three regular bodies, the *tetraedron*, the *octaedron*, the *icosaedron*. No other can be formed with eqilateral triangles; for six angles of such a triangle are equal to four right angles, and cannot form a solid angle.

2. If the faces are squares, their angles may be arranged by threes: hence results the *hexaedron* or *cube*. Four angles of a square are equal to four right angles, and cannot form a solid angle.

3. In fine, if the faces are regular pentagons, their angles likewise may be arranged by threes: the regular *dodecaedron* will result.

We can proceed no farther: three angles of a regular hexagon are equal to four right angles; three of a heptagon are greater.

Hence there can only be five regular polyedrons; three formed with equilateral triangles, one with squares, and one with pentagons.

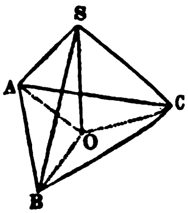
Scholium. In the following Proposition, we shall prove that these five polyedrons actually exist; and that all their dimensions may be determined, when one of their faces is known.

One of the faces of a regular polyedron being given, or only a side of it, to construct the polyedron.

This Problem subdivides itself into five, which we shall now solve in succession.

PROBLEM. I.

To construct a Tetraedron.



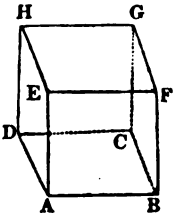
Let ABC be the equilateral triangle which is to form one face of the tetraedron. At the point O, the centre of this triangle, erect OS perpendicular to the plane ABC; terminate this perpendicular in S, so that AS=AB; join SB, SC: the pyramid SABC, will be the tetraedron required.

For, by reason of the equal distances OA, OB, OC, the oblique lines SA, SB, SC are equally removed from the perpendicular SO, and consequently equal. One of them SA=AB; hence the four faces of the pyramid SABC are triangles, equal to the given triangle ABC. And the solid angles of this pyramid are all equal, because each of them is formed by three equal plane angles: hence this pyramid is a regular tetraedron.

PROBLEM II.

To construct a hexaedron.

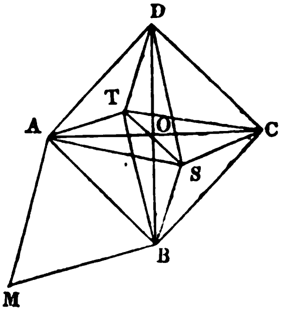
Let ABCD be a given square. On the base ACD, construct a right prism whose altitude AE shall be equal to the side AB. The faces of this prism will evidently be equal squares; and its solid angles all equal, each being formed with three right angles: hence this prism is a regular hexaedron or cube.



PROBLEM III.

To construct an octaedron.

Let AMB be a given equilateral triangle. On the side AB, describe a square ABCD; at the point O, the centre of this square, erect TS perpendicular to its plane, and terminating on both sides in T and S, so that OT=OS=OA; then join SA, SB, TA, &c.: you will have a solid SABCDT, composed of two quadrangular pyramids SABCD, TABCD, united together by their common base ABCD; this solid will be the required octaedron.



For, the triangle AOS is right-angled at O, and likewise the triangle AOD; the sides AO, OS, OD are equal; hence those triangles are equal, hence AS=AD. In the same manner, we could show that all the other right-angled triangles AOT, BOS, COT, &c., are equal to the triangle AOD; hence all the sides AB, AS, AT, &c., are equal. and therefore the solid SABCDT is contained by eight triangles, each equal to the given equilateral triangle ABM. We have yet to show that the solid angles of this polyedron are equal to each other; that the angle S, for example, is equal to the angle B.

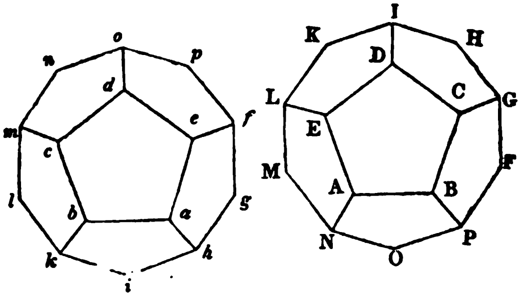
Now, the triangle SAC is evidently equal to the triangle DAC, and therefore the angle ASC is right; hence the figure SATC is a square equal to the square ABCD. But, comparing the pyramid BASCT with the pyramid SABCD, the base ASCT of the first may be placed on the base ABCD of the second; then, the point O being their common centre, the altitude OB of the first will coincide with the altitude OS of the second; and the two pyramids will exactly apply to each other in all points; hence the solid angle S is equal to the solid angle B; hence the solid SABCDT is a regular octaedron.

Scholium. If three equal straight lines, AC, BD, ST are perpendicular to each other, and bisect each other, the extremities of these straight lines will be the vertices of a regular octaedron.

PROBLEM IV.

To construct a dodecaedron.

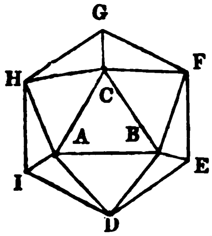
Let ABCDE be a given regular pentagon; let ABP, CBP be two plane angles, each equal to the angle ABC. With these plane angles form the solid angle B; and call the mutual inclination of those planes K. In like manner, at the points C, D, E, A, form solid angles equal to the solid angle B, and similarly situated: the plane CBP will be the same as the plane BCG, since both of them are inclined at an equal angle K to the plane ABCD: hence in the plane PBCG, we may describe the pentagon BCGFP, equal to the pentagon ABCDE. If the same thing is done in each of the other planes CDI, DEL, &c., we shall have a convex surface PEGH, &c., composed of six regular pentagons, all equal, and each inclined to

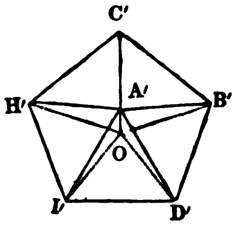


its adjacent plane by the same quantity K. Let *pfgh*, &c., be a second surface equal to PFGH, &c.; we assert that these two surfaces may be joined so as to form only a single continuous convex surface. For the angle *opf*, for example, may be joined to the two angles OPB, BPF, to make a solid angle P equal to the angle B; and in this junction, no change will take place in the inclination of the planes BPF, BPO, that inclination being already such as is required to form the solid angle. But whilst the solid angle P is forming, the side *pf* will apply itself to its equal PF, and at the point F will be found three plane angles PFG, *pfe*, *efg*, united, and forming a solid angle equal to each of the solid angles already formed: and this junction, like the former, will take place without producing any change either in the state of the angle P or in that of the surface *efgh*, &c.; for the planes PFG, *efp* already joined at P, have the requisite inclination K, as well as the planes *efg*, *efp*. Continuing the comparison, in this way, by successive steps, it appears that the two surfaces will adjust themselves completely to each other, and form a single continuous convex surface; which will be that of the regular dodecaedron, since it is composed of twelve equal regular pentagons, and has all its solid angles equal.

PROBLEM V.

To construct an icosaedron.





Let ABC be one of its faces. We must first form a solid angle with five planes, each equal to ABC, and each equally inclined to its adjacent one. To effect this, on the side B'C', equal to BC, construct the regular pentagon B'C'H'I'D'; at the centre of this pentagon, draw a line at right angles to its plane, and terminating in A', so that B'A'=B'C'; join A'C', A'H', A'I', A'D': the solid angle A' formed by the five planes B'A'C', C'A'H', &c., will be the solid angle required. For the oblique lines A'B', A'C', &c., are equal; one of them, A'B', is equal to the side B'C'; hence all the triangles B'A'C', C'A'H', &c., are equal to each other and the given triangle ABC.

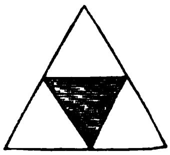
It is further manifest that the planes B'A'C', C'A'H', &c., are each equally inclined to their adjacent planes; for the solid angles B′, C′, &c., are all equal, being each formed by two angles of equilateral triangles, and one of a regular pentagon. Let the inclination of the two planes, in which are the equal angles, be named K; the angle K will at the same time be the inclination of each of the planes composing the solid angle A′ to their adjacent planes.

This being granted, if at each of the points A, B, C, a solid angle be formed equal to the angle A′, we shall have a convex surface DEFG, &c., composed of ten equilateral triangles, every one of which will be inclined to its adjacent triangle by the quantity K; and the angles D, E, F, &c., of its contour will alternately combine three angles and two angles of equilateral triangles. Conceive a second surface equal to the surface DEFG, &c., these two surfaces will adapt themselves to each other, if each triple angle of the one is joined to each double angle of the other; and, since the planes of these angles have already the mutual inclination K, requisite to form a quintuple solid angle equal to the angle A, there will be nothing changed in this junction by the state of either surface, and the two together will form a single continuous surface, composed of twenty equilateral triangles. This surface will be that of the regular icosaedron, since all its solid angles are likewise equal.

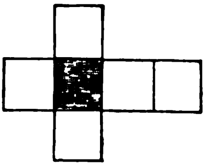
Formation of models of the five regular solids.

The learner will obtain far more definite ideas of the regular solids from models than from diagrams. These he can easily make for himself, from a card or a pice of pasteboard.

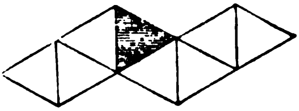
1. To make a *tetraedron*, draw an equilateral triangle on a card, and divide it into four equilateral triangles; trim off the card to the sides of the large triangle, and then cut about half through the card on the line between each of the triangles; finally, bend the partial triangles back, the part not cut serving as a hinge, fold them up, so that their edges will touch each other, and you will have a model of the regular solid required.



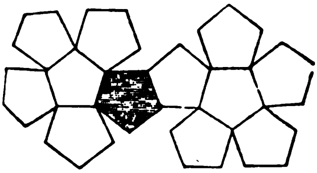
The shaded triangle in the diagram being the hinge, is designed to represent the base of the solid. The parts may be easily kept in place, by touching the edges with a little glue or paste.



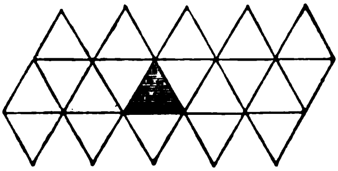
2. To make a *hexaedron*, draw on a card or pasteboard, six equal squres; trim the diagram; cut the lines between them, bend up the parts, &c., as before, and a model of the hexaedron will be completed.



3. To make the *octaedron*, draw a diagram containing eight equal and equilateral triangles; cut the boundary lines between them, and proceed as before.



4. To make the *dodecaegon*, draw a diagram of twelve equal pentagons, cut the boundary lines, &c., and proceed as before.



5. To make the *isosaedron*, draw a diagram of twenty equal and equilateral triangles, and proceed as before.

PROPOSITION VI. THEOREM.

Every polyedron may be divided into as many pyramids as the polyedron has faces, the faces of which shall form their bases; all referred to a common point within for their vertices, and the sum of the pyramids shall be equal to the whole polyedron.

For conceive lines to be drawn from each vertice of the polyedron, to a common point within; then let planes be made to pass through each plane angle made by the meeting of the faces with each other, the planes all meeting at the common point within, and terminated by the lines drawn from the virtices to this point. Then will the polyedron be divided into the same number of pyramids as the polyedron has faces, the bases of which are the faces of the polyedron, and their vertices the common point at which their planes of intersection all meet. For a pyramid (Def. 11.) includes all the space between a surface called the base and a point called the vertice of the pyramid, and this space is determined by planes passed from the edges of the base to this point. Now, as these planes are supposed to intersect each edge or plane angle on the surface of the polyedron, and as each plane forms a division between two portions of the solid and is common to them both, it follows that the solid is all divided by those planes into as many portions as the polyedron has faces; and as these planes all terminate in a point, it follows that it is all divided into portions included severally between surfaces and points, hence into pyramids.

Cor. 1. It matters not whether the polyedron is regular or irregular or whatever the number of faces, the whole polyedron would be equal to as many pyramids, with bases equal to those faces, and all with vertices referred to a common point within.

Cor. 2. If the polyedron is regular, the pyramids into which it is conceived to be divided, all have the same altitude; hence its solidity is equal to the sum of all its faces multiplied by $\frac{1}{3}$ of their common altitude.

GENERAL SCHOLIUM.

The chief propositions of this book relating to the surfaces and solidities of polyedrons may be exhibited in algebraic terms, and recapitulated in a brief manner.

1. Let P be the perimeter of a prism and H its altitude, and the convex surface will be $P \times H$ or PH.

2. Let B represent the base of a prism and H its altitude, then the solidity of the prism will be BH.

Or let F represent one of the rectangular faces of a triangular prism as its base and A the altitude of its opposite angle; then $F \times \frac{1}{2}A =$ its solidity.

Or when the equal bases of the prism are regular polygons, let $C =$ the convex surface and R the radius of the circle inscribed in the base, then will the solidity be $C \times \frac{1}{2}R$.

3. Let P be the perimeter of the base of a regular pyramid and S the slant height, and the convex surface will be $P \times \frac{1}{2}S$.

Or if $R =$ the radius of the circle inscribed in the base $C =$ the convex surface and $A =$ the area of the base, then $A : C :: R : S$.

Or let $C' =$ the convex surface of the circumscribed prism, and C= the convex surface of the pyramid, H= the vertical height and S the slant height, we have S : H : : C : $\frac{1}{2}$C′.

4. And the solidity of the pyramid would be B$\times\frac{1}{3}$H.

And if D= the distance of the centre of the base from one of its sides, C= convex surface then may we make the solidity =C$\times\frac{1}{3}$D.

5. Let H represent the altitude of a frustrum of a pyramid having parallel bases B and E, $\surd$BE will be the mean proportional bases between those bases; and the solidity of the frustum will be $\frac{1}{3}$H$\times$(A+B+$\surd$BE.)

Also, we may take (Prop. XXIX. Cor. 2.) C$\times\frac{1}{3}$D+E$\times\frac{1}{3}$H= the solidity of the frustum.

6. In fine, let Q and q represent two similar polyedrons, G and g two homologous edges, then we may have Q : q : : G^3 : g^3.

7. If S represent the sum of all the faces of a regular polyedron, and H the distance of each face to a common point within as its centre, $\frac{1}{3}$HS= its solidity.

BOOK III.

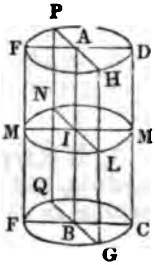
OF THE FOUR ROUND BODIES.

DEFINITIONS.

1. A *cylinder* is the solid generated by the revolution of a rectangle ABCD, conceived to turn about the immovable side AB.

In this movement, the sides AD, BC, continuing always perpendicular to AB, describe equal circles DHP, CGQ, which are called the *bases of the cylinder*, the side CD at the same time describing the *convex surface.*

The immoveable line AB is called the *axis of the cylinder*.



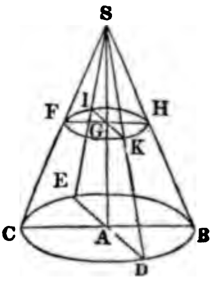
Every section KLM, made in the cylinder, at right angles to the axis, is a circle equal to either of the bases; for, whilst the rectangle ABCD turns about AB, the line KL, perpendicular to AB, describes a circle, equal to the base, and this circle is nothing else than the section made perpendicular to the axis at the point I.

Every section PQG, made through the axis, is a rectangle double of the generating rectangle ABCD.

The Archimedean cylinder is that which circumscribes a sphere, or whose altitude is equal to the diameter of its base.

2. A *cone* is the solid generated by the revolution of a right-angled triangle SAB, conceived to turn about the immoveable side SA.

In this movement, the side AB describes a circle ABCD, named the *base of the cone;* the hypothenuse SB describes the *convex surface of the cone.*



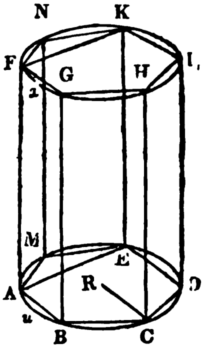
The point S is named the *vertex of the cone*, SA the *axis* or the *altitude*, and SB the *side* or the *apothem*.

Every section HKFI, at right angles to the axis, is a circle; every section SDE, through the axis, is an isosceles triangle, double of the generating triangle SAB.

3. If from the cone S-CDB, the cone S-FKH be cut off by a plane parallel to the base, the remaining solid CBHF is called a *truncated cone*, or the *frustum of a cone*.

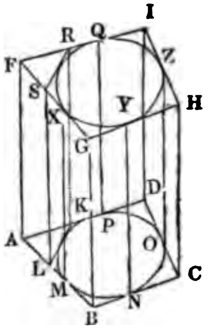
We may conceive it to be generated by the revolution of a peziod ABHG, whose angles A and G are right angles, out the side AG. The immoveable line AO is called the is or *altitude of the frustum*, the circles BDC, HFK, are its ses, and BH its *side*.

4. Two cylinders, or two cones, are *similar*, when their es are to each other as the diameters of their bases.



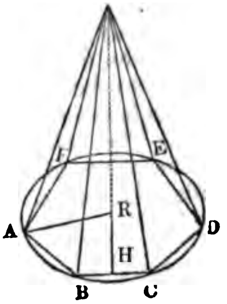
5. If in the circle ACD, which forms the se of a cylinder, a polygon ABCDE be scribed, a right prism, constructed on this se ABCDE, and equal in altitude to the linder, is said to be *inscribed in the cylin- r*, or the cylinder to be *circumscribed out the prism*.

The edges AF, BG, CH, &c., of the prism, ing perpendicular to the plane of the base, evidently included in the convex sur- e of the cylinder; hence the prism and cylinder touch one another along these ges.



6. In like manner, if ABCD is a polygon, cumscribed about the base of a cylin- r, a right prism, constructed on this base CD, and equal in altitude to the cylin- r, is said to be *circumscribed about the inder*, or the cylinder to be *inscribed in prism*.

Let M. N, &c., be the points of contact the sides AB, BC, &c.; and through the nts M, N, &c., let MX, NY, &c., be wn perpendicular to the plane of the e: these perpendiculars will evidently both in the surface of the cylinder, and hat of the circumscribed prism; hence they will be their s of contact.



7. If in the circle ACE, which forms base of a cone, any polygon ABCDEF inscribed, and from the vertices A, B, D, E, F, lines be drawn to S, the ver- of the cone, these lines may be re- ded as the edges of a pyramid whose e is the polygon ABCDEF and vertex The sides of this pyramid are in the vex surface of the cone, and the pyra- d is said to be *inscribed* in the cone.

8. A *conesected cone* is one containing a conical cavity of equal base, and less altitude than that of the solid.

9. A *conesected cylinder* is one containing a conical cavity of equal base and altitude with the cylinder.

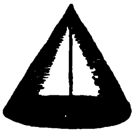
10. A *conesected frustum* is one containing a conical cavity whose base is one of the bases of the frustum, and whose altitude the altitude of the frustum.

11. A *polyedroid* is a solid formed by the revolution of a semipolygon about its axis.

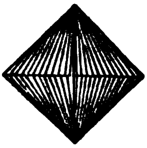
12. A regular *polyedroid* is one formed by the revolution of a regular *semipolygon* about its axis.

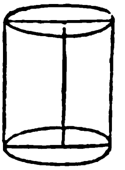
13. *Polyedroids* receive *particular names* according to the number of sides of the polygons by which they are supposed to be generated.

Thus the cone is a *tetredroid*, a vertical section of which through its axis being a triangle.

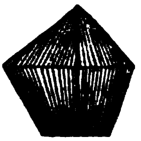


A *Quadredroid* may be a cylinder or a biconical solid. Thus the vertical section of a cylinder through its axis is a rectangle, so also the vertical section through a biconical solid may be a rectangle, and must necessarily be a quadrilateral.





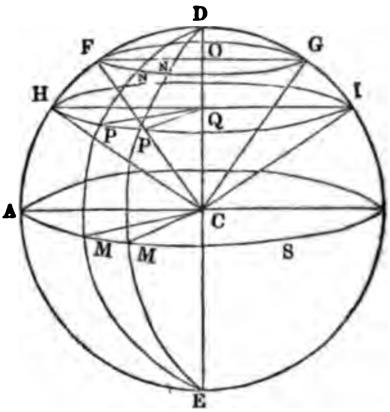
A *Pentadroid* is a solid formed by the revolution of a pentagon, about an axis perpendicular to one of its sides, and terminating in its opposite angle.



And generally any *polyedroid* formed by the revolution of a polygon of an odd number of sides, will always have a base or one plane face perpendicular to the axis, and one conical vertice.

And any *polyedroid* formed by the revolution of a polygon of an even number of sides, may have two plane faces perpendicular to the axis, one at each extremity; or the axis may terminate in a conical vertice at each extremity.

14. The *sphere* is a solid terminated by a curved surface, all the points of which are equally distant from a point within, called the *centre*.



The sphere may be conceived to be generated by the revolution of a semicircle DAE about its diameter DE: for the surface described in this movement, by the curve DAE, will have all its points equally distant from its centre C.

15. Whilst the semicircle DAE revolving round its diameter DE, describes the sphere; any circular sector, as DCF or FCH, describes a solid, which is named a *spherical sector.*

16. The *radius of a sphere* is a straight line drawn from the centre to any point of the surface; the *diameter* or *axis* is a line passing through the centre, and terminated on both sides by the surface.

All the radii of a sphere are equal; all the diameters are equal, and each double of the radius.

17. It will be shown that every section of the sphere, made by a plane, is a circle: this granted, a *great circle* is a section which passes through the centre; a *small circle*, is one which does not pass through the centre.

18. The *circumference* of a sphere is the same as that of a great circle.

19. A *plane* is *tangent* to a sphere, when their surfaces have but one point in common.

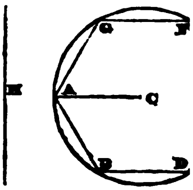
20. A *zone* is a portion of the surface of a sphere included between two parallel planes, which form its *bases.* One of these planes may be tangent to a sphere; in which case, the zone has only a single base.

21. A *spherical segment* is that portion of the solid sphere, included between two parallel planes which form its bases. One of these planes may be tangent to the sphere; in which case, the segment has only a single base.

22. The *altitude of a zone* or *of a segment* is the distance between the two parallel planes, which form the bases of the zone or segment.

PROPOSITION I. THEOREM.

The convex surface of a cylinder is equal to the circumfe of its base multiplied by its altitude.



Let CA be the radius of the given cylinder's base, and H its altitude: the circumference whose radius is CA being represented by *circ.* CA, we are to show that the convex surface of the cylinder is equal to *circ.* CA×H.

Inscribe in the circle any regular polygon, BDEFGA, and construct on this polygon a right p having its altitude equal to H, the altitude of the cylin this prism will be inscribed in the cylinder. The convex face of the prism is equal to the perimeter of the polygon, tiplied by the altitude H (Prop. VIII. B. II.). Let now arcs which subtend the sides of the polygon be contin bisected, and the number of sides of the polygon indefin increased: the perimeter of the polygon will then bec equal to *circ.* CA (Prop. XII. *Cor.* 2. *El. Geom.*), and the vex surface of the prism will coincide with the convex face of the cylinder. But the convex surface of the pris equal to the perimeter of its base multiplied by H, what be the number of sides: hence, the convex surface of cylinder is equal to the circumference of its base multiplie its altitude.

PROPOSITION II. THEOREM.

The solidity of a cylinder is equal to the product of its bas its altitude.

Let H (see diagram to Prop. I.) be the altitude of the g cylinder; CA the radius of its base; and let the area o base be represented by *area* CA; it is to be shown that solidity of the cylinder is equal to *area* CA×H.

Inscribe in the cylinder a right prism whose altitude is e to that of the cylinder. The solidity of the prism wil equal to the area of its base multiplied by its altitud (Prop. XVI. B. II.). Let the number of sides of the poly be indefinitely increased: the solidity of the prism will stil equal to its base multiplied by its altitude. But when the n ber of sides of the polygon is indefinitely increased, its is equal to the *area* CA, and its perimeter coincides with CA (Prop. XI. *Cor.* 2. and 3. B. V. *El. Geom.*), the inscr prism then coincides with the cylinder since their altitudes are equal, and their convex surfaces perpendicular to the common base; hence the two solids will be equal; therefore the solidity of a cylinder is equal to the product of its base by its altitude.

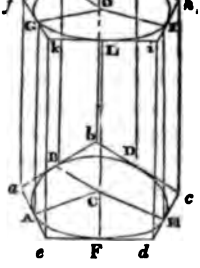
Cor. 1. Cylinders of the same altitude are to each other as their bases; and cylinders of the same base are to each other as their altitudes.

Cor. 2. Similar cylinders are to each other as the cubes of their altitudes, or as the cubes of the diameters of their bases. For the bases are as the squares of their diameters; and the cylinders being similar, the diameters of their bases are to each other as the altitudes (Def. 4.); hence the bases are as the squares of the altitudes; hence the bases, multiplied by the altitudes, or the cylinders themselves, are as the cubes of the altitudes.

Scholium. Let R be the radius of a cylinder's base; H the altitude; the surface of the base will be $\pi.R^2$ (Prop. XV. *Cor.* 2. *El. Geom.*); and the solidity of the cylinder will be $\pi.R^2 \times H$, or $\pi.R^2.H$.

PROPOSITION III. THEOREM.

The solidity of a cylinder is equal to the product of its convex surface multiplied by half the radius of its base.



Let ABDEF-G be the surface of the cylinder, and let AC be the radius of the circle forming the base; then the solidity will be equal to the product of the convex surface multiplied by one-half of the radius. For, circumscribe about the circle forming the base of the cylinder any regular polygon *abcde-f*, and construct on this polygon a right prism, having its altitude equal to the altitude of the cylinder. The solidity of this prism will be equal to the vertical surface of the prism, multiplied by a line drawn from the centre of the base of the polygon to the middle of one of its sides, as CA. (Prop. XVIII. *Cor.* 1. B. II.)

Let the number of the sides of the polygon be infinitely increased, the solidity of the prism will still be equal to the vertical surface multiplied by the half line CA, the radius.

But when the number of sides of the polygon become infinitely increased, its perimeter becomes equal to the circumference of the circle CA, and coincides with it (Prop. XV. B. V. *El. Geom.*) The circumscribed prism then coincides with the cylinder, since their altitudes are equal, and their convex surfaces perpendicular to the common base: hence the two solids will be equal. Therefore the solidity of a cylinder is equal to the product of the convex surface multiplied by half the radius of the base.

Cor. Hence any section of a cylinder made by planes passing through the axis, or meeting in the axis and forming a plane angle, is equal to the product of its convex surface, multiplied by half the radius of the base.

Thus, if the planes AGOC and BHOC meet in the axis OC of the cylinder, the section ABCGHO cut out by these planes, is equal to the convex surface ABHG, multiplied by half the radius CA of the base.

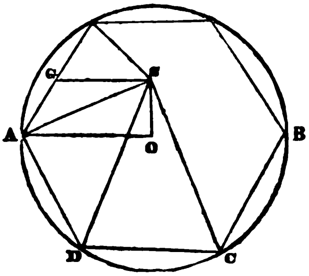
Scholium. As one of the triangular prisms constituting a polygonal prism is elementary to the polygonal prism (Prop. XVIII. *Schol.* 2. B. II.), so any section of the cylinder made by planes passing through the axis may be regarded as elementary portions of the cylinder; for similar portions of an adequate number constitute the cylinder: and we have seen that the section retains the same relation with regard to its solidity as the whole cylinder; viz: their solidities are both equal to their curve surfaces, multiplied by half the radius of curvature of those surfaces.

Let ABGH be the base of the elementary portion of a cylinder, and AC the vertical height of the opposite plane angle, then this will be equal to the triangular prism of the same vertical height and equivalent base, from the above demonstration and carollary, together with Prop. XVII. B. II.

Hence, as we have two orders of lines terminating a plane surface, which, by being multiplied by their distance from a certain point which may be called the vertice of the surface, will give the area of the surface (Prop. XVI. *Sch.* 3. B. V. *El. Geom.*), so we have two orders of surfaces, bounding a solid, which, by being multiplied by the distance from such surfaces, to an opposite vertical angle, will give the value of such solid. But as in the circle, so in this cylindric surface, the vertice to which this is referred must be in the centre of curvature of the cylindric surface.

PROPOSITION IV. THEOREM.

The convex surface of a cone is equal to the circumference of its base, multiplied by half its side.



Let the circle ADCB be the base of a cone, S the vertex, SO the altitude, and SA the side: then will its convex surface be equal to *circ.* $OA \times \frac{1}{2}SA$.

For, inscribe in the base of the cone any regular polygon ADCB, and on this polygon as a base conceive a pyramid to be constructed, having S for its vertex: this pyramid will be a regular pyramid, and will be inscribed in the cone.

From S, draw SG perpendicular to one of the sides of the polygon. The convex surface of the inscribed pyramid is equal to the perimeter of the polygon which forms its base, multiplied by half the slant height SG (Prop. XX. B. II.) Let now the number of sides of the inscribed polygon be indefinitely increased; the perimeter of the inscribed polygon will then become equal to *circ.* OA, the slant height SG will become equal to the side SA of the cone, and the convex surface of the pyramid to the convex surface of the cone. But whatever be the number of sides of the polygon which forms the base, the convex surface of the pyramid is equal to the perimeter of the base, multiplied by half the slant height: hence the convex surface of a cone is equal to the circumference of the base, multiplied by half the side.

Scholium. Let L be the side of a cone, R the radius of its base; the circumference of this base will be $2\pi.R$, and the surface of the cone will be $2\pi R \times \frac{1}{2}L$, or πRL.

PROPOSITION V. THEOREM.

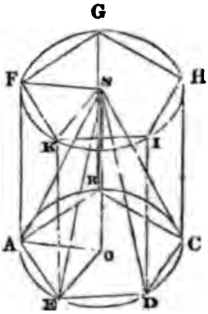
The convex surface of a right cone is to the area of its base, as the slant height of the cone is to the radius of the base.

For, since the area of the base is equal to its circumference, multiplied by half the radius, (Prop. XV. B. V. *El. Geom.*) and since the convex surface of the cone (Prop. IV.) is equal to the same circumference multiplied by its slant height, these surfaces must be proportional to each other, as the radius of the base and slant height of the cone; seeing the surface of the cone is equal to the circumference of the base, repeated half as many times as there are units in its slant height, and the area of the base is equal to the same circumference repeated as often as there are units in half the radius.

Thus, if R=the radius of the base, S=the slant height of the cone, C=the convex surface, and A=the area of the base, then C : A : : S : R.

PROPOSITION VI. THEOREM.

The convex surface of a cone exceeds half the convex surface of its circumscribing cylinder, in the same ratio that the length of the slant side of the cone exceeds its vertical height.

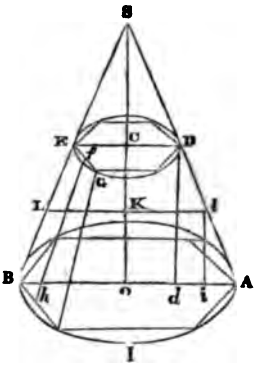


Let ABCDE-S be a cone, inscribed in the cylinder ABCDE-F, then will the convex surface of the cone ABCDE-S exceed half that of the cylinder ABCDE-F, in the same ratio that the slant height of the cone AS exceeds its vertical height OS.

Let there be a regular polygon ABCDE described in the base of the cylinder and cone; then, on this polygon as bases, describe a prism, whose vertical height is equal to that of the cylinder, and a pyramid, whose vertical height is equal to that of the cone, according to Prop. XXIII. B. II., the surface of the pyramid, exceeds half the surface of the prism, in the same ratio that the length of the line AS=the slant height of the pyramid exceeds the length of the line OS, its vertical height. Now, if the number of sides of the polygon is indefinitely increased, the prism erected on it as a base becomes identical with the cylinder, and the pyramid with the cone; hence the curve surface of the cone, is to half the curve surface of the cylinder as the length of the line AS=the slant height of the cone to the vertical height OS.

PROPOSITION VII. THEOREM.

The convex surface of the frustum of a cone is equal to its side, multiplied by half the sum of the circumferences of its two bases.



Let BIA-DE be a frustum of a cone; then will its convex surface be equal to $\mathrm{AD}\times\left(\frac{circ.\mathrm{OA}+circ.\mathrm{CD}}{2}\right)$

For, inscribe in the bases of the frustum two regular polygons, of the same number of sides, and having their homologous sides parallel, each to each. The lines joining the vertices of the homologous angles may be regarded as the edges of the frustum of a regular pyramid inscribed in the frustum of the cone. The convex surface of the frustum of the pyramid is equal to half the sum of the perimeters of its bases multiplied by the slant height fh (Prop. XX. *Cor.* B. II.)

Let now the number of sides of the inscribed polygons be indefinitely increased; the perimeters of the polygons will become equal to the circumferences BIA, EGD; the slant height fh will become equal to the side AD or BE, and the surfaces of the two frustums will coincide and become the same surface.

But the convex surface of the frustum of the pyramid will still be equal to half the sum of the perimeters of the upper and lower bases multiplied by the slant height; hence the surface of the frustum of a cone is equal to its side multiplied by half the sum of the circumferences of its two bases.

Cor. Through l, the middle point of AD, draw lKL parallel to AB, and li, Dd, parallel to CO. Then, since Al, lD, are equal, Ai, id, will also be equal (Prop. XIV. *Cor.* 2, B. IV. *El. Geom.*;) hence. Kl is equal to $\frac{1}{2}$(OA+CD.) But since the circumferences of circles are to each other as their radii (Prop. XIV. B. V. *El. Geom.*) the *circ.* $\mathrm{K}l=\frac{1}{2}(circ.\,\mathrm{OA}+circ.\,\mathrm{CD}$;) therefore, *the convex surface of a frustum of a cone is equal to its side multiplied by the circumference of a section at equal distances from the two bases.*

Scholium. If a line AD, lying wholly on one side of the line OC, and in the same plane, make a revolution around OC, the surface described by AD will have for its measure AD× $\left(\frac{\text{circ.AO}+\text{circ.DC}}{2}\right)$, or AD×circ.IK; the lines AO, DC, IK, being perpendiculars let fall from the extremities and from the middle point of AD, on the axis OC.

For, if AD and OC are produced till they meet in S, the surface described by AD is evidently the frustum of a cone having AO and DC for the radii of its bases, the vertex of the whole cone being S. Hence this surface will be measured as we have said.

This measure will always hold good, even when the point D falls on S, and thus forms a whole cone; and also when the line AD is parallel to the axis, and thus forms a cylinder. In the first case, DC would be nothing; in the second, DC would be equal to AO, and to IK.

PROPOSITION VIII. THEOREM.

The solidity of a cone is equal to its base multiplied by a third of its altitude.

Let SO (see diagram to Prop. IV.) be the altitude of a cone, OA the radius of its base, and let the area of the base be designated by *area* OA; it is to be proved that the solidity of the cone is equal to *area* OA×$\frac{1}{3}$SO.

Inscribe in the base of the cone any regular polygon ADCB, and join the vertices A, D, C, &c. with the vertex S of the cone; then will there be inscribed in the cone a regular pyramid having the same vertex as the cone, and having for its base the polygon ADCB. The solidity of this pyramid is equal to its base multiplied by one-third of its altitude, (Prop. XXVI. B. II.) Let now the number of sides of the polygon be indefinitely increased; the polygon will then become equal to the circle, and the pyramid and cone will coincide and become equal. But the solidity of the pyramid is equal to its base multiplied by one-third of its altitude, whatever be the number of sides of the polygon which forms its base: hence the solidity of the cone is equal to its base multiplied by a third of its altitude.

Cor. A cone is the third of a cylinder having the same base and the same altitude; whence it follows,

1. That cones of equal altitudes are to each other as their bases:

2. That cones of equal bases are to each other as their altitudes;

3. That similar cones are as the cubes of the diameters of their bases, or as the cubes of their altitudes.

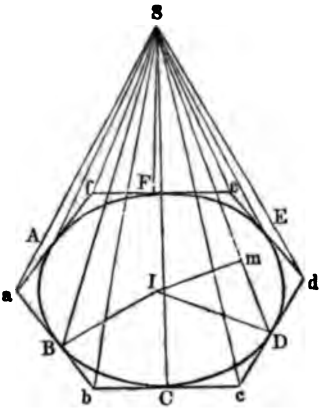
Cor. 3. The solidity of a cone is equivalent to the solidity of a pyramid having an equivalent base and the same altitude, (Prop. XXVI. B. II.)

Scholium. Let R be the radius of a cone's base, H its altitude; the solidity of the cone will $\pi R^2 \times \frac{1}{3}H$, or $\frac{1}{3}\pi R^2 H$.

PROPOSITION IX. THEOREM.

The solidity of a right cone is equal to its convex surface multiplied by one-third of a line drawn from the centre of its base, perpendicular to the surface of the cone.

Let ABCDEF-S be a cone: from the centre I of the base, draw the line I*m* perpendicular to the side, and the solidity of the cone will be equal to its curve surface multiplied by one-third of the line I*m*.

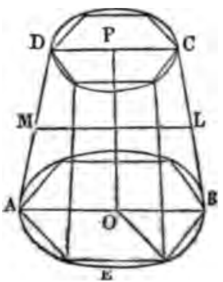


Circumscribe a regular polygon about the base of the cone; then, on this as a base, with the vertices, describe the pyramid *abcdef*-S, and the solidity of the pyramid, by Prop. XXVII. B. II., will be equal to its convex surface multiplied by one-third of the radius, I*m*, drawn from the centre of the base, perpendicular to one of its plane faces.

Now, if the number of the sides of the pyramid's base become infinitely increased, the pyramid will become assimilated to the cone. Hence, the solidity of a cone is equal to its curve surface multiplied by one-third of a line drawn perpendicular from the centre of the base to the lateral surface of the cone

PROPOSITION X. THEOREM.

The solidity of the frustum of a cone is equal to the sum of the solidities of three cones whose common altitude is the altitude of the frustum, and whose bases are, the upper bases of the frustum, the lower base of the frustum, and a mean proportional between them.



Let AEB-CD be the frustum of a cone, and OP its altitude; then will the solidity be equal to

$$\tfrac{1}{3}\pi \times OP \times (AO^2+DP^2+AO \times PD).$$

For, inscribe in the lower and upper bases two regular polygons having the same number of sides, and having their homologous sides parallel, each to each. Join the vertices of the homologous angles, and there will then be inscribed in the frustum of the cone, the frustum of a regular pyramid. The solidity of the frustum of the pyramid is equivalent to three pyramids having the common altitude of the frustum, and for bases, the lower base of the frustum, the upper base of the frustum, and a mean proportional between them, (Prop. XXVIII. B. II.)

Let now the number of sides of the inscribed polygon be indefinitely increased: the bases of the frustum of the pyramid will then coincide with the bases of the frustum of the cone, and the two frustums will coincide and become the same solid. Since the area of a circle is equal to $R^2.\pi$, (Prop. XV. *Cor.* B. V. *El. Geom.*) the expression for the solidity of the frustum will become

for the first pyramid $\tfrac{1}{3}OP \times OA^2.\pi$

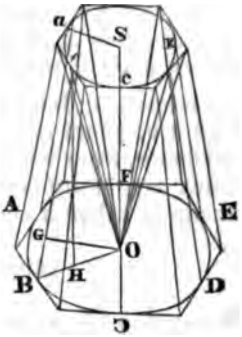
for the second $\tfrac{1}{3}OP \times PD^2.\pi$

for the third $\tfrac{1}{3}OP \times AO \times PD.\pi$; since $AO \times PD.\pi$ in a mean proportional between $AO^2.\pi$ and $PD^2.\pi$. Hence the solidity of the frustum of the cone is measured by $\tfrac{1}{3}\pi OP \times (AO^2+PD^2+AO \times PD)$.

PROPOSITION XI. THEOREM.

The solidity of the frustum of a cone is equal to its convex surface multiplied by one-third of a line drawn from the centre of the base perpendicular to the curve surface, plus the solidity of a cone whose base is the smaller base of the frustum, and whose vertical height is the vertical height of the frustum.

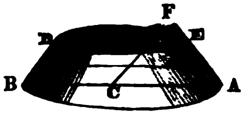
Let ABCDF-*a* be the frustum of a cone, OS its altitude, G a line drawn from the centre of its larger base perpen-icular to the surface; then will its lidity be equal to the convex surface ultiplied by one-third OG plus the one *ac*K-O. For circumscribe about e larger and smaller bases regular olygons having the same number of des, and their homologous sides par-lel each to each. Join the vertices f the homologous angles, and there ill then be circumscribed about the ustum of the cone the frustum of a re-ular pyramid. The solidity of the frus-um of the pyramid is equivalent to two yramids, one having the smaller base f the circumscribed pyramid for its base, and the vertical height f the frustum for its vertical height; and the other having he sum of the several exterior vertical surfaces for its base, nd a line drawn from the centre of the larger base perpendi-ular to one of its facial sides for its vertical height, (Prop. XIX. B. II.)



Now let the number of sides of the circumscribed polygon e indefinitely increased; then the bases of the frustum of the yramid will coincide with the bases of the cone; and the wo frustums will coincide and become the same solid, since he area of a circle coincides with the area of the circumscrib-d polygon, when the number of sides of the polygon become ndefinitely increased.

Cor. Let the cone *ac*KO be taken from the curve frustum ABCDEF*ac*K, and the part remaining will be a conesected rustum of a cone. Hence the solidity of a conesected frustum f a cone is equal to its external convex surface multiplied by ne-third of its distance from the vertice of the conical cavity ithin.

Scholium. 1. The line OG, is to the radius of the base OB as S or H*f*, the vertical height of the frustum, is to its slant eight B*f*. For from the point *f* the centre of one of the sides f the polygon of the smaller base draw the line *f*H parallel o OS, and at right angles to the radius OB, then we have the wo right angled triangles *f*HB and OGB with the angle at B ommon. Therefore, the two triangles are similar; hence Prop. XVII. Cor. 1. and 6. B. IV. *El. Geom.*) OG : OB : : S or H*f* : B*f*.

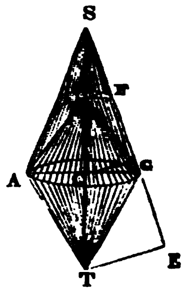


Scholium 2. If the frustum ABDE is such that a perpendicular cannot be drawn from AE to meet the centre C of the base, then it will be necessary to take the line CF perpendicular to AE produced.

Scholium. 3. Let the smaller base of the frustum be increased till it becomes equal to the larger, and the frustum becomes a cylinder, from which, if a cone be taken whose base and vertical height are equal to those of the cylinder, the part that remains is called a conesected cylinder.

PROPOSITION XII. THEOREM.

Let any point be assumed in the axis of a conic surface, or the axis produced downward from the base, and if a line be drawn from that point to meet the convex surface perpendicularly, at the point of contact, or that surface produced, if necessary, then the solid included between the surface and the point will be equal to one-third of the product of the surface into the line so drawn perpendicular to such surface.



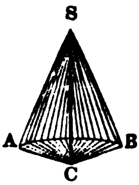
Let AGS be a conic surface, and let I be some point assumed in the axis CS, drawn perpendicular to the surface at the point F. Then the solid SAIGS will be equal to one-third of the product of the conical surface, multiplied by the distance IF.

For, let us suppose that, about the conical surface ASG, a pyramidal surface be described, then it is evident that each of the lateral faces of this surface would form the base of a pyramid, whose vertice would be referred to the point I, and IF would be its altitude. Hence the solidity of each would be equal to one-third of the product of such lateral face as a base into its altitude IF, on such base. Therefore, the sum of the solidities of those pyramids would be equal to the sum of their bases, or the whole convex surface, multiplied by one-third of IF, their common altitude.

Let the number of lateral faces be indefinitely increased, and the surface of the pyramid would become equal to, and coincide with, the conical surface: hence, then, the conesected cone AIGS is equal to one-third of the product of the conical surface into IF.

Again—let the axis SC be produced to T, and let T be the point assumed: produce SG to E, so as to form with ET a right angle: join AT and GT, forming the double cone or spindle, or conical quadredroid, SATGS, and this solid so described will be equal to the conical surface ASG, multiplied by one-third of the perpendicular TE, drawn to such surface produced, since the same reasons as have been applied to the former conditions may be applied to this also. Hence the proposition as enunciated is general.

Cor. Hence a cone, if instead of having a plane, has a conical base, as ACB, such that the slant side of SB is perpendicular to CB, then the solidity of the biconical solid, or conical spindle, will be equal to one-third of the product of the curve surface ASB, into the perpendicular CB, or is equal to one-third of the product of the convex surface ACB into SB, being perpendicular to such surface at the point of contact.



PROPOSITION XIII. THEOREM.

The solidity of a conesected cylinder is equal to the product of its external convex surface, multiplied by one-third of the radius of its base.

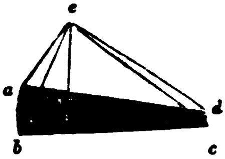
Let ABCDE-F (see diagram to Prop. VI.) be a conesected cylinder; then will its solidity be equal to its convex surface, multiplied by one-third of the radius AO of its base.

For (Prop. XI.) the solidity of the frustum of a cone is equal to the convex surface of the frustum, multiplied by one-third of the distance of this surface from the centre of the base, *plus* the solidity of a cone whose base is equal to the upper base of the frustum, and of a vertical height equal to that of the frustum: hence, if this cone is removed, the residual portion will be equal to its convex surface, multiplied by its distance from the centre of the base. And (Prop. XI. *Sch.* 3.) if the smaller base of the conical frustum is increased so as to be equal to the larger, the frustum becomes a cylinder. Hence, &c., according to the proposition.

Scholium 1. As (Prop. III. *Scholium*) sections of a cylinder cut out by planes parallel to and passing through the axis, may be regarded as elementary portions of the cylinder, so may portions cut out by similar planes from a conesected cylinder be regarded as elementary to the conesected cylinder.

Thus, the portion AESFK (see diagram to Prop. VI.) is an elementary portion of the conesected cylinder, it being one of a series of similar portions of which the conesected cylinder is composed.

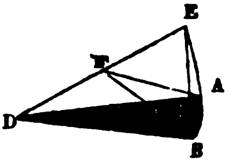
Scholium 2. And if a cone whose base is the smaller base of a conical frustum, and whose vertical height is equal to that of the frustum, be taken therefrom, we should then have a conesected frustum of a cone; which, if it be divided in like manner as the conesected cylinder (*Schol.* 1.) we shall have the elementary portions of the conesected frustum of a cone.



Thus, the portion *abcde* is an elementary portion of this solid, it being one of a series of similar portions forming the conesected frustum.

Scholium 3. And if a regular cone be divided by planes passing through its axis in a similar manner, it will also be divided into elementary portions of the cone.

Thus, ABDE is an elementary portion of the cone, the radius of whose base is equal to AE or EB, and vertical height equal to ED.

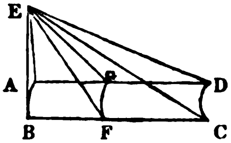


Scholium 4. Or, if a portion, ABDF, be taken, which is evidently an elementary portion of a conesected cone, the circumstances would be similar, and this portion would be a regular pyramidal.

Scholium 5. We have shown that each of these solids are equal to their curve surfaces, multiplied by one-third of their vertical heights on their curve surfaces as bases; and as, in this respect, they agree with regular pyramids, but their bases are cylindrical, or conical surfaces, I shall, to distinguish them from regular pyramids, and from each other, call them pyramidal elements of the bodies to which they belong. Thus, the pyramidal element of a conesected cylinder, the pyramidal element of a conesected frustum, and the pyramidal element of a cone. Hence they may be called cylindrical pyramidals, or conical pyramidals, as their bases are cylindrical or conical surfaces.

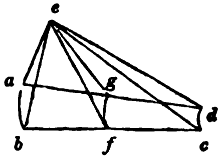
PROPOSITION XIV. THEOREM.

If any elementary portion, ABCDE, abcde, or ABCD (Prop. XII. *Sch.* 4.), *be divided in any manner by planes, or conical surfaces passing through their bases and vertices, each portion so cut off will still be an elementary portion of the solid from whence it is derived—viz: it will still be equal to the curve surface of its base, multiplied by one-third of the common vertical height.*



First—If the element of ABCDE, of the conesected cylinder, be divided by conical surface FGE, so as to make the portions of the base equal, then will the solid be divided in the same ratio.

For, conceive the base to become a portion of the surface of a prism of an indefinite number of sides, then we should have a solid composed of an indefinite number of pyramids, whose bases are indefinitely narrow, and equal in length to the base AD: and each of these pyramids may be divided by planes passing through their bases and vertices; and each of the new pyramids formed by thus dividing them, would be equal to the value of the base, multiplied by one-third of its vertical height, AE (Prop. XXVI. B. II.); and if these pyramids are all divided, and in whatever ratio, the sum of all of the parts of the series of bases included in the division ABFG, or CDFG, multiplied by one-third of the common altitude AE, will give the solidity of the portion included in the solid on such base. But the surface of a prism of an indefinite number of sides may be regarded as the surface of a cylinder. Hence, as in the proposition.



Second—In the section *abcde* of the conesected frustum, let the base *abcd* be divided by the conical surface *fge*, from the base to the vertex *e*, and the solid will be divided in the same ratio, for the same reasons as given for the section of a conesected cylinder; or because, when so divided, the section *abfge* becomes a similar portion of a new conesected frustum, with the same vertical distance. Or if, instead of the conical surface *fge*, a plane be made to pass through the vertice and base, the base *abcd* and solid *abcde* will be divided in the same ratio, because if the base is conceived to consist of a portion of the convex surface of a pyramid of an indefinite number of sides, those sides may be divided in any ratio, and the solid which they subtend will be divided in the same ratio. Now, however the base may be divided, that portion which fall one side of the line of division, multiplied by the altitude, give the solidity included between such portion and the ve Therefore, as in the proposition.

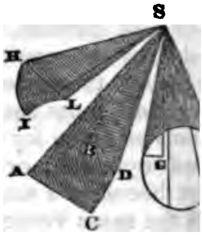
And, if we take the elementary portion of a cone, we arrive at the same result, through the same means.

Cor. Hence any portion of a cylinder, cone, or c frustum, included between any given portion of the cylind or conical surface as a base, and any point in the axis of cylinder, cone, or conic frustum, as a vertice, is a regular p midal, and is elementary to the body from whence it is deri

Scholium. From what has been said in reference to elementary pyramidals, either with plane, cylindrical, or nical bases, those pyramidals may be divided indefinitely out losing their elementary character, if the line of divi passes direct from each point of division on the base to vertice; for the indefinitely small pyramids into which may be conceived to be divided, are all perfect: but if portion of the solid is cut between any portion of the base vertice, in a right line, then the elementary character of solid is disturbed, and the portions so cut are no longer p midals.

PROPOSITION XV. THEOREM.

The sum of two or more pyramids or cones of the same vert height, is equal to the sum of their bases multiplied by third of their common altitude.



Let ABDC and GE be the bases of a pyramid and cone on the same plane, whose vertices both centre in the same point S, and their common altitude will be S G.

Let HIL be the base of another pyramidal not in the same plane with the other bases but with the same vertice S and at the same vertical distance SH=SG; then will sum of the solidities, of the cone and pyramidals be equa the sum of their bases DE+ABDC+HIL multiplied by third of their common altitude SG or SH.

For, let E be the area of the base of the cone EG D= area of the base of the pyramidal ABCD-S and let L=the of the pyramidal HIL-S, and if H=the common altitude, the solidity of the cone GE-S will be

$$E\times\tfrac{1}{3}H=\tfrac{1}{3}H.E.$$

The solidity of the pyramidal ABC-S will be

$$D\times\tfrac{1}{3}H=\tfrac{1}{3}H.D,$$

and the solidity of the pyramidal HIL-S, is equal to

$$L\times\tfrac{1}{3}H=\tfrac{1}{3}H.L,$$

hence the sum of their solidities will be

$$\tfrac{1}{3}H.E\times\tfrac{1}{3}H.D\times\tfrac{1}{3}H.L,$$

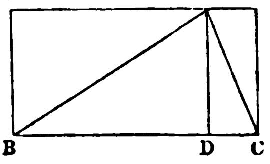
which is equal to $(E\times D\times L)\times\tfrac{1}{3}H$, viz: the sum of their solidities is equal to the sum of their bases multiplied by one-third of their common altitude.

Cor. Hence if a polyedron be composed of any number of pyramidals whose bases are the faces of the polyedron, and whose vertices all centre in one point, and if this point is equidistant from each of the bases of the pyrimidals, then the solidity of the polyedron, will be equal to the surface of its faces, multiplied by one-third of the distance of the faces from the common vertice, and this will be true, without regard to the number or figure of the planes, comprising the faces of the polyedron.

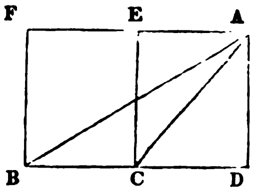
PROPOSITION XVI. THEOREM.

If a triangle and a rectangle, having the same base and the same altitude turn simultaneously about the common base, the solid described by the triangle, will be a third of the cylinder described by the rectangle.

Let ACB be the triangle, and BE the rectangle.



On the axis, let fall the perpendicular AD: the cone described by the triangle ABD is the third part of the cylinder described by the rectangle AFBD (Prop. VIII. Cor.;) also the cone described by the triangle ADC is the third part of the cylinder described by the rectangle ADCE; hence the sum of the two cones, or the solid described by ABC, is the third part of the two cylinders taken together, or of the cylinder described by the rectangle BCEF.



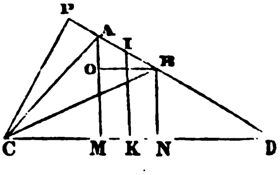
If the perpendicular AD falls without the triangle; the solid described by ABC will, in that case, be the difference of the two cones described by ABD and ACD; but at the same time, the cylinder described by BCEF will be the difference of the two cylinders described by AFBD and AECD. Hence the solid, described by the revolution of the triangle, will still be a third part of the cylinder described by the revolution of the rectangle having the same base and the same altitude.

Schloium. The circle of which AD is radius, has for its measure $\pi\times AD^2$; hence $\pi\times AD^2\times BC$ measures the cylinder described by BCEF, and $\frac{1}{3}\pi\times AD^2\times BC$ measures the solid described by the triangle ABC.

PROPOSITION XVII. THEOREM.

If a triangle be revolved about a line drawn at pleasure through its vertex, the solid described by the triangle will have for its measure, the area of the triangle multiplied by two-thirds of the circumference traced by the middle point of the base.

Let CAB be the triangle, and CD the line about which it revolves.



Produce the side AB till it meets the axis CD in D; from the points A and B, draw AM, BN, perpendicular to the axis, and CP perpendicular to DA produced.

The solid described by the triangle CAD is measured by $\frac{1}{3}\pi\times AM^2\times CD$ (Prop. XVI. Sch.); the solid described by the triangle CBD is measured by $\frac{1}{3}\pi\times BN^2\times CD$; hence the difference of those solids, or the solid described by ABC, will have for its measure $\frac{1}{3}\pi(AM^2—BN^2+CD.)$

To this expression another form may be given. From I, the middle point of AB, draw IK perpendicular to CD; and through B, draw BO parallel to CD: we shall have $AM+BN=2IK$ (Prop.IX.B.IV.*El.Geom.*;) and $AM—BN=AO$; hence $(AM+BN)\times(AM—NB)$, or $AM^2—BN^2=2IK\times AO$(Prop. XII.B.IV. *El. Geom.*) Hence the measure of the solid in question is expressed by

$$\frac{2}{3}\pi\times IK\times AO\times CD.$$

P being drawn perpendicular to AB, the triangles ABO
vill be similar, and give the proportion

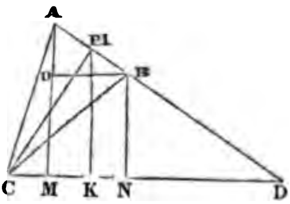
$$AO : CP : : AB : CD;$$

$$\text{hence } AO\times CD=CP\times AB;$$

’$\times AB$ is double the area of the triangle ABC, hence
ve

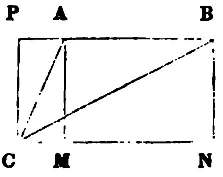
$$AO\times CD=2ABC;$$

the solid described by the triangle ABC is also mea-
ɔy $\frac{2}{3}\pi\times ABC\times IK$, or which is the same thing, by $ABC\times$
IK, *circ.* IK being equal to $2\pi\times IK$. Hence *the solid des-*
by the revolution of the triangle ABC, *has for its mea-*
ıe area of this triangle multiplied by two-thirds of the cir-
·ence traced by I, *the middle point of the base.*



. If the side AC=CB, the line
l be perpendicular to AB, the
ıBC will be equal to $AB\times\frac{1}{2}CI$,
ıe solidity $\frac{2}{3}\pi\times ABC\times IK$ will
e $\frac{2}{3}\pi\times AB\times IK\times CI$. But the
les ABO,CIK, are similar, and
ıe proportion AB : BO or MN
: IK; hence $AB\times IK=MN$
hence the solid described by
ısceles triangle ABC will have for its measure $\frac{2}{3}\pi\times CI^2\times$
that is, *equal to two-thirds of* π *into the square of the per-*
ular let fall on the base, into the distance between the two
ıdiculars let fall on the axis.

olium. The general solution appears to include the sup-
ın that AB produced will meet the axis; but the results
be equally true, though AB were parallel to the axis.
ıs the cylinder described by AMNB
al to $\pi.AM^2.MN$; the cone described
:M is equal to $\frac{1}{3}\pi.AM^2.CM$, and the
lescribed by BCN to $\frac{1}{3}\pi AM^2CN$. Add
st two solids and take away the third;
all have the solid described by ABC
to $\pi.AM^2.(MN+\frac{1}{3}CM-\frac{1}{3}CN)$: and
$CN-CM=MN$, this expression is reducible to $\pi.AM^2.\frac{2}{3}$
ɔr $\frac{2}{3}\pi.CP^2.MN$; which agrees with the conclusion found



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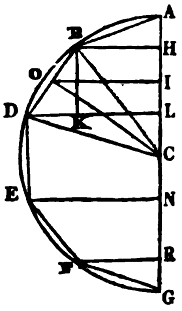
PROPOSITION XVIII. THEOREM.

The surface of a regular polyedroid whose axis terminates in two opposite veritices is equal to the circumference of its inscribed sphere, multiplied by the diameter of its circumscribed sphere.

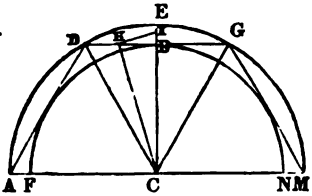


Let AB, BD, DE, &c. be the several sides of a regular semipolygon inscribed in a semicircle ADG whose axis is AG. From the several angles of the semipolygon draw BH, DL, EN and FR perpendicular to AG. Now if the whole be revolved about the axis, AG, the semipolygon will describe the polyedroid (Def. 11.) and the semicircle will describe a sphere (Def. 14). The polyedroid may be divided into the sections described by AHB which will be a cone, HLDB which will be a conic frustum, LNED which will be a cylinder, &c. Bisect the side BD of the conic frustum in O, and draw OI perpendicular to AG, from the centre C draw also CO, the radius of an inscribed circle, and draw BK parallel to AG. Now the convex surface of this frustum is equal to BD×*circumference* OI. (Prop. VII. Cor). And since CO is perpendicular to BD, the triangles BDK, COI are similar (Prop. XX. B. IV. *El. Geom.*) consequently (Prop. XVII. B. IV. *El. Geom.*) BD : BK or HL : : OC : OI : or as *circ.* OC : *circ.* OI.— Hence, BD×*circ.* OI=HL×*circ.* OC. That is the convex surface of the frustum produced by the revolution of BD is equal to the product of its altitude HL into the *circ.* OC, the circumference of an inscribed circle. In like manner it may be proved that the convex surfaces produced by the revolution of DE, EF, &c. the sides of the inscribed semipolygon about the axis AG are respectively equal to the altitude of each, as LN, NR, &c. multiplied by the *circ.* OC, the circumference of the inscribed circle. Hence the convex surface of the whole polyedroid produced by the revolution of the semipolygon about the axis AG is equal to (AH×HL×LN×NR×RG,) or AG×*circ.* CO; that is, the convex surface of the polyedroid produced by the revolution of a semipolygon of any number of sides about an axis AG terminating in two opposite vertices, is equal to the product of its axis or the diameter of its circumscribing sphere, into the circumference of an inscribed sphere.

Cor. Since it has been proved that the surface described revolution of BD about AG is equal to the altitude HL ied by the *circ.* OC. Hence the surface of a zone of a roid is equal to the product of its altitude into the cir- ence of its inscribed sphere.

PROPOSITION XIX. THEOREM.

rface of a regular polyedroid, whose axis terminates in equal plane bases, is equal to the product of the circum- ice of its inscribed sphere into the sum of its diameter + urth proportional whose first term is the diameter of the ribed sphere, the two means are the radii of the circle ing one of the bases.



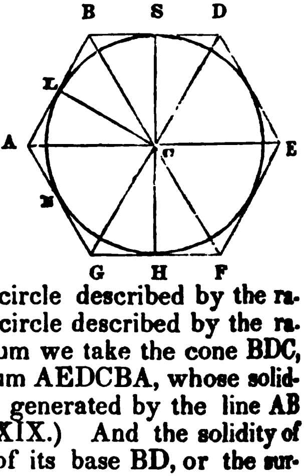
ADB be a quadrant of a n, and ADE a quadrant of cumference of its circum- g circle, and FB a quadrant ıscribed circle, if the whole lved about the axis EC, the nt of the hexagon will des- polar semipolyedroid, and the circular quadrants will e hemispheres. But the solid described by DBCA, will a conic frustum ACMGD, and the surface described by olution of the line DF we have shown (Prop. XVIII.) = the product of the circumference of the inscribed into the altitude BC of the frustum. Now the surface ed by DB will evidently be equal to the product of the ference described by H, the middle point of DB into DB.

n H draw HI perpendicular to HC to meet the axis en, since CHI is a right angled triangle, having a right t H, and HB being perpendicular to CI, we have BC : BH : BI, or as the *circumference* CB or CF : *circ.* BH. BI× *circ.* CB=DB× *circ.* BH. That is, the plane base polyedroid, formed by the revolution of DB, is equal product of the circumference of an inscribed sphere × urth proportional to the radius CB, and two mean terms Now, if a semipolygon is revolved, it will generate a polyedroid, the axis will be =2EC, and we shall also vo plane bases, in which case we shall have the ratio 2BH or BD) : : BD : 2BI, hence as in the proposition.

Hence the surface of a polyedroid, whose generating n has an odd number of sides, will be equal to the pro- the circumference of its inscribed sphere into an arithmetical mean, between the diameter of its circumscribed sphere, and the sum of the diameter of the inscribed sphere + a fourth proportional having this diameter as the first term, and radii of the plane base as the two means, since (Def. 12.) a polyedroid generated by the revolution of a polygon having an odd number of sides, has one plane base and one vertice in which its axis terminates.

PROPOSITION XX. THEOREM.

The solidity of a regular polyedroid is equal to its surface multiplied by one-third of the radius of its inscribed sphere.



Let the regular semipolygon SBAGH, and also its inscribed semicircle be revolved simultaneously about the axis SH, and the semipolygon will generate a polyedroid, (Def. 11.) and the semicircle will generate a sphere, (Def. 15.) Moreover, the section SBAC of the semipolygon would describe the conic frustum AEDB, whose greater base is the circle described by the radius AC, and its smaller base the circle described by the radius SB. Now, if from this frustum we take the cone BDC, we shall have the conesected frustum AEDCBA, whose solidity is equal to its convex surface generated by the line AB multiplied by one-third CS, (Prop XIX.) And the solidity of the BDC is equal to the product of its base BD, or the surface generated by the line SB multiplied by CS or CL. Hence the hemisphere AEDBA of the polyedroid is equal to the product of its convex surface multiplied by one-third of the radius of its inscribed sphere, which is also true of the whole polyedroid.

Cor. Hence any polyedroidal sector is equal to the product of its polyedroidal surface multiplied by one-third of the radius of the inscribed sphere of one-third of the altitude of its inscribed spherical sector.

Scolium. 1. Let $R =$ the radius of a sphere inscribed in a regular polyedroid having two conical vertices, and D' the diameter of the circumscribed sphere ; $2\pi RD'$ will express the surface and $\frac{2}{3}\pi R^2 D'$ will express the solidity of the polyedroid.

2. If the axis of the polyedroid terminates in two plane

ses ; then if r=the radius of the circle forming one of its ual bases, its surface will be$=\left(2R+\frac{r^2}{2R}\right)\times 2\pi R=4\pi R^2+\pi r$. nd its solidity will$=\frac{4}{3}\pi R^3+\frac{1}{3}\pi Rr^2$.

3. If the polyedroid have one conical vertice and one plane ase, and if A=its axis, then $\left(A+\frac{\frac{1}{4}r^2}{R}\right)\times 2\pi R=2\pi RA+\frac{1}{2}\pi r^2=$ e surface. And $\frac{2}{3}\pi R^2A+\frac{1}{6}\pi r^2R$=the solidity where π represents the ratio of the circumference to the diameter.

PROPOSITION XXI. THEOREM.

very regular polyedroid may be inscribed in a sphere, and circumscribed about one.

For, conceive a semicircle inscribed in a semipolygon, and nother semicircle circumscribed about the same polygon, then Prop. III. B. V. *El. Geom.*) that polygon will be regular ; and we conceive those semicircles and semi-polygon to revolve bout the diameter as an axis, the semipolygon will describe regular polyedroid, and the two semicircles will describe pheres, one inscribed within, and the other circumscribed about e polyedroid : hence as enunciated above.

Cor. Here it becomes evident that every section of a sphere s a circle, for, as the number of sides of the polyedroids are in- efinitely increased, they become equal to, and coincide in sur- ce with the sphere ; hence, as it follows that the sphere may e generated by the revolution of a semipolygon of an indefi- ite number of sides, and because as the polygon is revolved bout its axis in common with the sphere, every vertice may be upposed to lie in the surface of the sphere, and also to form he circumference of a circle, it evidently follows that every ection of a sphere is a circle.

PROPOSITION XXII. THEOREM.

The surface of a sphere is equal to the product of its circumference multiplied by its diameter.

For it has been shown (Prop. XVIII.) that the surface of a egular polyedroid whose axis terminates in two opposite ver- ices is equal to the circumference of its inscribed sphere mul- iplied by the diameter of its circumscribed sphere. Whatev- r be the number of the sides of the generating polygon, let he number of the sides of the generating polygon be indefi- itely increased and its surface is still equal to the circumference of its inscribed sphere multiplied by the diameter of its circumscribed sphere. But if the number of the sides are indefinitely increased, the circumscribed and inscribed spheres approach to an equality with the polyedroid, which in such case becomes a sphere ; and since lines and surfaces are supposed to possess no thickness, they ultimately become identical, or there is ultimately no difference in the inscribed and circumscribed spheres, hence the surface of the sphere, which if possible is between the two, is equal to its circumference multiplied by its diameter.

Cor. 1. Since the surface of a great circle of the sphere is measured by the product of its circumference into half the radius, or by a fourth of the diameter (Prop. XV. B. V. *El. Geom.*) the surface of a sphere is four times the area of its great circle.

Cor. 2. It has been shown (Prop. XIX. *Cor.*) that the surface of a zone of the polyedroid is equal to its altitude multiplied by the circumference of its inscribed sphere. Hence the surface of a zone, or any segment of a sphere, is equal to the product of its altitude into the circumference of the sphere.

Cor. 3. The surfaces of two zones or two segments taken in the same sphere, or in equal spheres, are to each other as the altitudes of the zones or segments, and the surface of any zone or segment is to that of the whole sphere as the altitude of the zone or segment to the attitude or diameter of the sphere.

PROPOSITION XXIII. THEOREM.

The solidity of a sphere is equal to the product of its circumference multiplied by one-third of its radius.

For it has been shown (Prop. XXII.) that the solidity of a regular poledroid, whatever be the number of the sides of its generating polygon, is equal to its surface, multiplied by one-third of the radius of the inscribed sphere. Let the number of the sides of the polyedroid be indefinitely increased, and the polyedroid at length becomes identical with the sphere about which it is supposed to be described, and the proposition is still true. Therefore the solidity of a sphere is equal to the product of its surface multiplied by one-third of its radius.

Scholium 1. This Proposition, as well as Proposition XXI. might be deduced directly from Proposition XVI. and XIV. Scholium 5, by supposing the sphere or polyedroid to consist of an indefinite number of indefinitely small pyramidal portions whose bases are in the surface and whose vertices are all in the centre of the sphere.

Scholium 2. From the above it appears that any pyramidal portion of a sphere whose base is a portion of the spherical surface, is equal to the product of that spherical base multiplied by one-third of its altitude on that base. Hence we have four classes of pyramidals, viz: those with spherical, cylindrical, conical, and plain bases, all of which are equal to their bases multiplied by one-third of their altitudes, when their vertices are situated in the centre of curvature of their bases.

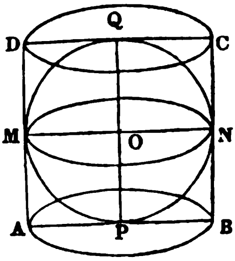
Cor. 1. And any spherical sector is also equal to one third of the product of its spherical surface multiplied by the radius of the sphere.

Cor. 2. The surfaces of spheres being as the squares of their radii, these surfaces multiplied by one-third their radii, are as the cubes of the radii. Therefore the solidities of two spheres are as the cubes of their radii, or as the cubes of their diameters, and the same is also true of similar polyedroids.

Scholium 3. Let R be the radius of a sphere, its surface will be $4\pi R^2$, and its solidity $4\pi R^2 \times \frac{1}{3}R$, or $\frac{4}{3}\pi R^3$. If we call D the diameter we shall have $R=\frac{1}{2}D$, and $R^3=\frac{1}{8}D^3$; therefore the solidity will also be expressed by $\frac{4}{3}\pi \times \frac{1}{8}D^3$ or $\frac{1}{6}\pi D^3$.

PROPOSITION XXIV. THEOREM.

The surface of a sphere is to the whole surface of the circumscribed cylinder, including its bases, as 2 is to 3; and the solidities of these two bodies are to each other in the same ratio.



Let MPNQ be a great circle of the sphere; ABCD the circumscribed square: if the semicircle PMQ and the half square PADQ are at the same time made to revolve about the diameter PQ, the semicircle will generate the sphere, while the half square will generate the cylinder circumscribed about the sphere.

The altitude AD of the cylinder is equal to the diameter PQ; the base of the cylinder is equal to the great circle, since its diameter AB is equal to MN; hence, the convex surface of the cylinder is equal to the circumference of the great circle multiplied by its diameter (Prop. I.). The measure is the same as that of the surface of the sphere (Prop. XXII.): hence *the surface of the sphere is equal to the convex surface of the circumscribed cylinder.*

But the surface of the sphere is equal to four great circles; hence the convex surface of the cylinder is also equal to four great circles: and adding the two bases, each equal to a great circle, the total surface of the circumscribed cylinder will be equal to six great circles; hence the surface of the sphere is to the total surface of the circumscribed cylinder as 4 is to 6, or as 2 is to 3; which was the first branch of the Proposition.

In the next place, since the base of the circumscribed cylinder is equal to a great circle, and its altitude to the diameter, the solidity of the cylinder will be equal to a great circle multiplied by its diameter (Prop. II.) But the solidity of the sphere is equal to four great circles multiplied by a third of the radius (Prop. XXIII.); in other terms, to one great circle multiplied by $\frac{4}{3}$ of the radius, or by $\frac{2}{3}$ of the diameter; hence the sphere is to the circumscribed cylinder as 2 to 3, and consequently the solidities of these two bodies are as their surfaces.

PROPOSITION XXV. THEOREM.

The sphere is equal to a vertical hexadroid whose axis and lateral diameter are each equal to the diameter of the sphere, and the side of whose generating hexagon parallel to the axis, is equal to the radius of the sphere.

For since it has been proved (Prop. XXIV.) that the sphere is equal to two-thirds its circumscribing cylinder, and since a vertical quadredroid or double cone is equal to one-third of its circumscribing cylinder, therefore the sphere is equal to half the circumscribing cylinder + half the inscribed vertical quadredroid. But the hexadroid generated by the revolution of the hexagon whose vertical axis and conjugate diameter are each equal to the diameter of the sphere, and whose sides parallel to the axis is equal to the radius of the sphere, may be resolved into a cylinder = to half the cylinder circumscribing a sphere, and two cones, which together form a vertical quadredroid = to half the vertical quadredroid inscribed in the sphere.

Cor. Since a cone is equivalent to a vertical quadredroid, when the base of the cone is equal to the conjugate section of the quadredroid, and if its altitude is equal to the vertical axis of the quadredroid, it follows that a sphere is equal to a cylinder of equal diameter therewith whose altitude is equal to the radius + a cone of equal base and altitude with the cylinder.

General Scholium.

Let R be the radius of a cylinder's base, H its altitude;
nvex surface will be $2R\times\pi\times H=2\pi RH$.
d its solidity will be $\pi\times R^2\times H=\pi R^2H$. Or if we let C
e convex surface, R the radius of its base, the solidity
e $\frac{1}{2}R\times C$, or $\frac{1}{2}RC$.
Let R be the radius of a cone's base, H its altitude, and
slant height, then will its convex surface be
$\pi\times\frac{1}{2}S=\pi RS$.
d its solidity will be $\pi R^2\times\frac{1}{3}H$, or $\frac{1}{3}\pi R^2H$. Or let C be
onvex surface, and P a perpendicular drawn to such sur-
from the centre of the base, then the solidity will be
P or $\frac{1}{3}CP$.
Let R and r be the radii of the bases of a truncated cone,
altitude, and S its slant height, then will its convex sur-
be $(\pi\times 2R+\pi\times 2r)\times\frac{1}{2}S=\pi RS+\pi rS$.
d its solidity will be $\frac{1}{3}\pi H(R^2+r^2+Rr.)$
Let R=the radius of a sphere inscribed in a regular poly-
d having two conical vertices, and D′ the diameter of the
mscribed sphere; $2\pi RD'$ will express the surface, and
D′ will express the solidity of the polyedroid.
the axis of the polyedroid terminates in two plane bases;
if r=the radius of the circle forming one of its equal
, its surface will be$=\left(2R+\frac{r^2}{2R}\right)\times 2\pi R=4\pi R^2+\pi r$. And
lidity will$=\frac{4}{3}\pi R^3+\frac{1}{3}\pi Rr^2$.
the polyedroid have one conical vertice and one plane
and if A=its axis, then$\left(A+\frac{\frac{1}{4}r^2}{R}\right)\times 2\pi R=2\pi RA+\frac{1}{2}\pi r^2=$
urface. And $\frac{2}{3}\pi R^2A+\frac{1}{6}\pi r^2R$=the solidity when π repre-
the ratio of the circumference to the diameter.
Let R be the radius of a sphere, its circumference will be
R, or $2\pi R$, its surface will be $2\pi R\times 2R=4\pi R^2$. And its
ty will be $4\pi R^2\times\frac{1}{3}R=\frac{4}{3}\pi R^3$.
Let R be the radius of a spherical sector, and H the alti-
of the zone which forms its spherical surface, then will
herical surface of the sector be $2\pi R\times H$ or $2\pi RH$. And
lidity will be $2\pi RH\times\frac{1}{3}R=\frac{2}{3}\pi R^2H$.
Let P and Q be the two bases of a spherical segment, H
itude; its surface will be $2\pi R\times H=2\pi RH$. And if p and
he altitudes of the external segments, its solidity will be
$\times\frac{1}{3}R+P\times\frac{1}{3}(R-p)+Q\times\frac{1}{3}(R-q)$.

BOOK IV.

ISOPERIMETRY OF SOLIDS.

PROPOSITION I. THEOREM.

Of all prisms of the same altitude, whose base is given in
nitude and species, or figure, the right prism has the sm
surface.

For, the area of each face of the prism is proportion
its height; therefore the area of each face is the sm
when its height is the smallest: that is to say, when it is
to the altitude of the prism itself: and in that case the
is evidently a right prism.

PROPOSITION II. THEOREM.

Of all prisms whose base is given in magnitude and sp
and whose lateral surface is the same, the right prism h
greatest altitude, or the greatest capacity.

This is the converse of the preceding theorem, and
readily be proved after the manner of Prop. II. B. V. *El.*

PROPOSITION III. THEOREM.

Of all right prisms of the same altitude, whose bases are
in magnitude, and of a given number of sides, that
base is a regular figure has the smallest surface.

For, the surface of a right prism of given altitude, an
given in magnitude, is evidently proportional to the peri
of its base. But (Prop. VIII. B. VI. *El. Geom.*) the base
given in magnitude, and having a given number of sid
perimeter is smallest when it is regular: whence, the tr
the proposition is manifest.

PROPOSITION IV. THEOREM.

Of two right prisms of the same altitude, and with irregular bases equal in surface, that whose base has the greatest number of sides has the smallest surface; and, in particular, the right cylinder has a smaller surface than any prism of the same altitude and the same capacity.

The demonstration is analogous to that of the preceding theorem, being at once deducible from Propositions VII. and XII. B. VI. *El. Geom.*

PROPOSITION V. THEOREM.

Of all right prisms whose altitudes and whose whole surfaces are equal, and whose bases have a given number of sides, that whose base is a regular figure is the greatest.

Let P, P′ be two right prisms of the same name, equal in altitude, and equal whole surface, the first of these having a regular, the second an irregular base: then is the base of the prism P less than the base of the prism P′.

For, let P″ be a prism of equal altitude, and whose base is equal to that of the prism P′, and similar to that of the prism P. Then the lateral surface of the prism P″ is smaller than the lateral surface of the prism P′ (Prop. III.): hence, the total surface of P″ is smaller than the total surface of P′, and therefore (by hypothesis) smaller than the whole surface of P. But the prisms P″ and P have equal altitudes and similar bases; therefore the dimensions of the base of P″ are smaller than the dimensions of the base of P. Consequently the base of P″, or that of P′, is less than the base of P; or the base of P greater than that of P′.

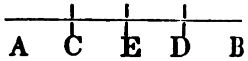
PROPOSITION VI. THEOREM.

Of two right prisms, having equal altitudes, equal total surfaces, and regular bases, that whose base has the greatest number of sides, has the greatest capacity. And, in particular, a right cylinder is greater than any right prism of equal altitude and equal total surface.

The demonstration of this is similar to that of the preceding proposition, and flows from Prop. IV.

PROPOSITION VII. THEOREM.

The greatest parallelopipedon which can be contained under the three parts of a given line, any way taken, will be that constituted of equal length, breadth, and depth.

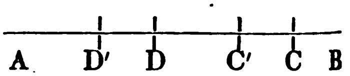


For, let AB be the given line, and, if possible, let two parts AE, ED, be unequal. Bisect AD in C, then will the rectangle under AE ($=AC+CE$) and ED ($=AC-CE$), be less than AC^2, or than $AC . CD$, by the square of CE, (Prop. XII. B. IV. *El. Geom.*) Consequently, the solid $AE . ED . DB$, will be less than the solid $AC . CD . DB$; which is repugnant to the hypothesis.

Cor. Hence, of all the rectangular parallelopipedons, having the sum of their three dimensions the same, the cube is the greatest.

PROPOSITION VIII. THEOREM.

The greatest parallelopipedon that can possibly be contained under the square of one part of a given line, and the other part, any way taken, will be when the former part is the double of the latter.



Let AB be a given line, and $AC=2CB$, then is $AC^2 . CB$ the greatest possible.

For, let AC′ and C′B be any other parts into which the given line AB may be divided; and let AC, AC′, be bisected in D, D′, respectively. Then shall $AC^2 . CB=4AD . DC . CB$ (Prop. X. *Cor.* 1. B. IV. *El. Geom.*), $>4AD' . D'C . CB$, or greater than its equal $C'A^2 . C'B$, by the preceding theorem.

PROPOSITION IX. THEOREM.

Of all right parallelopipedons given in magnitude, that which has the smallest surface has all its faces squares, or is a cube. And reciprocally, of all parallelopipedons of equal surface, the greatest is a cube.

For, by Propositions III. and V., the right parallelopipedon having the smallest surface with the same capacity, or the greatest capacity with the same surface, has a square for its base. But, any face whatever may be taken for a base: therefore, in the parallelopipedon whose surface is smallest with the same capacity, or whose capacity is the greatest with the same surface, any two opposite faces, whatever, are squares: consequently, this parallelopipedon is a cube.

PROPOSITION X. THEOREM.

The capacities of prisms circumscribing the same right cylinder, are respectively as their surfaces, whether total or lateral.

For, the capacities are respectively as the bases of the prisms; that is to say, (Prop. IX. B VI. *El. Geom.*,) as the perimeters of their bases; and these are manifestly as the lateral surfaces: whence the proposition is evident.

Cor. The surface of a right prism circumscribing a cylinder, is to the surface of that cylinder as the capacity of the former to the capacity of the latter.

PROPOSITION XI. THEOREM.

The Archimedean cylinder has a smaller surface than any other right cylinder of equal capacity; and it is greater than any other right cylinder of equal surface.

Let C and C′ denote two right cylinders, of which the first is Archimedean, the other not: then,

1st, if . . . . $C=C'$, $\textit{surf.}\ C<\textit{surf.}\ C$.
2dly, if $\textit{surf.}\ C=\textit{surf.}\ C'$, $C>C'$.

For, having circumscribed about the cylinders, C, C′, the right prisms P, P′, with square bases, the former will be a cube, the latter not: and the following series of equal ratios will obtain, viz: $C : P :: \textit{surf.}\ C : \textit{surf.}\ P :: \textit{base}\ C : \textit{base}\ P :: \textit{base}\ C' : \textit{base}\ P' :: C' : P' :: \textit{surf.}\ C' : \textit{surf.}\ P'$.

Then, 1st: when $C=C'$. Since $C : P :: C' : P'$, it follows that $P=P'$; and therefore (Prop. IX.) $\textit{surf.}\ P<\textit{surf.}\ P'$. But, $\textit{surf.}\ C : \textit{surf.}\ P :: \textit{surf.}\ C' : \textit{surf.}\ P'$; consequently $\textit{surf.}\ C< \textit{surf.}\ C'$.

2dly: when $\textit{surf.}\ C=\textit{surf.}\ C'$. Then, since $\textit{surf.}\ C : \textit{surf.}\ P :: \textit{surf.}\ C' : \textit{surf.}\ P'$, it follows that $\textit{surf.}\ P= \textit{surf.}\ P'$; and therefore (Prop. IX.) $P>P'$. But $C : P :: C' : P'$; consequently $C>C'$.

PROPOSITION XII. THEOREM.

Of all right prisms whose bases are circumscribable about circles, and given in species, that whose altitude is double the radius of the circle inscribed in the base, has the smallest surface with the same capacity, and the greatest capacity with the same surface.

This may be demonstrated exactly as the preceding theorem, by supposing cylinders inscribed in the prisms.

Scholium. If the base cannot be circumscribed about a circle, the right prism which has the minimum surface or the maximum capacity, is that whose lateral surface is quadruple of the surface of one end, or that whose lateral surface is two-thirds of the total surface. This is manifestly the case with the Archimedean cylinder; and the extension of the property depends solely on the mutual connection subsisting between the properties of the cylinder, and those of circumscribing prisms.

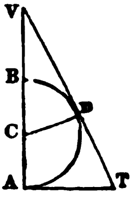
PROPOSITION XIII. THEOREM.

The surfaces of right cones circumscribed about a sphere, are as their solidities.

For, it may be demonstrated, in a manner analogous to the demonstrations of Prop. IX. B. VI. *El. Geom.*, and Prop. X., that these cones are equal to right cones whose altitude is equal to the radius of the inscribed sphere, and whose bases are equal to the total surfaces of the cones: therefore the surfaces and solidities are proportional.

PROPOSITION XIV. THEOREM.

The surface or the solidity of a right cone circumscribed about a sphere, is directly as the square of the cone's altitude, and inversely as the excess of that altitude over the diameter of the sphere.



Let VAT be right-angled triangle, which, by its rotation upon VA as an axis, generates a right cone; and BDA the semicircle which, by a like rotation upon VA, forms the inscribed sphere: then, the surface or solidity of the cone varies as $\frac{VA^2}{VB}$

For, draw the radius CD to the point of contact of the semicircle and VT. Then, because the triangles VAT, VDC, are similar, it is $AT : VT : : CD : VC$.

And, by compos., $AT : AT+VT : CD : CD+CV=VA$;

Therefore $AT^2 : (AT+VT)\ AT : : CD : VA$, by multiplying the terms of the first ratio by AT.

But, because VB, VD, VA, are continued proportionals, it is $VB : VA : : VD^2 : VA^2 : : CD^2 : AT^2$ by similar triangles.

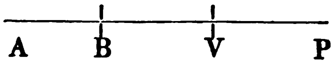
But $CD : VA : : AT^2 : (AT+VT)\ AT$ by the last; and these mult. give $CD . VB : VA^2 : : CD^2 : (AT+VT)\ AT$,

$$\text{or } VB : CD : : VA^2 : (AT+VT)\ AT=CD.\frac{VA^2}{VB}$$

But the surface of the cone, which is denoted by $S . AT^2 + S . AT . VT$, is manifestly proportional to the first member of this equation, is also proportional to the second member; or, since CD is constant, it is proportional to $\frac{AV^2}{VB}$ or to a third proportional to BV and AV. And, since the capacities of these circumscribing cones are as their surfaces (Prop. XIII.) the truth of the whole proposition is evident.

PROPOSITION XV. THEOREM.

The difference of two right lines being given, the third proportional to the less and the greater of them is a minimum when the greater of those lines is double the other.



Let AV and BV be two right lines, whose difference AB is given, and let AP be a third proportional to BV and AV; then is AP a minimum when $AV.=2BV$.

For, since $AP : AV \quad : : AV : BV$;
By division $AP : AP—AV : : AV : AV—BV$;
That is, $AP : VP \quad : : AV : AB$.
Hence $VP . AV=AP . AB$.

But $VP . AV$ is either = or $<\frac{1}{4}AP^2$ (Prop. X. *Cor.* 1. B. IV. *El. Geom.*, and Prop. VII. of this book.)

Therefore $AP . AB \overset{=}{<} \frac{1}{4}AP^2$: whence $4AB \overset{=}{<} AP$, or $AP \overset{=}{>} 4AB$. Consequently, the minimum value of AP is the quadruple of AB; and in that case $PV=VA=2AB$.

PROPOSITION XVI. THEOREM.

Of all right cones circumscribed about the same sphere, the smallest is that whose altitude is double the diameter of the sphere.

For (Prop. XIV.) the solidity varies as $\frac{VA^2}{VB}$ (see diagram to that Prop.), and (Prop. XIV.) since $VA—VB$ is given, the third proportional $\frac{VA^2}{VB}$ is a minimum when $VA=2AB$.

Cor. 1. Hence, the distance from the centre of the sphere to the vertex of the least circumscribing cone, is triple the radius of the sphere.

Cor. 2. Hence, also, the side of such cone is triple the radius of its base.

PROPOSITION XVII. THEOREM.

The whole surface of a right cone being given, the inscribed sphere is the greatest when the slant side of the cone is triple the radius of its base.

For, let C and C' be two right cones of equal whole surface, the radii of their respective inscribed spheres being denoted by R and R'; let the side of the cone C be triple the radius of its base, the same ratio not obtaining in C'; and let C'' be a cone similar to C, and circumscribed about the same sphere with C'. Then, (Prop. XV.) *surf.* $C''<$ *surf.* C'; therefore *surf.* $C''<$ *surf.* C. But C'' and C are similar, therefore all the dimensions of C'' are less than the corresponding dimensions of C: and consequently the radius R' of the sphere inscribed in C'' or in C', is less than the radius R of the sphere inscribed in C, or $R>R'$.

Cor. The capacity of a right cone being given, the inscribed sphere is the greatest when the side of the cone is triple the radius of its base.

For the capacities of such cones vary as their surfaces, (Prop. XIII.)

PROPOSITION XVIII. THEOREM.

ll right cones of equal whole surface, the greatest is that ose side is triple the radius of its base : and reciprocally, all right cones of equal capacity, that whose side is triple radius of its base has the least surface.

r (Prop. XII.) the capacity of a right cone is in the com- d ratio of its whole surface and the radius of its inscribed re. Therefore, the whole surface being given, the capa- is proportional to the radius of the inscribed sphere: and equently is a maximum when the radius of the inscribed re is such: that is, (Prop. XVII.) when the side of the is triple the radius of the base.

gain, reciprocally, the capacity being given, the surface is inverse ratio of the sphere inscribed: therefore, it is mallest when the radius is the greatest; that is, (Prop. I.) when the side of the cone is triple the radius of its

PROPOSITION XIX. THEOREM.

urfaces, whether total or lateral, of pyramids circumscribed out the same right cone, are respectively as their solidities. d, in particular, the surface of a pyramid circumscribed out a cone, is to the surface of that cone, as the solidity of pyramid is to the solidity of the cone ; and these ratios equal to those of the surfaces or the perimeters of the bases.

r the capacities of the several solids are respectively as bases ; and their surfaces are as the perimeter of those : so that the proposition may manifestly be demonstrated chain of reasoning exactly like that adopted in Prop. B. VI. *El. Geom.*

PROPOSITION XX. THEOREM.

base of a right pyramid being given in species, the capaci- of that pyramid is a maximum with the same surface, and the contrary, the surface is a minimum with the same capa- y, when the height of one face is triple the radius of the circle cribed in the base.

t P and P' be two right pyramids with similar bases, the t of one lateral face of P being triple the radius of the circle inscribed in the base, but this proportion not obtaining with regard to P' : then

First. If *surf.* $P =$ *surf.* P', $P > P'$.

Secondly. If. . . $P =$ . . . P', *surf.* $P, <$ *surf.* P'.

For, let C and C' be right cones inscribed within the pyramids P and P' : then in the cone C, the slant side is triple the radius of its base, while this is not the case with respect to the cone C'. Therefore, if $C = C'$, *surf.* $C <$ *surf.* C' and if *surf.* $C =$ *surf.* C' . $C > C'$ (Prop. XVIII.)

But 1st. *surf.* P : *surf.* C : : *surf.* P' : *surf.* C';
whence, if *surf.* $P =$ *surf.* P' *surf.* $C =$ *surf.* C' ;
therefore $C > C'$. But $P : C :: P' : C'$. Therefore $P > P'$.

2dly, $P : C :: P' : C'$. Therefore if $P = P'$, $C = C'$: consequently *surf.* $C <$ *surf.* C'. But, *surf.* P : *surf.* C : : *surf.* P' : *surf.* C'. Whence, *surf.* $P <$ *surf.* P'.

Cor. The regular tetraedron possesses the property of the minimum surface with the same capacity, and of the maximum capacity with the same surface, relatively to all right pyramids with equalateral triangular bases, and, *a fortiori*, relatively to every other triangular pyramid.

PROPOSITION XXI. THEOREM.

If a regular polyedroid be formed by the revolution of a polygon about an axis which terminates in two opposite vertices; and if another regular polyedroid be formed by the revolution of the same polygon about an axis bisecting its opposite sides, the former polyedroid will be greater than the latter, both in surface and solidity.

For, (Prop. XVIII., B. III.,) the surface of the vertical polyedroid is equal to the circumference of its inscribed sphere multiplied by the diameter of its circumscribed sphere or the axis of the polyedroid, and (Prop. XIX., B. III.) the surface of a polyedroid whose axis terminates in two opposite bases, is equal to the circumference of its inscribed sphere into the sum of its diameter *plus* a fourth proportional, whose first term is the diameter of the inscribed sphere; the two means are the radii of the circle forming one of the bases.

Hence the surface of the vertical polyedroid is greater than one whose axis terminates in two opposite plane bases; and since (Prop. XX. B. III.) the solidity of a regular polyedroid is equal to its surface multiplied by one-third of the radius of the inscribed sphere, it follows that the solidity of the former is laso greater than that of the latter.

PROPOSITION XXII. THEOREM.

lyedroids of equal surfaces, the greatest is that which
lar, or which may be inscribed within a sphere, and
scribed about one.

nay be inferred from Proposition VI., B. VI., *El.*

PROPOSITION XXIII. THEOREM.

ities of regular polyedroids, and also of regular poly-
, circumscribed about the same or equal spheres, are
surfaces.

hatever be the surfaces or the number of sides of a
d, or a polyedron circumscribed about a sphere, its
equal to its surface multiplied by one-third of the
the inscribed sphere: hence, whatever be the ratio
rfaces of polyedroids, or polyedrons, circumscribed
same or equal spheres must also be the ratio of their

Hence, also, the solidities of polyedroids and polye-
cumscribed about a sphere, are to the solidity of the
their respective surfaces to the surface of the sphere.

PROPOSITION XXIV. THEOREM.

gular polyedroids, the sum of the diameters of whose
scribed and inscribed spheres are equal, that, the differ-
whose inscribed and circumscribed spheres is the least,
reatest, both in surface and solidity.

the polyedroid is a vertical polyedroid, its surface is
rop. XIX., B. III.) to the circumference of the in-
phere, multiplied by the diameter of the circumscribed

fore, when this product is the greatest, then will the
e the greatest.
$=$the circumference of the sphere whose diameter is
$\pi=$the ratio of the circumference to the diameter;
$1+a=$the diameter of the circumscribed sphere, and
e diameter of the inscribed sphere; then will $(1-a)$
circumference of the inscribed sphere, and the surface
yedroid will be$=(\pi-\pi a)\times(1+a)=\pi-\pi a^2$, which val-

ue is evidently greater as the value of *a* is smaller, and is smaller as *a* is greater; and since the solidity of a polyedroid circumscribed about a sphere, is as its surface, it follows that the solidity increases as *a* diminishes, and diminishes as *a* increases: this may also be shown in relation to a polyedroid whose axis terminates in two opposite bases.

Cor. 1. Hence, also, of polyedroids, the sum of the diameters of whose circumscribed and inscribed spheres are equal; that, the difference of whose circumscribed and inscribed spheres is the least, is the greatest both in surface and solidity, and reciprocally.

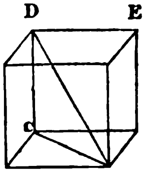
Cor. 2. And, since it is sufficiently evident that the greater the number of sides of a polyedroid, or of a polyedron, the sum of the diameters of whose inscribed and circumscribed spheres is given, the less is the difference of the diameters of the inscribed and circumscribed spheres; it hence becomes evident that the greater the number of sides of a polyedroid, or of a polyedron, the sum of the diameters of whose inscribed and circumscribed spheres is given, the greater is the surface and solidity of each.

Cor. 3. Hence, where the circumscribed and inscribed spheres become identical, or where the polyedroid becomes a sphere, its surface as well as its solidity is a maximum.

PROPOSITION XXV. THEOREM.

If a regular polyedroid and a regular polyedron be both inscribed in, and circumscribed about, the same spheres, the polyedron will be greater than the polyedroid both in surface and solidity.

For, let any zone of a polyedroid, made by a plane perpendicular to its axis, be compared with a corresponding zone of the polyedron, and since the perimeter of a polygon is greater than the circumference of its inscribed circle, it is sufficiently manifest, that the surface of the zone of the polyedron is greater than the corresponding zone of the polyedroid, since they are both inscribable in and circumscribable about the same spheres by hypothesis; and as this may also be shown in reference to any corresponding zones made by planes perpendicular to the axis of the polyedroid, it follows that the whole surface of the polyedron is greater than the whole surface of the polyedroid. And, since their surfaces are as their solidities, (Prop. XXIII.,) hence the solidity of the polyedroid is also greater than that of the polyedron.



Scholium. 1. In a hexaedron, or cube EB, the diameter of the inscribed sphere is evidently equal to the linear edge AB of the hexaedron, and the diameter of the circumscribed sphere is the diagonal $AD=\sqrt{(AC^2+CD^2)}=\sqrt{(AB^2+BC^2+CD^2)}=\sqrt{3AB^2}$; hence, if the diameter of the circumscribed sphere=10, and its circumference =31.416, then will the diameter of the circumscribed sphere $=\sqrt{3\times10^2}=17.3$, and the surface of a polyedroid similarly inscribed, would be $31.416\times17.3=543.4868$, (Prop. XVIII. B. III.) but the surface of the hexaedron is $10\times10\times6=600$, which is greater than that of the polyedroid.

Scholium 2. Perhaps it may be necessary to observe, that a regular polyedroid and a regular polyedron circumscribing and inscribed in the same spheres is impossible; but this will, nevertheless, not vitiate our reasoning in relation to their properties when compared with each other, for whether they are possible or impossible, the law of their developments is nevertheless manifest.

PROPOSITION XXVI. THEOREM.

A sphere is to any circumscribing solid, bounded by plane surfaces, as the surface of the sphere to that of the circumscribing solid.

For, since all planes touch the sphere, the radius drawn to each point of contact will be perpendicular to each respective plane. So that, if planes be drawn through the centre of the sphere and through all the edges of the body, the body will be divided into pyramids whose bases are the respective planes, and their common altitude the radius of the sphere. But the capacity of the sphere is equal to the surface multiplied by one-third of its radius. Consequently, the capacity of the sphere, is to that of the circumscribing solid, as the surface of the former to that of the latter.

Cor. 1. All circumscribing cylinders, cones, polyedroids, &c., are to the sphere they circumscribe as their respective surfaces. For (Scholium 2. Prop. XXIII. B. III.) the solidities of cylinders, cones, and polyedroids, are respectively equal to their surfaces multiplied by one-third of their respective vertical or central distance, which, in case of their circumscribing a sphere, is the radius of the sphere.

Cor. 2. All bodies bounded by plane surfaces, or by surfaces formed by the revolutions of rectilineal figures circumscribing the same sphere, are respectively as their surfaces.

NOTES.

BOOK I.

As plane angles may be measured by the arc of a circle, so solid angles may be measured by the portion of a spherical surface, intercepted by the planes forming the solid angles, since solid angles bear the same relation to the surfaces which comprise them, as plane angles do to the lines by which they are included; so that, as in the latter, it is not the magnitude of the lines, but their mutual inclination, which determines the angles.

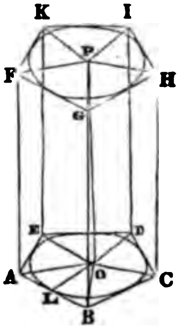
BOOK II.

In addition to the symmetrical polyedrons of Legendre, we have discussed the subject of elementary prisms and pyramids; and as all polyedrons may be divided into triangular pyramids, these I have regarded as the elements of all: but more particularly the pyramids of equal altitudes, and whose bases are of a similar character, are regarded as elementary to the same constituted pyramid or polyedron. And when these elementary pyramids are all equal and similar, then, in their associated capacity, they may be regarded as symmetrical by association, and equal by symmetry with each other. It should, however, be borne in mind, that the designations *equal* figures, *symmetrical* figures, and *equivalent* figures, are designed to express different qualities, and ought not to be confounded with each other. The term equal, when applied to solid as well as plane figures, is designed to express such as being applied to each other, would coincide in all their parts.

Now symmetrical solids may be similar in every respect, according to symmetrical similarity, yet incapable of such coincidence.

The term equivalent should be applied to such figures, as well planes as solids, as are equal in surface or solidity, but do not admit of coincidence.

Let there be a prism, AH, whose base is a regular polygon, and if this prism is divided into the several triangular prisms AEOKFP, EDOIKP, &c., those triangular prisms may be regarded as elementary to the prism AH, and each of the similar triangular prisms, in their associated capacity, may be regarded as symmetrical by association, and equal by symmetry.



It is frequently necessary to speak of *the product of a line by a line*, or of a *surface by a line*, &c., by which is understood the product of the numbers representing those lines, surfaces, &c. In this sense of the word, there is no impropriety in its use, when applied to magnitudes of any kind which admit of numerical representation.

BOOK III.

In this book is introduced some new terms and definitions, as applied to some new solids, which are here introduced as elementary solids, since they are capable of being classified and generalized to the same extent as others which have usually been regarded as the only elementary solids of revolution.

First, we have introduced the term *conesected*, as applied to a cone, conical frustum, or a cylinder, having a conical cavity within. Thus we have a *conesected cone*, a *conesected frustum of a cone*, and a *conesected cylinder*. This, therefore, supplies a general designation for a certain class of solids not heretofore named; and which, from their obvious connection with elementary geometry, as appears in the text, they should not only be included in the mathematical nomenclature, but their properties should also be investigated.

The subject of pyramidals, as small elementary portions of conesected cylinders, cones, or conical frusta, and elementary portions of other solids of revolution, is amply discussed in this book.

The definition of the class of solids formed by the revolution of a semipolygon about an axis, is rendered polyedroid, following the analogy of polyedron, as applied to solids terminated by plane surfaces; and a solid formed from the revolution of any particular polygon designated by name, receives a particular designation, following the analogy; thus we have the *tetredroid*, or cone, the *quadredroid*, which may be either a cylinder or double cone, a *pentadroid*, a *hexadroid*, a *heptadroid*, &c.

BOOK IV.

The isoperimetry of solids is here discussed and rendered more interesting, by the comparison of the polyedroid with the polyedron and sphere. The subject might be a considerably extended, but it is deemed best, in this place, to compare solids only; as their relations to each other depend on elementary principles, we shall have occasion to resume this subject again in a subsequent volume, in connection with other classes of solids, where we shall be able to bring into use a greater variety of principles in the discussion.

THE APPLICATION OF THE FOREGOING PRINCIPLES TO THE MENSURATION OF SOLIDS.

The common *measuring unit* of solids is a *cube*, whose sides are squares of the same name. The sides of a cubic inch are square inches; of a cubic foot, square feet, &c. Finding the *capacity*, *solidity*, or *solid contents* of a body, is finding the number of cubic measures, of some given denomination contained in the body.

In solid measure.

1728 cubic inches = 1 cubic foot,
27 cubic feet = 1 cubic yard,
4492¼ cubic feet = 1 cubic rod,
32763000 cubic rods = 1 cubic mile,
282 cubic inches = 1 ale gallon,
231 cubic inches = 1 wine gallon,
2150.42 cubic inches = 1 bushel,
1 cubic foot of pure water weighs 1000 avoirdupois ounces, or 62½ pounds.

PROBLEM I.

To find the solidity of a prism.

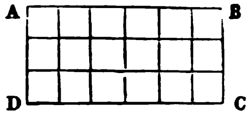
Rule.—Multiply the area of the base by the height.

This is a general rule, applicable to parallelopipedons, whether right or oblique, cubes or prisms.

But if the base of the prism is a regular polygon, *multiply the vertical or convex surface by one-half the radius of the circle inscribed in the base.* (Prop. XVIII. *Cor.* 1. B. II.)

Or, if the prism has a triangular base, *multiply either of its rectangular sides by half the distance from such side, or such side produced, as a base, to its vertical angle.*

As *surfaces* are measured by comparing them with some given square as a unit, (Prop. VI. *Sch.* B. IV. *El. Geom.*,) so solids are measured by comparing them with some given cube as a unit.



If ABCD be the base of a right parallelopipedon, as a stick of timber standing erect, it is evident that the number of *cubic* feet contained in one foot of the height is equal to the number of *square* feet in the area of the base. And if the solid be of any other height, instead of one foot, the contents must have the same ratio. For parallelopipedons of the same base are to each other as their heights. (Prop. XIII. B. II.)

The solidity of a right parallelopipedon, therefore, is equal to the *product of its length, breadth, and thickness.*

And an *oblique* parallelopipedon being equal to a right one of the same base and altitude, (Prop. X. B. II.,) is equal to the area of the base, multiplied into the perpendicular height. This is true, also, of prisms, whatever be the form of their bases. (Prop. XVI. B. II.)

As the sides of a *cube* are all *equal,* the solidity is found by *cubing one of its edges.* On the other hand, if the solid contents be given, the length of the edges may be found.

Ex. 1. What are the solid contents of a stick of timber, which is 31 feet long, 1 foot 3 inches wide, and 9 inches thick?

Here, $31\times1\frac{1}{4}\times\frac{3}{4}=29$ feet 108 inches, the solidity required.

Ex. 2. What is the solidity of a wall which is 22 feet long, 12 feet high, and 2 feet 6 inches thick? Ans. 660 cubic feet.

Ex. 3. What is the capacity of a cubical vessel which is 2 feet 3 inches deep? Ans. 11 feet 675 inches.

Ex. 4. If the base of a prism be 108 square inches, and the height 36 feet, what are the solid contents?

Ans. 27 cubic feet.

Ex. 5. If the height of a square prism be $2\frac{1}{4}$ feet, and each side of the base $10\frac{1}{3}$ feet, what is the solidity?

The area of the base$=10\frac{1}{3}\times10\frac{1}{3}=106\frac{7}{9}$ square feet.

And the solid contents$=106\frac{7}{9}\times2\frac{1}{4}=240\frac{1}{4}$ cubic feet.

Ex. 6. If the height of a prism be 23 feet, and its base a regular pentagon, whose perimeter is 18 feet, what is the solidity? Ans. 512.84 cubic feet.

Ex. 7. What is the solidity of a prism having a regular base, the convex surface being 28 feet, and the radius of the circle inscribed in its base 4 feet? Ans. 21 cubic feet.

Ex. 8. In a triangular prism whose altitude is 12 feet, one of the sides of the triangle of the base is $2\frac{1}{2}$ feet in length, and its distance from the opposite angle of the triangle is 3 feet; what is its solidity? Ans. $37\frac{1}{2}$.

Scholium. The number of *gallons* or *bushels* which a vessel will contain, may be found by calculating the capacity in inches, and then dividing by the number of inches in one gallon or one bushel.

The *weight of water* in a vessel of given dimensions, may also be calculated; as it is found, by experiment, that a cubic foot of pure water weighs 1000 oz., or 62½ lbs. avoirdupois. For the weight in ounces, then, multiply the cubic feet by 1000; or, for the weight in pounds, multiply by 62½.

Ex. 9. How many ale gallons are there in a cistern which is 11 feet 9 inches deep, and whose base is 4 feet 2 inches square?

The cistern contains 352500 cubic inches; which, being divided by 282, the number of inches in a gallon, gives 1250, the number of gallons required.

Ex. 10. How many wine gallons will fill a ditch 3 feet 11 inches wide, 3 feet deep, 115½ feet long? Ans. 10152.

Ex. 11. What weight of water can be put into a cubical vessel 4 feet deep? Ans. 4000 lbs.

PROBLEM. II.

To find the lateral or convex surface of a right prism.

Rule.—Multiply the length of the prism into the perimeter of the base (Prop. VIII. B. II.)

If the areas of the two bases be added to the lateral surface, the sum will be the whole surface of the prism.

Ex. 1. If the base of a right prism be a regular hexagon whose sides are each 2 feet 3 inches, and if the height be 16 feet, what is the convex surface? Ans. 216 square feet.

Ex. 2. If the base of a prism be an equilateral triangle whose perimeter is 6 feet, and if the height of the prism be 17 feet, what is the whole surface?

The area of the triangle is	1.732
The lateral surface	103.732
The whole surface	105.464

Ex. 3. What is the superfices of a cube the length of whose side is 5½ feet? Ans. 181½ square feet.

PROBLEM. III.

To find the lateral or convex surface of a regular pyramid.

Rule 1.—Multiply the perimeter of the base into half the slant height (Prop. XX. B. II.)

2.—Or if the base of the pyramid is a regular polygon, find the area of its base, which, increased in the ratio of the radius of the circle inscribed in the base to the slant height, will give the convex surface (Prop. XXI. B. II.)

Or multiply the area of the base by its slant height, and divide that product by the radius of the circle inscribed in the base, the quotient will be the vertical surface.

3.—Or its convex surface may be found by first finding the convex surface of its circumscribing prism, then say, as the virtical height of the pyramid is to its slant height, so is half the vertical surface of the circumscribing prism to the convex surface of the pyramid (Prop. XXIII. B. II.)

Or multiply half the convex surface of the prism by the slant height of the pyramid, and divide this product by the vertical height of the pyramid.

Scholium. 1. If the area of the base be added to the lateral surface, the sum will be the whole surface. The lateral surface of an oblique pyramid may be found by taking the sum of the areas of the unequal triangles which form its sides.]

Ex. 1. Required the lateral surface of a regular hexagonal pyramid whose axis is 20 feet, and the sides of whose base are each 8 feet?

Here $\sqrt{8^2-4^2}=\sqrt{48}$=the radius of the inscribed circle.

And $\sqrt{48+20^2}=21.16$=the slant height. Hence the lateral surface$=21.16\times4\times6=507.84$ square feet.

Ex. 2. What is the lateral surface of a regular hexagonal pyramid, the radius of the circle inscribed in its base being 8 feet, and the slant height being 20 feet?

Let r=the radius of the inscribed circle.

And let x=the side of the hexagon.

Then x=also the radius of the inscribed circle.

Then $x^2-\frac{1}{4}x^2=r^2,=\frac{3}{4}x^2=64$.

And $x^2=\frac{1}{3}r^2=85\frac{1}{3}$.

Hence $x=9.238$.

The area of the base by Prop. VII.$=9.238\times4=36.752$.

By rule 2nd, 8 : 20 : : 36.752 : 91.88 the lateral surface.

Ex. 3. In a square prism whose altitude is 10 feet, and the side of the square of the base is 5 feet, it is proposed to cut out a pyramid of the same base and altitude, required the convex surface.

Here $10\times5\times4=200$=the lateral surface of the prism.

And $\sqrt{10^2+2\frac{1}{2}^2}=10.307$=the slant height of the pyramid.

Then by rule 3, 10 : 10.307 : : 200÷2 : 103.07 the lateral surface of the pyramid.

4. The base of a regular pyramid is 6 square feet, and the of its inscribed circle is to the slant height of the pyra- 1 to 4¼, what is the whole surface of the pyramid?

Ans. 33 square feet.

lium. 2. If through a pyramid and parallel to its axis, n be passed cutting out a portion of the base, and also of eral surface, the portion of the lateral surface cut out by rism, will be to that cut out from the base as the slant of the pyramid is to the radius of the base (Prop. XXII. . II.)

5. If through a regular triangular pyramid whose base ns 3 square feet, and whose slant height is four times r than the radius of the circle inscribed in the base, three tic holes, whose section is 4 square inches each, be cut l to the axis; required, the lateral surface of the pyramid ing.

Ans. $14\frac{7}{12}$ square feet.

PROBLEM. IV.

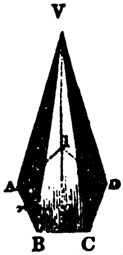
To find the solidity of a pyramid.

GENERAL RULE.

Multiply the area of the base by one-third of the height, . XXVI. B. II.)

f the base of the pyramid is a regular polygon, its solidi- qual to the convex surface multiplyed by one-third of stance of the centre of the base from the lateral surface.

olium. Since similar pyramids are to each other as the of their altitudes, or the cubes of their like dimensions, . XXXV. B. II.); if a pyramid be cut by a plane paral- its base, the portion cut off toward the vertex, will be to hole pyramid as the cube of the altitude of the portion so , to the cube of the altitude of the whole pyramid.



1. What is the solid contents of a penta- pyramid ABCDIV, whose altitude OV is feet, and each side AB of the base=2 feet?

e tabular area of the base, (Rule II. Table I. Prob. *Mens. El. Geom.*)

$^2\times1.7204774=4\times1.7204774=6.8819096$ feet.

Here by the rule $6.8819096\times\frac{12}{3}=27.5277384$ cubic feet, the solid content of the pyramid required.

Ex. 2. The convex surface of a regular pyramid is 24 square inches; the distance of the centre of the base is 2 inches from one of its lateral faces, what is the solidity?

Ans. $24\times2\times\frac{1}{3}=16$ solid inches.

Ex. 3. What is the solid content of a hexagonal pyramid whose perpendicular height is 6.4 feet, and each side of the base 6 inches. Ans. 1.38564064 cubic feet.

PROBLEM V.

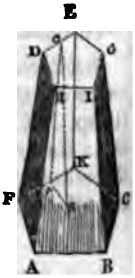
To find the lateral surface of the frustum of a regular pyramid.

Rule.—Multiply half the sum of the perimeters of the two ends by the slant height.

For each side of a frustum of a regular pyramid is a trapezium, as AFDH, and the area of each of the sides is evidently equal to half the sum of the parallel sides, as AF, DH, multiplied by its slant height *ro*, and the whole convex surface being made up of all the lateral faces, is evidently equal to half the sum of the perimeters of the two bases multiplied by the slant height.

If to the lateral surface of a frustum, the areas of the two bases be added, the sum will be the whole surface.

Ex. 1. Required the lateral surface of the frustum of a pentagonal pyramid, its slant height $ro=11$ feet 8 inches, each side of the greater end AB=2 feet 5 inches, and each side of the less end HI=1 foot 8 inches.



$$\frac{AB+HI}{2}\times5\times ro=\tfrac{1}{2}(2\tfrac{1}{2}+1\tfrac{2}{3})\times11\tfrac{2}{3}=121.527777$$

square feet, the surface required.

Ex. 2. If the slant height of a frustum of a regular octagonal pyramid be 42 feet, the sides of one end 5 feet each, and the sides of the other end 3 feet each, required the lateral surface.

Ans. 1344 square feet.

Scholium. If the slant height is not given, it may be found by the properties of the right angled triangle $\sqrt{os^2+rs^2}=$the slant height, rs being equal to the difference of the radii of circles inscribed in the two bases.

PROBLEM VI.

To find the solidity of the frustum of a pyramid.

Rule 1. Add into one sum the areas of the two ends and square root of the product of these areas, and multiply sum by one-third of the altitude of the frustum, (Prop. [VIII. B. II.)

2. Or, multiply each face of the frustum, including its bases, one-third of their several distances from any point within frustum, (Prop. XXIX. *Sch.* 1, B. II.)

Scholium. If, in the frustum of a pyramid whose base is a ular polygon, any point in the axis is assumed, the lateral es will all be equidistant from such point; and if the point ıssumed in the centre of either base, then,

Multiply the convex surface by one-third of its distance n the centre of either base, or, by one-third of a line drawn n such centre perpendicular to the lateral surface, to which a pyramid formed on the opposite base, and whose altie is equal to that of the frustum, and this sum will be the dity of the frustum.

x. 1. Required the solidity of a frustum of a pengtagonal amid, whose perpendicular height (see diagram to Prob. *os*=5 feet, each side of the greater end AB=18 inches, each side of the less end HI=6 inches.

y rule II., *Mens.*, *El. Geom.*, the tabular area of the lower e=557.43467; the upper base=61.93718.

ıen, by the ıule, $557.43467 \times 61.93718 = 34525.9355323351$

$\sqrt{34525.9355323351} = 185.81155$, the mean area.

nd $557.43467 + 61.93718 + 185.81155 = 805.18342$ square hes.

Hence, $805.18342 \times 60 \div 3 = 16103.66840$ cubic inches= 1925 cubic feet, the solidity of the frustum required.

What is the solidity of the frustum of a pyramid having drilateral bases 12 and $1\frac{1}{2}$ inches square, its vertical height ng 7 inches, its slant height $8\frac{3}{4}$ inches, and from a certain nt assumed in the axis at a distance of 4 inches above the larger base, the lateral faces are each $2\frac{2}{5}$ inches distant?

Iere, since the assumed point is 4 inches from the lower e, it must be 3 inches from the upper base.

Then $\frac{12+1\frac{1}{2}}{2} \times 8.75 = 59.0625$=one of the lateral faces; $59.0625 \times 2\frac{2}{5} \div 3 = 47.25$=the pyramid formed on such as a base whose vertex is the point assumed; and 47.25

$\times 4 = 189 =$ the four similar pyramids formed on the four lateral sides as bases. To which add $(12\times12\times\frac{4}{3})+(1\frac{1}{2}\times1\frac{1}{2}\times\frac{3}{3})=192+2\frac{1}{4}=$ the two pyramids formed on the two bases, and whose vertices are in the same point, and we have the whole solid$=189+192+2\frac{1}{4}=383\frac{1}{4}$.

Ex. 3. What is the solidity of the frustum of a pyramid with quadrilateral bases, the surface of the upper base being 36 square inches, each of the lateral faces 45 inches, and their distances from the centre of the base being each $4\frac{4}{5}$ inches, the altitude of the frustum being 4 inches?

Here $44\times4\times4\frac{4}{5}\div3=288=$ the pyramids formed on the lateral faces;

And $36\times4\div3 \quad = \quad 48=$ a pyramid formed on upper base.

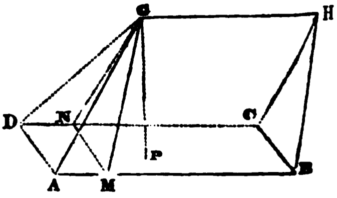
Hence $288+48=336$ solid inches, the content.

PROBLEM VII.

To find the solidity of a wedge.

Rule.—Add half the length of the edge to the length of the base, and multiply this sum by one-third of the product of the height of the wedge and the breadth of the base, (Proposition XXXIII., B. II.)

Ex. 1. If the base AC of the wedge ABCDGH be 20 by 6 inches, the edge GH 16 inches, and the perpendicular height PG 10 inches, what is the solidity?



$(8+20)\times\frac{10\times6}{3}=560$ solid inches.

Ex. 2. The length and breadth of the base of a wedge are 35 and 15 inches, the length of the edge is 55 inches, and the perpendicular height 17.14508 inches: what is the solidity?

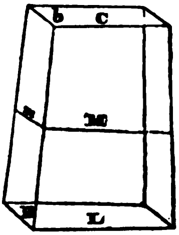
Ans. 3.1006 cubic feet.

Ex. 3. How many solid feet are there in a wedge whose perpendicular height is 6 feet 4 inches, the length of the edge is 2 feet 8 inches, the length of the base 3 feet 5 inches, and its breadth 1 foot 1 inch? Ans. $10\frac{373}{432}$ cubic feet.

PROBLEM VIII.

To find the solidity of a prismiod.

Rule.—To the areas of the two ends, add four times the area of a parallel section equally distant from the ends, and multiply this sum by one-sixth of the height, (Prop. XXXIV. B. II.)



Ex. 1. How many cubic feet are there in a stone whose ends are rectangles, the length L of the greater end $=14$ inches, the breadth $B=12$ inches, the corresponding sides C, *b*, of the less end $=6$ and 4 inches, the perpendicular height being $30\frac{1}{2}$ feet.

The area of the larger end $=14\times12=168$
The area of the smaller end $=\ 6\times\ 4=\ 24$
The area of the middle section$=10\times\ 8=\ 80$

Hence the solidity$=(168+24+4\times80)30\frac{1}{2}\div6=2602\frac{2}{3}$, which divided by 144, gives $18\frac{1}{7}$ cubic feet for the content.

Ex. 2. What is the solid content of a pyramid whose greater end measures 12 inches by 8, the lesser end 8 inches by 6, and the length, or height, 60 inches? Ans. 2.453 feet.

Ex. 3. What is the solid content of a waggon whose inside dimensions are as follows: the length of the top $81\frac{1}{2}$ inches, its breadth 55 inches, the length of the bottom 41 inches, its breadth $29\frac{1}{2}$ inches, the perpendicular height being $47\frac{1}{2}$ inches?
Ans. 73.11376953125 cubic feet.

PROBLEM IX.

To find the surface of a regular polyedron.

Rule.—Multiply the area of one of the sides by the number of sides.

Or multiply the square of one of the edges by the surface of a similar solid whose edges are 1.

Since the faces of a regular polyedron are all equal, it is evident that the area of one side multiplied by the number of sides, will give the area of the surface.

Hence, if we have a table exhibiting the surfaces of the several regular polyedrons whose linear edges are unity, we may from this find the surfaces of all other similar polyedrons; since the areas of the similar polygons composing the faces of each are as the squares of their homologous sides, (Prop. XXIV. Cor. E. B. IV. *El. Geom.*) Such a table may be formed by multiplying the area of one of the faces by the number of them. Thus the area of one equalateral triangle whose side is 1, is$=0{,}4330127$ by table under Problem IX. ***Mens.*** *El. Geom.* Therefore, the surface

of a regular tetraedron $=.4330127\times4=1.7320508$
of a regular octaedron $=4330127\times8=3.4641016$
of a regular icosaedron $=4330127\times20=8.6602540$

Hence the surfaces in the following table may be in like manner computed.

A table of the surfaces and solidities of the regular polyedrons whose edges are 1.

NAMES.	NO. OF SIDES.	SURFACES.	SOLIDITIES.
Tetraedron.	4	1.7320508	0.1178513
Hexaedron.	6	6.0000000	1.0000000
Octaedron.	8	3.4641016	0.4714045
Dodecaedron.	12	20.6457288	7.6631189
Icosaedron.	20	8.6602540	2.1816950

Ex. 1. What is the surface of a regular dodecaedron whose edges are each 25 inches?

The area of one of the sides is 1075.3. And the surface of the whole solid$=1075.3\times12=12903.6$.

Ex. 2. What is the surface of a regular icosaedron whose edges are each 102? Ans. 90101.3.

PROBLEM X.

To find the solidity of a regular polyedron.

Rule 1.—Multiply the surface by one-third of he perpendicular distance from the centre to one of its sides. (Prop. VII. Cor. 2. B. II. Sec. 2.)

2. Or multiply the cube of one of the edges by the solidity of a similar solid whose edges are one.

As the solidity is made up of a number of equal pyramids, whose bases are the sides, and whose height is the perpendicular distance of the sides from the centre, (Prop. XIV. B. III.), the solidity of the whole must be equal to the areas of the sides multiplied into one-third of this perpendicular, (Prop. VI. Cor. 2. B. II. Sec. 2.)

If the contents of the several regular solids, whose edges are one, be arranged in a *table*, this may be used to measure other similar solids. For two similar regular solids contain the same number of similar pyramids, and these are to each other as the cubes of their linear sides or edges.

Thus, by reference to the table of the surfaces and solidities of polyedrons under the last problem, the solidity of a tetraedron whose linear edge is 1, is found$=.1178513$, and if this is multiplied by $2\times2\times2$, the product will be the solidity of a tetraedron whose edge is 2.

Ex. 1. What is the solidity of a regular octaedron, whose edges are each 32 inches? Ans. 15447 inches.

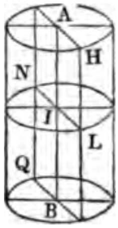
. 2. What is the solidity of a *dodecaedron* whose linear is 2 feet? Ans. 61,3049512 cubic feet.

. 3. Required the solidity of an *icosaedron* whose linear equals 2 feet? Ans. 17,45356 cubic feet.

PROBLEM XI.

To find the convex surface of a right cylinder.

LE.—Multiply the circumference of the base by the h? (Prop. I. B. III.)



c. 1. What is the convex surface of the right der AB, the circumference of whose base is t 4 inches, and its altitude 14 feet?
ere $14\times8\frac{1}{3}=116.666666\times$feet the convex surface required.

x. 2. What is the convex surface of a right cylinder whose h is 20 feet, the diameter of the base being 2 feet?
Ans. 125.664.

PROBLEM XII.

To find the solidity of a cylinder.

ULE 1.—Multiply the area of the base by the height. p. II. B. III.)

Or multiply its convex surface by half the radius of the . (Prop. III. B. III.)

x. 1. What is the solidity of a cylinder, the surface of the being 9 feet, and its altitude 5 feet?
lere $9\times5=45$ feet the solidity required.

x. 2. The circumference of the base of an oblique cylin- is 20 feet, and the perpendicular height 19,318; what is solidity? Ans. 614,926 feet.

x. 3. What is the solidity of a cylinder, the convex surface g 20 feet, and the radius of the base 6 inches? Ans. 5 feet.

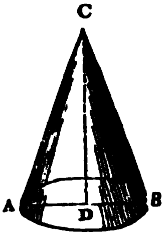
PROBLEM XIII.

To find the convex surface of a cone.

ULE 1.—Multiply the circumference of the base by half slant height. (Prop. IV. B. III.)

2. Or multiply the area of the base by the slant height, and divide this product by the radius of the base. (Prop. V. B. III.)

3. Or multiply half the convex surface of the circumscribing cylinder by the slant height, and divide the product by the vertical height of the cone.



Ex. 1. What is the convex surface of the cone ABCD, the diameter of the base AB being 5 feet, and the slant height AC, 18 feet?

By rule 2 $(.7854 \times 5^2 \times 18) \div 2\frac{1}{2} = 141.372$ square feet, the convex surface of the required.

Ex. 2. What is the convex surface of a cone whose slant height is 20 feet, and the circumference of its base 9 feet?

Here $\frac{20}{2} \times 9 = 90$ square feet, the convex surface required.

Ex. 3. What is the convex surface of a right cone, the convex surface of its circumscribing cylinder being 301,5936, the slant side 10 and the vertical height 8?

Here $301,5936 \div 2 \times 10 \div 8 = 188,496$ the surface required.

Ex. 4. Required the whole surface of a cone whose perpendicular height $16\frac{1}{2}$ feet, and the diameter of its base $26\frac{1}{2}$ inches. Ans. 61,1944 square feet.

Ex. 5. If the axis of a right cone be 16, and the circumference of the base 75.4: what is the whole surface? Ans. 1206.4.

PROBLEM XIV.

To find the solidity of a cone.

Rule 1.—Multiply the area of the base into one-third of the height. (Prop. VII. B. III.)

2. Or multiply the convex surface of the cone by one-third of its distance from the centre of the base. (Prop. VIII. B. III.

Ex. 1. What is the solidity of a right cone whose height is 663, and the diameter of whose base 101?

Here $\overline{101}^2 \times .7854 \times 22\dot{1} = 177062.$

Ex. 2. What is the solidity of a cone whose convex surface is 188.496, and the length of a line drawn from the centre of its base perpendicular to the convex surface $4\frac{4}{5}$?

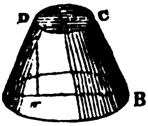
Here $188.496 \times \frac{24}{15} = 301,593.$ the solidity required.

Ex. 3. Required the solidity of a conical church spire, whose perpendicular height is 68.1 yards, and the diameter of its base 10 yards. Ans. 1782.858 cubic yards.

PROBLEM XV.

To find the convex surface of the frustum of a cone.

Rule.—Multiply half the sum of the circumferences of the two bases by the slant height, (Prop. VII. B. III.)



Ex. 1. What is the convex surface of the frustum of a cone, the circumference of the greater base, AB, being 9 feet, the smaller, CD, 2, and the slant height 3.

Here $((9\times2)\div2)\times3=16.5$ the required surface.

Ex. 2. Required the whole surface of the frustum of a cone, the slant height being $10\frac{1}{2}$ feet, the circumference of the greater end $5\frac{1}{3}$ feet, and that of the less $2\frac{1}{4}$ feet.

Ans. 42,47898 square feet.

PROBLEM XVI.

To find the solidity of the frustum of a cone.

Rule 1.—Add into one sum the areas of the two ends of the frustum, with the square root of their product; multiply this sum by one-third of the altitude of the frustum, (Prop. X. B. III.)

2. Or multiply the convex surface of the frustum by one-third of its shortest distance from the centre of the greater base, and to this product add one-third the product of the smaller base into the vertical height of the frustum, (Prop. XI. B. III.)

Ex. 1. What is the solidity of the frustum of a cone whose perpendicular height, rD, (see last diagram,) is 30 inches, the greater diameter, AB, 32 inches, and the less diameter, DC, 24 inches?

By rule 1, $\overline{32}^2\times.7854$ . . . . . . $=804.2496$
the area of the greater end,

And $\overline{24}^2\times.7854$ . . . . . . $=452.3904$
the area of the less end.

Then $\left.\begin{array}{l}\sqrt{804.2496\times452.3904}=\\ \sqrt{363834.79824384}\cdots\end{array}\right\}$ . . . $=603.1872$
the mean area,

The sum $=1859.8272$

Now, $1859.8272\times(30\div3)=18598.272$ cubic inches
$=10.7628$ cubic feet the solid content of the conical frustum required.

Ex. 2. What is the solidity of the frustum of a cone, whose vertical surface is 186.6109, the smaller base 1.1309, the distance from he centre of the base to the conical surface being $4\frac{4}{5}$, and its perpendicular height $7\frac{1}{5}$.

Here $186.6109 \times (5\frac{4}{5} \div 3)$ 298.9107,
And $1.1309 \times (7\frac{1}{5} \div 3)$ 2.7143,

Hence the solidity required= 301,625.

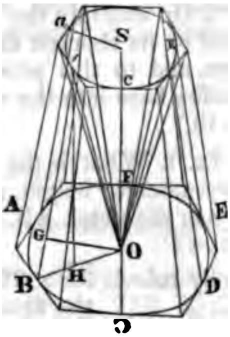
Ex. 3. What is the solidity of the frustum of a cone, the circumference of the greater end being 40, that of the smaller 20, and its perpendicular height 50? Ans. 3713.64.

PROBLEM XVII.

To find the solidity of a conesected cylinder, or of a conesected frustum of a cone.

Rule 1.—Multiply the convex surface by one-third of its shortest distance from the centre of the base in which the conical vertice terminates, (Prop. XIII. B. III.)

2. Or from the solidity of the cylinder or conic frustum, subtract the solidity of a cone, whose base is the wanting base of the cylinder, or frustum, and whose altitude is the altitude of the cylinder or of the frustum.



Ex. 1. What is the solidity of the conesected frustum of a cone OBAFEDCc, whose convex surface is 30, and the length of the line OG drawn from the centre of the base to the convex surface 4?

Here $30 \times (4 \div 3) = 40$ the solidity required.

Ex. 2. What is the solidity of a conesected cylinder whose convex surface is 80, and the radius of the base 5? Ans. 133.333.

PROBLEM XVIII.

To find the surface of a regular polyedroid.

There are three conditions:

First. If the polyedroid has two conical vertices.

RULE.—Multiply the circumference of its inscribed sphere by its axis or vertical diameter, (Prop. XVIII. B. III.)

Second. If the polyedroid has vertical bases, or if the axis terminates in a plane base at each extremity, then

Divide the square of the radius of one of the equal bases by the diameter of the inscribed sphere; and add the quotent to that diameter. And the product of this sum, multiplied by the circumference of the inscribed sphere, is the surface, (Prop. XIX. B. III.)

Third. If the polyedroid has one conical vertice, and one plane base, then

Divide the square of half the radius of the base by the radius of the inscribed sphere, and add this quotent to the axis of the polyedroid. And this sum multiplied by the circumference of the inscribed sphere, will be the surface, (Prop. XIX. Cor B. III.)

Ex. 1. If the circumference of a sphere inscribed in a polyedroid with two vertices be 20 inches, and the diameter of the circumscribing sphere be 8 inches, what is the surface of the polyedroid? Ans. $20\times8=160=$the surface required.

2. If the radii of the two equal bases of a polyedroid be 5 inches, and the radius of the inscribed sphere be 10 inches, what is the surface of the polyedroid?

Ans. $(20+5\times5\div20)\times6.283=1331.99=$the surface of the polyedroid.

3. What is the surface of a regular pentedroid, the radius of whose base is 5 inches?

By the table of polygons under (Prop. IX. *Mens. El. Geom.*) we find the radius of the circumscribed circle of a pentagon whose side is $1=.85065$, and the radius of the inscribed circle 68819, the sum of which is evidently=the axis of the pentaedroid, the radius of whose base is $\frac{1}{2}$; and since the sides of similar figures are proportional,

$(85065+68819)\times5\div\frac{1}{2}=15.3884=$the axis of the pentaedroid-

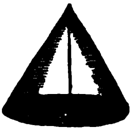
Thus $(15.3884+\frac{5}{2}\times\frac{5}{2}\div6.8819\times43.24=607.26$ the surface required.

PROBLEM XIX.

To find the solidity of a regular polyedroid.

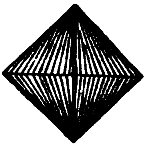
Rule.—Multiply the surface by one-third of the radius of the inscribed sphere, (Prop. XX. B. III.)

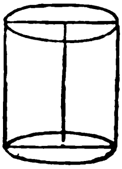
Ex. 1. What is the solidity of a regular tetredroid, the diameter of whose inscribed sphere is 10?



Here the radius of the base will be found$=18.32$, and the axis of the tetredroid will be found$=26.32$; hence, by Problem XVIII., Rule 3d, we have the surface of the tetredroid$=[(17.32\div2)^2+27.32]\times31.416=952.2189$, and, by the rule, $952.2189\times\frac{5}{3}=1587.0128$, the solidity.

Ex. 2. What is the difference in solidity of a quadredroid with conical vertices, and one whose axis terminates in two planes, the diameter of the inserted sphere in each case being 10 inches?

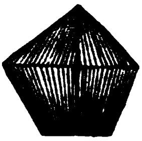




The diameter of their circumscribed spheres will be found $=14.142$, which is also $=$ the axis of the vertical quadredroid, and $31.416\times14.142\times\frac{5}{3}=740.475$, its solidity.

The axis of the quadredroid, with two equal plane bases, is $=$the diameter of the inscribed sphere$=10$, and its surface is $(5^2\div10+10)\times31.415=392.7$, and its solidity$=392.7\times\frac{5}{3}=654.5$. The difference is$=85.975$.

Ex. 3. What is the solidity of a pentedroid, the diameter of whose inscribed sphere is 10 inches?



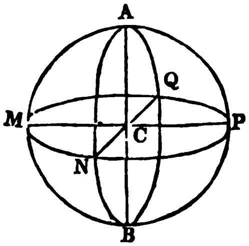
Here $320.5=$the surface; hence $320.5\times\frac{5}{3}=534.15=$the solidity required.

PROBLEM XX.

To find the surface of a sphere.

Rule 1.—Multiply the diameter by the circumference, (Prop. XXII. B. III.)

2. Or multiply the area of a circle of the same diameter by 4, (Prop. XXII. Cor. 1. B. III.)



Ex. 1. What is the surface of a globe whose diameter AB is 12, and the circumference MNPQ 37.6991?

Here $37.6991 \times 12 = 452.3892$ the convex surface.

Ex. 2. How many square inches will cover a globe 6 inches in diameter? Ans. 226.1952 square inches.

POBLEM XXI.

To find the solidity of a sphere.

Rule 1.—Multiply the surface by one-third of the radius, or one-sixth of the diameter, (Prop. XXIII. B. III.)

2. Or multiply the square of the diameter by one-sixth of the circumference, (Prop. XXIII. *Sch.* 3. B. III.)

3. Or multiply the cube of the diameter by .5236, which is the solidity of a sphere whose diameter is 1.

Ex. 1. What is the solidity of a sphere whose diameter is $1\frac{1}{3}$ the convex surface, being 5.58506?

Here $5.58506 \times (1\frac{1}{3} \div 6) = 1.24112$ the solidity required.

Ex. 2. Required to find the solid content of the earth, supposing its circumference to be 25,000 miles.

Ans. 263857437760 cubic miles.

PROBLEM XXII.

To find the convex surface of a segment, or zone of a sphere.

Rule.—Multiply the height of the segment or zone into the circumference of the sphere, (Prop. XXII. *Cor.* 2, B. III.)

Ex. 1. Required the convex surface of the segment of a sphere whose height is 9 inches, and the diameter of the whole sphere is 42 inches.

Here $42 \times 3.1416 = 131.4472$ inches, the circumference of the sphere.

Hence $131.9472 \times 9 = 1187.5248$ square inches, the convex surface of the segment required.

Ex. 2. What is the convex surface of a spherical segment, the diameter of the base being 17.25 feet and its height 4.5 feet? Ans. 297.3229875 square feet.

PROBLEM XXIII.

To find the solidity of a spherical sector.

RULE.—Multiply its spherical surface by one-third of the radius of the sphere, (Prop. XXIII. *Cor.* 1, B. III.)

Ex. 1. What is the solidity of a spherical sector, the diameter of whose sphere is 12, its convex surface being 7?

Here $7\times2=14$, the solidity required.

Ex. 2. Supposing the earth to be a sphere, 7930 miles in diameter, and the polar circle to be 23° 28′ from the pole, what is the solidity of the spherical sector whose base is bounded by this polar circle? Ans. 10,799,867,000 miles.

PROBLEM XXIV.

To find the solidity of a spherical segment.

RULE.—Find the solidity of the spherical sector having the same spherical surface; find, also, the solidity of the cone having the same base as the segment, and whose vertice is in the centre of the sphere; subtract the solidity of the cone from that of the sector, which will give the solidity of the segment, if the segment is less than a hemisphere; and add the solidity of the cone to that of the sector, if the segment be greater than a hemisphere.

Ex. 1. What is the solidity of a spherical segment whose convex surface is 326.7264, the surface of the base of the segment being 310.834, the height of the segment 2, and the diameter of the sphere 52?

Here $336.7264\times7=2287.0648$, the solidity of the sector.
and $310.834\times6\frac{1}{3}=1968.607$ " " " cone,
318.4578 " " " segment.

Ex. 2. What is the solidity of a spherical segment whose height is .15051, the diameter of the base being 19.69897, and the radius of the sphere 10?

PROBLEM XXV.

To find the solidity of a spherical zone, or frustum.

RULE.—Multiply the spherical surface of the zone by one-third of the radius of the sphere, and each end of the the zone by one-third of its distance from the centre of the sphere; the sum of these products will be the solidity.

For, if from the zone there be taken two cones, whose bases are the ends of the zone, and whose altitudes are the distances of those bases from the centre, the portion remaining will be equal to the spherical surface multiplied by one-third of the radius; since the portions required to complete the sphere in such case, are two spherical sectors, whose solidities are in the same ratio in regard to their spherical surfaces as the whole sphere.

Ex. 1. What is the solidity of a spherical zone, the diameter of whose sphere is 12, the convex surface of the zone 301.594, the surface of each base 15.708, and their perpendicular distances from the centre 4?

Here $301.594 \times 2 = 603.188$

and $15.708 \times 1\frac{1}{3} \times 2 = 41.888$

645.076 the solidity required.

Ex. 2. What is the solidity of a spherical frustum, the diameter of the sphere being 9, one end of frustum 2, and the other 3?

EXAMPLES FOR PRACTICE.

Ex. 1. How many solid feet in a square prism whose altitude is $5\frac{1}{2}$ feet, and each side of its base $1\frac{1}{3}$ feet?
Ans. $9\frac{7}{9}$ solid feet.

Ex. 2. What is the solidity of a quadrangular pyramid each side of whose base is 30 feet, the altitude of each face being 25 feet? Ans. 6000 solid feet.

Ex. 3. What is the solidity of the frustum of a quadrangular pyramid, the side of the inferior base being 15 inches, that of the superior 6 inches, and the altitude 24 feet?
Ans. 19.5 solid feet.

Ex. 4. The dimensions of the great pyramid, Geeza, in Egypt, as derived from the accurate measurement of M. Coutelle, are as follows:—Side of the base, 763.62 feet; entire altitude, 476.01; altitude of the frustum, 456.43 feet. Required the area of the base, the whole surface and the solid contents of the frustum. Ans. Area of the base, 583115.5 square feet. Whole surface, 1517757.05 square feet. Solidity, 92516502 solid feet.

Ex. 5. What is the solidity of a cylinder whose altitude, and the circumference of whose base are each 20 feet?
Ans. 636.62 solid feet.

Ex. 6. What is the convex surface of a right triangular prism whose altitude is 20 feet, and each side of the base 18 inches? Ans. 90 square feet.

Ex. 7. What is the surface of a cube whose side is 20 feet? Ans. 2400 square feet.

Ex. 8. What is the convex surface of a cylinder whose altitude is 20 feet, and the diameter of its base 2 feet? Ans. 125.664 square feet.

Ex. 9. What is the whole surface of a cylinder whose altitude is 10 feet, and the circumference of its base 3 feet? Ans. 31.4324 square feet.

Ex. 10. What is the convex surface of a quadrangular pyramid whose altitude is 20 feet, and each side of the base 30 feet? Ans. 1500 square feet.

Ex. 11. What is the convex surface of a cone whose side is 20 feet, and the circumference of its base 9 feet? Ans. 90 square feet.

Ex. 12. How many square feet in the surface of the frustum of a quadrangular pyramid, the altitude of each face being 10 feet, each side of the inferior base 3 feet 4 inches, and of the superior 2 feet 3 inches? Ans. 110 square feet.

Ex. 13. The earth being supposed a perfect sphere, whose diameter is 7912 miles, what is its surface? Ans. 196660672 square miles.

Ex. 14. What is the convex surface of the frigid zone of the earth, the altitude of this zone being 327 miles? Ans. 8127912 square miles.

Ex. 15. What is the convex surface of the torrid zone, the altitude being 3150.6 miles? Ans. 78311313.6 square miles.

Ex. 16. What is the convex surface of either temperate zone, the altitude being 2053.7 miles? Ans. 51046767.2 square miles.

Ex. 17. What is the solidity of the earth? Ans. 259328406532 solid miles nearly.

Ex. 18. A prismatic pile of wood whose base is 4 feet square, and whose altitude is 8 feet, comprises a cord, what part of a cord then is contained in a pile 7½ feet long, 3½ feet broad, and 4 feet high? Ans. $\frac{105}{128}$.

Ex. 19. What is the weight of a round bar of wrought iron 12 feet long, 1⅛ inches in diameter; a cubic inch of the metal weighing 4½ ounces, as it does very nearly? Ans. 40.2 lbs.

Ex. 20. The channel of a certain river is 80 yards wide, and averages 8½ feet deep at a certain place, and the current at such place sets at the rate of 2½ miles per hour, what is its velocity at another place where the channel is contracted to 60 yards in width and 9 feet deep, supposing the same quantity of water to pass through each place, and how many cubic feet pass per minute? Ans. 3.148 miles per hour, and 59.840 cubic feet per minute.

Ex. 21. It is required to find the thickness of the lead of a pipe of an inch and quarter bore, which weighs 14lb. per yard in length; the cubic foot of lead weighing 11325 ounces?

Ans. ·20738 inches.

Ex. 22. What will the diameter of a globe be, when the solidity and superficial content are expressed by the same number?

Ans. 6.

Ex. 23. A sack that would hold 3 bushles of corn, is 22½ inches broad when empty; what will that sack contain which, being of the same length, has twice its breadth or circumference?

Ans. 12 bushels.

Ex. 24. Seven men bought a grinding-stone, of 60 inches diameter, each paying one-seventh part of the expense; what part of the diameter must each grind down for his share?

Ans. the 1st, 4.4508; 2d, 4.8400; 3d, 5.3535; 4th, 6.0765;
5th, 7.2079; 6th, 9.3935; 7th, 22.6778 inches.

Ex. 25. How many three-inch cubes may be cut out of a twelve-inch cube?

Ans. 64.

Ex. 26. To divide a cone into three equal parts by sections parallel to the base, and to find the altitudes of three parts, the height of the whole cone being 20 inches?

Ans. The upper part 13.867, the middle part 3.604,
the lower part 2.528.

Ex. 26. A gentleman has a bowling-green 300 feet long, and 200 feet broad, which he would raise one foot higher, by means of the earth to be dug out of a ditch that goes round it; to what depth must the ditch be dug, supposing its breadth to be every where 8 feet?

Ans. $7\frac{23}{86}$ feet.

Ex. 27. How high above the earth must a person be raised, that he may see one-third of its surface?

Ans. To height of the earth's diameter.

Ex. 28. A cubic foot of brass is to be drawn into a wire of 1-40th of an inch in diameter; what will the length of the wire be, allowing no loss in the metal?

Ans. 97784.797 yards, or 55 miles 984.646 yards.

Ex. 29. How much water can be put into a cubical vessel 3 feet deep, which has been previously filled with cannon-balls of the same size, 2, 4, 6, or 9 inches in diameter, regularly arranged in tiers, one directly above another?

Ans. 96⅕ wine gallons.

Ex. 30. What is the solidity of the greatest square prism which can be cut from a cylindrical stick of timber, 2 feet 6 inches in diameter, and 56 feet long?

Ans. 175 cubic feet.

Ex. 31. How many such globes as the earth are equal in bulk to the sun, if the former is 7930 miles in diameter, and the latter 890,000? Ans. 1,413,678.

Ex. 32. How many cubic feet of wall are there in a conical tower 66 feet high, if the diameter of the base be 20 feet from outside to outside, and the diameter of the top 8 feet; the thickness of the wall being 4 feet at the bottom, and decreasing regularly, so as to be only 2 feet at the top? Ans. 7188.

Ex. 33. If a conical cistern be 3 feet deep, 7½ feet in diameter at the bottom, and 5 feet at the top, what will be the depth of a fluid occupying half its capacity? Ans. 14.535 inches.

Ex. 34. If a globe 20 inches in diameter be perforated by a cylinder 16 inches in diameter, the axis of the latter passing through the centre of the former; what part of the solidity, and the surface of the globe, will be cut away by the cylinder?

Ans. 3284 inches of the solidity, and 502,655 of the surface.

Ex. 35. What is the solidity of the greatest cube which can be cut from a sphere 3 feet in diameter? Ans. 5⅕ feet.

Ex. 36. Required the surface and the solidity of the greatest parallelopipedon, whose length, breadth, and depth, together make 18?

Ex. 37. The surface of a square prism is 546: what is its solidity when a maximum?

Ex. 38. The content of a cylinder is 159.645968: what is its surface when a minimum?

Ex. 39. The whole surface of a right cone is 201.061952: what is its solidity when a maximum?

Ex. 40. The surface of a triangular pyramid is 43.30127: what is its capacity when a maximum?

Ex. 41. The radius of a sphere is 10. Required the solidities of this sphere, of its circumscribed equilateral cone, and of its circumscribed cylinder.

Ex. 41. The surface of a sphere is 28.274337, and of an irregular polyedron circumscribed about it 35: what are their respective solidities?

Ex. 42. The solidity of a sphere, equilateral cone, and Archimedean cylinder, are each 500: what are the surfaces and respective dimensions of each?

Ex. 43. If the surface of a sphere be represented by the number 4, the circumscribed cylinder's convex surface and whole surface will be 4 and 6, and the circumscribed equilateral cone's convex and whole surface, 6 and 9 respectively. Show how these numbers are deduced.

Ex. 44. The solidity of a sphere, circumscribed cylinder, and circumscribed equilateral cone, are as the numbers 4, 6, and 9. Required the proof.

JARES, CUBES, AND THEIR PROPERTIES.

:N it is proposed to find the length of the side of a given
or the edge of a given cube, we find it necessary to
an arithmetical process. But neither geometry nor
etic furnishes us with accurate modes of reasoning,
will lead us to the true determination of such lines, in-
ent of mechanical considerations or trials. It is true,
re rules, founded on mathematical investigations, which
t us to the required data; but, in all cases, it is neces-
assume a quantity, or magnitude, as the value of the
d line, and then ascertain by trial whether this is the
e; and if this is too small, we assume some other quan-
ich we add to this, and after having ascertained that
n of the quantities assumed are too small, we add again,
on, till we come at the true value of the line, or till we
imate sufficiently near to it for our purpose.

ANALYSIS OF POWERS AND ROOTS.

is been shown that the surface of a square is equal to
oduct of its side into itself; and that the solidity of
is equal to the continued product of the edge twice
self; therefore, the finding the side of a square, or the
f a cube, consists in dividing the given square, or cube,
ich factors as compose them; and this, in the case of the
, consists in dividing it into two equal factors; these
s are, in arithmetic, termed roots, and the process by
they are found is called the extraction of roots. The
ty from which such root is to be extracted is called the
of the root, and is further designated according to the
er of times the root is multiplied into itself to produce
ower. Thus the square, or second power, consists of
nple product of the root into itself; thus $3\times3=9$, is the
d power, or square of 3. The cube, or third power,
ts of the continued product of the root twice into itself:
$3\times3\times3=27$, which is the third power of 3. These two

powers are the only ones which elementary geometry furnishes us, but arithmetic and algebra conduct us to many other higher powers, such as the *biquadrate,* or fourth power, the fifth, sixth, &c. Algebra furnishes us with methods of dividing a given power into its constituent factors, when such power is perfect, and under an algebraic expression.

From our common methods of notation, it becomes necessary to consider any given power to consist of more than one term, unless it is resolvable immediately, by a single trial, into its constituent factors; and, in this view of the case, we will suppose the second power to be put under the algebraical form $(a+b)^2=a^2+2ab+b^2=a^2+(2a+b)b$, the square of the two terms, which is exhibited geometrically in Prop. X. B. IV. *El. Geom.*, where it appears that a is the first term of the root, and b is the second. Let a be the first divisor, thus,

$$\begin{array}{rl}
a) & a^2+2ab+b^2(a+b \\
 & \underline{a^2} \\
2a+b) & 2ab+b^2 \\
 & \underline{2ab+b^2}
\end{array}$$

Let the second divisor be $2a+b$, or double the first term, increased by the second; whence we have the root required, $a+b$.

Again, let this power consist of the squares of three terms: thus, $(a+b+c)^2=a^2+2ab+b^2+2ac+2bc+c^2$

$$=a^2+(2a+b)b+(2a+2b+c)c$$

(for a geometrical construction see Prop. X. B. IV. *Cor.* 4. *El. Geom.*) where a is the first term in the root, b the second, and c the third term; hence, we may make a the first divisor, $2a+b$ the second, and $2a+2b+c$ the third, and the mode of extraction is as follows:

$$\begin{array}{rl}
\text{1st divisor, } a) & a^2+2ab+b^2+2ac+2bc+c^2(a+b+c \\
 & \underline{a^2} \\
\text{2d divisor, } 2a+b) & 2ab+b^2 \\
 & \underline{2ab+b^2} \\
\text{3d divisor, } 2a+2b+c) & 2ac+2bc+c^2 \\
 & \underline{2ac+2bc+c^2}
\end{array}$$

whence we have the root $a+b+c$.

Now, since it is found by observation that the square of a single figure never consists of more than two places or terms in our decimal notation, and the square of two figures, of not more than four nor less than three; and the square of three figures, of not more than six places, and not less than five, and so on: we may, from this law, determine the number of figures which we are to have in the root.

Hence, from the foregoing, we may deduce the following le for the

EXTRACTION OF THE SQUARE ROOT.

RULE.

Divide the given number into periods of two figures each setting a point over the place of units, another over the ace of hundreds, and so on over every second figure from e right to the left hand in integers, and toward the right in cimals.

Find the greatest square in the first period on the left hand, d set its root on the right hand of the given number as for quotient in division.

Subtract the square thus found from the said period, and the remainder annex the two figures of the next following pe-d, for a new dividend.

Double the root, above mentioned, for a new divisor, and d how often it is contained in the new dividend, exclusive of right hand figure, and set that quotent figure both in the otent and divisor.

Multiply the whole augmented divisor by this last quotent ure, and subtract the product from the said dividend, bring- down to the remainder, the next period of the given num-r for a new dividend.

Repeat the same process as before, viz.: find another new visor by doubling its last figure, from which, and the last vidend, find the next figure in the root as before, and so on rough all the periods to the last.

If, when the periods are all exhausted, and there is still a mainder, whose root is not accurately attainable, cyphers ay be annexed, and the process may be continued at plea-ire, or till the true root is found.

Examples.

To find the square oot of 29506624. }

. . . .
29506624(5432 *root*
25
104)450
416
1083)3466
3249
10862)21724
21724

Scholium. When the root is to be extracted to many places of figures, and where it is interminable, the work may be considerably shortened, thus—

Having proceeded in the extraction after the common method, till there be found half the required number of figures in the root, or one figure more than half; then, for the rest, divide the last remainder by its corresponding divisor, dropping one figure from the right hand of the divisor at each operation.

To find the square root of 2 to 9 places of figures.

```
                         2  (1.41421356
                         1
                        ---
                     24)1.00
                          96
                         ---
                     281)400
                         281
                        ----
                   2824)11900
                        11296
                        -----
                    2828)60400
                         56564
                         -----
                    28284)3836
by contraction       ····· 1008 remainder
                            160 remainder
                             19 remainder
                              2 remainder.
```

Here 1.41421356 is the root required.

EXTRACTION OF THE CUBE ROOT.

When a cube is given, whose root is to be extracted, we may at once determine the number of figures or digits in the root, by considering that the cube of one figure never amounts to more than three figures, and the cube of two figures never amounts to more than six, nor less than four; and so on. Therefore, if we divide the given cube into periods of three figures each, beginning at the right hand, and proceed toward the left, the number of periods will indicate the number of digits in the root. And, in like manner, if the nth root is to be extracted, the power is to be divided into periods of n digits.

These things being premised, we may proceed in our in-tigations to determine what those digits actually are.

f we cube the binomial $a+b$, we shall have

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$

Or, if we cube an integer, consisting of two digits, for ex-ple, 26, we may have

$$(20+6)^3=20^3+3\times20^2\times6+3\times20\times6^2+6^3=26^3$$
$$\therefore\ 26^3=20^3+3\times20\times6\times(20+6)+6^3$$
$$=20^3+3\times20\times6(21+5)+6^3$$
$$=20^3+(3\times20\times21+3\times20\times5+6^2)\times6\ldots(a)$$

Again, let us cube a trinomial $a+b+c$, and we have

$$a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2+b^3+3b^2c+3b^2+c^3$$

Or, if we cube an integer of three digits, 2, 6, 5, we may ve, in the first instance, by resolving it into two terms, $5^3=(260+5)^3=260^3+(3\times260\times261+3\times260\times4+5^2)\times5$ by oceeding as above.

But, $260^3=26^3\times10^3$, and formula (a) gives

$$26^3=20^3+(3\times20\times21+3\times20\times5+6^2)\times6$$

Hence,

$$\left.\begin{array}{l}265^3=200^3+(3\times20\times21+3\times20\times5+6^2)\times6\times10^3\\ +(3\times260\times261+3\times260\times4+5^2)\times5\ldots\ldots\ldots\ldots\end{array}\right\}\ldots(b)$$

Thus we may proceed with an integer of four, or any num-r of digits. But the above is sufficient to show that every be may be transformed so as to consist of as many terms its root has digits; and that each of these terms, except the st, is the product of three factors; two of which are the gits of the root and the cube of unity, with cyphers annexed, d continued as far as indicates the place of this digit in the ot. The third factor consists of three parts, one of which is e square of the digit to which the term belongs; another rt is the product of 3 into the whole of the root up to that git, followed by a cypher, and multiplied by the whole of e root up to that digit, followed by a unit; and the remain-g part is the product of 3 into the whole of the root up to at digit, multiplied by 1 less than that digit.

From the above considerations, it may be inferred that, to d any digit of the root, we must divide the known part of e term of the power which belongs to the digit, required by e product of 3, into the root already known, followed by a ypher, multiplied by that root, followed by unity.

It is also evident that, having found the digit, it is then easy make up the whole of the term of the cube which corres-onds to that digit.

Let it be required to extract the cube root of 17576.

First. Let the number be divided into periods as directed above.

$\dot{1}75\dot{7}6$(26 *root*
subtrahend $2^3=$ 8
divisor $3\times20\times21=1260$)9576 *resolvend*
$3\times20\times5=$ 300
$6^2=$ 36
1596
6
subtrahend 9576
9576

By comparing this operation with *formula* (*a*), the first subtrahend, viz.: 8 corresponds with the term 20^3, or 8000, when the cyphers are neglected, these are neglected on account of its location; since, if they should be retained, its value would not be altered.

Again, the divisor for finding the second digit, is that part of the compound factor of the second term of *formula* (*a*), which does not involve the required digit.

And the second subtrahend is the entire second term of (*a*), of which the formation is easy and expeditious in practice, viz.: first, adding to the divisor the product of its two first factors multiplyed by one less than the new digit; then adding the square of the new digit, and multiplying the aggregate also by this new digit.

However numerous may be the digits in the root, they may each be found by the same operation.

To find the cube root of 18609625.

$\dot{1}86\dot{0}96\dot{2}5$(265 *root*
subtrahend $2^3=$ 8
divisor $3\times20\times21=1260$)10609 *resolvend*
$60\times5=$ 300
$6^2=$ 36
1596
6
subtrahend 9576
1033625 *resolvend*

$$\begin{array}{rr}
\textit{divisor}\ 3\times 260\times 261= & 203580)1033625\ \textit{resolvend} \\
780\times 4= & 3120 \\
6^2= & 25 \\
\hline
 & 206725 \\
 & 5 \\
\hline
\textit{subtrahend} & \underline{1033625}
\end{array}$$

This operation may be compared with *formula* (*b.*)
Enough has been exhibited to make evident the following:

GENERAL RULE.

Divide the given power or number into periods of three figures each, begining at the units place, and point off every third figure toward the left, and if there be decimals, also toward the right.

Find the greatest cube in the left hand period, place its root on the right hand, and subtract its cube from the first period. To the right of the remainder bring down the next period, thereby forming a second resolvend.

Then take three times the product of the root already found followed by a cypher, into that root followed by a unit, which makes a divisor: divide the resolvend by this divisor, and the quotient will be the next digit in the root required, requiring corrections if it is found too great or too small.

Then form a subtrahend by adding to the divisor the product of the two first factors of the divisor, multiplied by one less than the new digit of the root, and also the square of this new digit, and then by multiplying the aggregate of these three by the new digit.

Lastly, take the subtrahend from the resolvend, and bring down to the right of the remainder the next period. This forms a new resolvend; from which, by repeating the operation, the next digit of the root may be found, and so on till the whole root is perfected, or till its approximation is sufficiently extended.

Note.—In finding the second digit, the quotient arising from the division, must generally be taken one less than would appear to be the true quotient, and in some instances two less; for if it be taken too great, the subtrahend will likewise be too great. With respect to the subsequent digits, this remark will not apply.

Let the cube root of the following number be required.

28113875.829398(304.07 *root*
27

divisor $3\times300\times301=270900$)1113875 *resolvend*
$900\times3=$ 2700
$4^2=$ 16

273616
4

subtrahend 1094464

div. $3\times3040\times3041=27733920$)19411829 *resolvend*

div.$3\times30400\times30401=2772571200$)19411829398
$91200\times6=$ 547200
$7^2=$ 49

2773118449
7

19411829143

255 *resolvend.*

By annexing cyphers, the extractions may be continued indefinitely if the series should not terminate.

GENERAL EXTRACTION OF ROOTS.

By means of the extractions of the square and cubic roots, the higher roots may be found whenever the index of the power consists of integeral factors.

Thus to find the biquadratic or fourth root of an integer or decimal.

First find the square root, then take the square root of the result, and this last root will be the biquadratic root required.

For, if a^4 be any integer or decimal, its square root is a^2, since $a^4=a^2\times a^2$, and the square root of a^2 is a.

So, if the 8th root is required, we have only to repeat the extraction of the square root three times.

$$\text{Thus, } a^8=a^4\times a^4$$
$$a^4=a^2\times a^2$$
$$\text{and } a^2=a\times a$$

And in the same manner may the 16th, 32d, &c., be obtained.

Also, the 6th root may be obtained by first extracting the square root, and then the cube root of that root. The 9th root may be found by two successive extractions of the cube root, the 12th root by the square root twice, and the cube root once.

And so on for any root when the index of the power can be divided into integral factors.

If we were required to find the fifth, seventh, eleventh, or any other root, the index of which, however, is a prime number, rules might be devised according to principles already explained, but the process would be so laborious, and their application of such rare occurrence, that it will not be necessary to give them here.

I shall here give a table of squares and cubes, square roots, and cube roots, of numbers from 1 to 1000.

INTRODUCTION TO THE TABLE OF POWERS AND ROOTS.

In the following table of powers and roots of numbers, the first column contains the natural numbers, and opposite thereto in the second and third, are found the squares and cubes answering to each of those numbers respectively, and in the fourth and fifth are found the square roots and cube roots of the same numbers.

Thus suppose it were required to find the square, and also the cube of 20. We shall find in the column of squares opposite to 20 in the columns of numbers, 400, hence 400 is the square of 20; and in the same manner will the cube be found=8000; hence also if the square root and cube root of 20 is required; here in the fourth column opposite to 20 in the column of numbers will be found 4.472136=the square root of 20, and the cube root is in like manner found=2.714418.

In this table not only may the regular series of powers and roots for the given series of numbers be found, but also the roots of many higher, and the powers of many lower, and intermediate numbers may be found; thus the regular series of numbers whose powers and roots are computed, extend only to 1000, but since the number from which a power is raised is the root of that power, if the root of any number corresponding to any power, either square or cube in the table is required, the root may be taken from the column of numbers.

Thus, if the cube root of 7880599 is required, it will be found in the column of numbers opposite to 7880599, in the column of cubes=199.

Also, if the square or cube of any number corresponding with any number found in the column of roots is required, it may be found in the column of numbers opposite thereto.

Thus let it be required to find the square of 13.96424; this we shall find in the column of numbers opposite 13.96424 in the column of square roots=195.

Also, if the root of any decimal number less than unity is required, we may regard the decimal as an integral number, and proceed to take out the power or root answering thereto; and then place a decimal point before it, in such position as to give the power or root its proper value in reference to the decimal number, prefixing cyphers when necessary.

Thus, let it be required to find the square of .16, here in the table opposite 16 will be found 256, to which, if we prefix a cypher, we shall have .0256=the square of .16; since the square of any decimal number contains double the number of decimal places of its prime number.

And the cube of 16 will be found=4096, and since the cube of any decimal number contains three times as many decimal places as the given number, if to this we prefix two cyphers with a decimal point, we shall have the cube of .16=.004096.

Also, if it were required to find the square root of .16, we find in the column of roots opposite 16 in the column of numbers 4; to which, if we prefix a decimal point, we shall have the square root of .16=.4, and the cube root of the same number, we shall find=.259842.

Whenever it is necessary to find the power or root of any number less than 1000 consisting of integers and decimals which cannot be found in the table, such power or root may be approximately found by taking a proportional value between the power or root of the next higher and next lower number in the table, always observing that this will give the power a little in excess and the root a little in defect, as in the following proportion.

As the difference of the next higher and next lower number is to the difference of their powers or roots, so is the difference of the next lower, and the given number to the correction to be added to the power or root of the next lower number to produce the power or root required. Thus, if the square of 70.85 is required, we shall find the square of 70=4900, and the square of 71=5041; hence, we have 71—70=1 : 5041—4900=141 : : 70.85—70=.85 : 11985 the correction, which, if we add to 4900, we have 5019.85, the power required. And if the square root of 70.85 is required, it may be found in a similar manner. Thus the square root of 70 is 8.3666003, and that of 71 is 8.4261498, the difference is .0595495, hence we have 1 : 0595495 : : .85 : .0500617. the correction to be added to 8.3666003, therefore the root =8.416662, which will be found a little in defect, the true root being 8.417184.

NO.	SQUARE.	CUBE.	SQUARE ROOT.	CUBE ROOT.
1	1	1	1.0000000	1.000000
2	4	8	1.4142136	1.259921
3	9	27	1.7320508	1.442250
4	16	64	2.0000000	1.587401
5	25	125	2.2360680	1.709976
6	36	216	2.4494897	1.817121
7	49	343	2.6457513	1.912933
8	64	512	2.8284271	2.000000
9	81	729	3.0000000	2.080084
10	100	1000	3.1622777	2.154435
11	121	1331	3.3166248	2.223080
12	144	1728	3.4641016	2.289428
13	169	2197	3.6055513	2.351335
14	196	2744	3.7416574	2.410142
15	225	3375	3.8729833	2.466212
16	256	4096	4.0000000	2.519842
17	289	4913	4.1231056	2.571282
18	324	5832	4.2426407	2.620741
19	361	6859	4.3588989	2.668402
20	400	8000	4.4721360	2.714418
21	441	9261	4.5825757	2.758923
22	484	10648	4.6904158	2.802039
23	529	12167	4.7958315	2.843867
24	576	13824	4.8989795	2.884499
25	625	15625	5.0000000	2.924018
26	676	17576	5.0990195	2.962496
27	729	19683	5.1961524	3.000000
28	784	21952	5.2915026	3.036589
29	841	24389	5.3851648	3.072317
30	900	27000	5.4772256	3.107232
31	961	29791	5.5677644	3.141381
32	1024	32768	5.6568542	3.174802
33	1089	35937	5.7445626	3.207534
34	1156	39304	5.8309519	3.239612
35	1225	42875	5.9160798	3.271066
36	1296	46656	6.0000000	3.301927
37	1369	50653	6.0827625	3.332222
38	1444	54872	6.1644140	3.361975
39	1521	59319	6.2449980	3.391211
40	1600	64000	6.3245553	3.419952
41	1681	68921	6.4031242	3.448217
42	1764	74088	6.4807407	3.476027
43	1849	79507	6.5574385	3.503398
44	1939	85184	6.6332496	3.530348
45	2025	91125	6.7082039	3.556893
46	2116	97336	6.7823300	3.583048
47	2209	103823	6.8556546	3.608826
48	2304	110592	6.9282032	3.634241
49	2401	117649	7.0000000	3.659306
50	2500	125000	7.0710678	3.684031

A TABLE OF

NO.	SQUARE.	CUBE.	SQUARE ROOT.	CUBE ROOT.
51	2601	132651	7.1414284	3.708430
52	2704	140608	7.2111026	3.732511
53	2809	148877	7.2801099	3.756286
54	2916	157464	7.3484692	3.779763
55	3025	166375	7.4161985	3.802953
56	3136	175616	7.4833148	3.825862
57	3249	185193	7.5498344	3.848501
58	3364	195112	7.6157731	3.870877
59	3481	205379	7.6811457	3.892996
60	3600	216000	7.7459667	3.914867
61	3721	226981	7.8102497	3.936497
62	3844	238328	7.8740079	3.957892
63	3969	250047	7.9372539	3.979057
64	4096	262144	8.0000000	4.000000
65	4225	274625	8.0622577	4.020726
66	4356	287496	8.1240384	4.041240
67	4489	300763	8.1853528	4.061548
68	4624	314432	8.2462113	4.081656
69	4761	328509	8.3066239	4.101566
70	4900	343000	8.3666003	4.121285
71	5041	357911	8.4261498	4.140818
72	5184	373248	8.4852814	4.160168
73	5329	389017	8.5440037	4.179339
74	5476	405224	8.6023253	4.198336
75	5625	421875	8.6602540	4.217183
76	5776	438976	8.7177979	4.235824
77	5929	456533	8.7749644	4.254321
78	6084	474552	8.8317609	4.272659
79	6241	493039	8.8881944	4.290841
80	6400	512000	8.9442719	5.308870
81	6561	531441	9.0000000	4.326749
82	6724	551368	9.0553851	4.344481
83	6889	571787	9.1104336	4.362071
84	7056	592704	9.1651514	4.379519
85	7225	614125	9.2195445	4.396830
86	7396	636056	9.2736185	4.414005
87	7569	658503	9.3273791	4.431047
88	7744	681472	9.3808315	4.447960
89	7921	704969	9.4339811	4.464745
90	8100	729000	9.4868330	4.481405
91	8281	753571	9.5393920	4.497942
92	8464	778688	9.5916630	4.514357
93	8649	804357	9.6436508	4.530655
94	8836	830584	9.6953597	4.546836
95	9025	857375	9.7467943	4.562903
96	9216	884736	9.7979590	4.578857
97	9409	912673	9.8488578	4.594701
98	9604	941192	9.8994949	4.610436
99	9801	970299	9.9498744	4.626065
100	10000	1000000	10.0000000	4.641589

NO.	SQUARE.	CUBE.	SQUARE ROOT.	CUBE ROOT.
101	10201	1030301	10.0498756	4.657010
102	10404	1061208	10.0995049	4.672330
103	10609	1092727	10.1488916	4.687548
104	10816	1124864	10.1980390	4.702669
105	11025	1157625	10.2469508	4.717694
106	11236	1191016	10.2456301	4.732624
107	11449	1225043	10.3940804	4.747459
108	11664	1259712	10.3923048	4.762203
109	11881	1295029	10.4403065	4.776856
110	12100	1331000	10.4880885	4.791420
111	12321	1367631	10.5356538	4.805896
112	12544	1404928	10.5830052	4.820284
113	12769	1442897	10.6301458	4.834588
114	12996	1481544	10.6770783	4.848808
115	13225	1520875	10.7238053	4.862944
116	13456	1560896	10.7703296	4.876999
117	13689	1601613	10.8166538	4.890973
118	13924	1643032	10.8627805	4.904868
119	14161	1685159	10.9087121	4.918685
120	14400	1728000	10.9544512	4.932424
121	14641	1771561	11.0000000	4.946088
122	14884	1815848	11.0453610	4.959675
123	15129	1860867	11.0905365	4.973190
124	15376	1906624	11.1355287	4.986631
125	15625	1953125	11.1803399	5.000000
126	15876	2000376	11.2249722	5.013298
127	16129	2048383	11.2694277	5.026526
128	16384	2097152	11.3137085	5.039684
129	16641	2146689	11.3578167	5.052774
130	16900	2197000	11.4017543	5.065797
131	17161	2248091	11.4455231	5.078753
132	17424	2299968	11.4891253	5.091643
133	17689	2352637	11.5325626	5.104469
134	17956	2406104	11.5758369	5.117230
135	18225	2460375	11.6189500	5.129928
136	18496	2515456	11.6619038	5.142563
137	18769	2571353	11.7046999	5.155137
138	19044	2628072	11.7473444	5.167649
139	19321	2685619	11.7898261	5.180101
140	19600	2744000	11.8321596	5.192494
141	19881	2803221	11.8743421	5.204828
142	20164	2863288	11.9163753	5.217103
143	20449	2924207	11.9582607	5.229321
144	20736	2985984	12.0000000	5.241482
145	21025	3048625	12.0415946	5.253588
146	21316	3112136	12.0830460	5.265637
147	21609	3176523	12.1243557	5.277632
148	21904	3241792	12.1655251	5.289572
149	22201	3307949	12.2065556	5.301459
150	22500	3375000	12.2474487	5.313293

NO.	SQUARE.	CUBE.	SQUARE ROOT.	CUBE ROOT.
151	22801	3442951	12.2882057	5.325074
152	23104	3511808	12.3288280	5.336803
153	23409	3581577	12.3693169	5.348481
154	23716	3652264	12.4096736	5.360108
155	24025	3723875	12.4498996	5.371685
156	24336	3796416	12.4899960	5.383213
157	24649	3869893	12.5299641	5.394690
158	24964	3944312	12.5698051	5.406120
159	25281	4019679	12.6095202	5.417501
160	25600	4096000	12.6491106	5.428835
161	25921	4173281	12.6885775	5.440122
162	26244	4251528	12.7279221	5.451362
163	26569	4330747	12.7671453	5.462556
164	26896	4410944	12.8062485	5.473703
165	27225	4492125	12.8452326	5.484806
166	27556	4574296	12.8840987	5.495865
167	27889	4657463	12.9228480	5.506879
168	28224	4741632	12.9614814	5.517848
169	28561	4826809	13.0000000	5.528775
170	28900	4913000	13.0384048	5.539658
171	29241	5000211	13.0766968	5.550499
172	29584	5088448	13.1148770	5.561298
173	29929	5177717	13.1529464	5.572054
174	30276	5268024	13.1909060	5.582770
175	30625	5359375	13.2287566	5.593445
176	30976	5451776	13.2664992	5.604079
177	31329	5545233	13.3041347	5.614673
178	31684	5639752	13.3416641	5.625226
179	32041	5735339	13.3790882	5.635741
180	32400	5832000	13.4164079	5.646216
181	32761	5929741	13.4536240	5.656652
182	33124	6028568	13.4907376	5.667051
183	33480	6128487	13.5277493	5.677411
184	33856	6229504	13.5646600	5.687734
185	34225	6331625	13.6014705	5.698019
186	34596	6434856	13.6381817	5.708267
187	34969	6539203	13.6747943	5.718479
188	35344	6644672	13.7113092	5.728654
189	35721	6751269	13.7477271	5.738794
190	36100	6859000	13.7840488	5.748897
191	36481	6967871	13.8202750	5.758965
192	36864	7077888	13.8564065	5.768998
193	37249	7189057	13.8924440	5.778996
194	37636	7301384	13.9283883	5.788960
195	38025	7414875	13.9642400	5.798890
196	38416	7529536	14.0000000	5.808786
197	38809	7645373	14.0356688	5.818648
198	39204	7762392	14.0712473	5.828476
199	39601	7880599	14.1067360	5.838272
200	40000	8000000	14.1421356	5.848035

NO.	SQUARE.	CUBE.	SQUARE ROOT.	CUBE ROOT.
201	40401	8120601	14.1774469	5.857765
202	40804	8242408	14.2126704	5.867464
203	41209	8365427	14.2478068	5.877130
204	41616	8489664	14.2828569	5.886765
205	42025	8615125	14.3178211	5.896368
206	42436	8741816	14.3527001	5.905941
207	42849	8869743	14.3874946	5.915481
208	43264	8998912	14.4222051	5.924991
209	43681	9123329	14.4568323	5.934473
210	44100	9261000	14.4913767	5.943911
211	44521	9393931	14.5258390	5.953341
212	44944	9528128	14.5602198	5.962731
213	45369	9663597	14.5945195	5.972091
214	45796	9800344	14.6287388	5.981426
215	46225	9938375	14.6628783	5.909727
216	46656	10077696	14.6969385	6.000000
217	47089	10218313	14.7309199	6.009244
218	47524	10360232	14.7648231	6.018463
219	47961	10503459	14.7986486	6.027650
220	48400	10648000	14.8323970	6.036811
221	48841	10793861	14.8660687	6.045943
222	49284	10941048	14.8996644	6.055048
223	49729	11089567	14.9331845	6.064126
224	50176	11239424	14.9666295	6.073177
225	50625	11390625	15.0000000	6.082201
226	51076	11543176	15.0332964	6.091199
227	51529	11697083	15.0665192	6.100170
228	51984	11852352	15.0996689	6.109115
229	52441	12008989	15.1327460	6.118032
230	52900	12167000	15.1657509	6.126925
231	53361	12326391	15.1986842	6.135792
232	53824	12487168	15.2315462	6.144634
233	54289	12649337	15.2643375	6.153449
234	54756	12812904	15.2970585	6.162239
235	55225	12977875	15.3297097	6.171005
236	55696	13144256	15.3622915	6.179747
237	56169	13312053	15.3948043	6.188463
238	56644	13481272	15.4272486	6.197154
239	57121	13651919	15.4596248	6.205821
240	57600	13824000	15.4919334	6.214464
241	58081	13997521	15.5241747	6.223083
242	58564	14172488	15.5563492	6.231678
243	59049	14348907	15.5884573	6.240251
244	59536	14526784	15.6204994	6.248800
245	60025	14706125	15.6524758	6.257324
246	60516	14886936	15.6843871	6.265826
247	61009	15069223	15.7162336	6.274304
248	61504	15252992	15.7480157	6.282760
249	62001	15438249	15.7797338	6.291194
250	62500	15625000	15.8113883	6.299604

NO.	SQUARE.	CUBE.	SQUARE ROOT.	CUBE ROOT.
251	63001	15813251	15.8429795	6.307992
252	63504	16003008	15.8745079	6.316359
253	64009	16194277	15.9059737	6.324704
254	64516	16387064	15.9373775	6.333025
255	65025	16581375	15.9687194	6.341325
256	65536	16777216	16.0000000	6.349602
257	66049	16974593	16.0312195	6.357859
258	66564	17173512	16.0623784	6.366095
259	67081	17373979	16.0934769	6.374310
260	67600	17576000	16.1245155	6.382504
261	68121	17779581	16.1554944	6.390676
262	68644	17984728	16.1864141	6.398827
263	69169	18191447	16.2172747	6.406958
264	69696	18399744	16.2480768	6.415068
265	70225	18609625	16.2788206	6.423157
266	70756	18821096	16.3095064	6.431226
267	71289	19034163	16.3401346	6.439275
268	71824	19248832	16.3707055	6.447305
269	72361	19465109	16.4012195	6.455314
270	72900	19683000	16.4316767	6.463304
271	73441	19902511	16.4620776	6.471274
272	73984	20123648	16.4924225	6.479224
273	74529	20346417	16.5227116	6.487153
274	75076	20570824	16.5529454	6.495064
275	75625	20796875	16.5831240	6.502956
276	76176	21024576	16.6132477	6.510829
277	76729	21253933	16.6433170	6.518684
278	77284	21484952	16.6733320	6.526519
279	77841	21717639	16.7032931	6.534335
280	78400	21952000	16.7332005	6.542132
281	78961	22188041	16.7630546	6.549911
282	79524	22425768	16.7928556	6.557672
283	80089	22665187	16.8226038	6.565415
284	80656	22906304	16.8522995	6.573139
285	81225	23149125	16.8819430	6.580844
286	81796	23393656	16.9115345	6.588531
287	82369	23639903	16.9410743	6.596202
288	82944	23887872	16.9705627	6.603854
289	83521	24137569	17.0000000	6.611488
290	84100	24389000	17.0293864	6.619106
291	84681	24642171	17.0587221	6.626705
292	85264	24897088	17.0880075	6.634287
293	85849	25153757	17.1172428	6.641851
294	86436	25412184	17.1464282	6.649399
295	87025	25672375	17.1755640	6.656930
296	87616	25934336	17.2046505	6.664443
297	88209	26198073	17.2336879	6.671940
298	88804	26463592	17.2626765	6.679419
299	89401	26730899	17.2916165	6.686882
300	90000	27000000	17.3205081	6.694328

NO.	SQUARE.	CUBE.	SQUARE ROOT.	CUBE ROOT.
301	90601	27270901	17.3493516	6.701758
302	91204	27543608	17.3781472	6.709172
303	91809	27818127	17.4068952	6.716569
304	92416	28094464	17.4355958	6.723950
305	93025	28372625	17.4642492	6.731316
306	93636	28652616	17.4928557	6.738065
307	94249	28934443	17.5214155	6.745997
308	94864	29218112	17.5499288	6.753313
309	95481	29503629	17.5783958	6.760614
310	96100	29791000	17.6068169	6.767899
311	96721	30080231	17.6351921	6.775168
312	97344	30371328	17.6635217	6.782422
313	97969	30664297	17.6918060	6.789661
314	98596	30959144	17.7200451	6.796884
315	99225	31281875	17.7482393	6.804091
316	99856	31554496	17.7763888	6.811284
317	100489	31855013	17.8044938	6.818461
318	101124	32157432	17.8325545	6.825624
319	101761	32461759	17.8605711	6.832771
320	102400	32768000	17.8885438	6.839903
321	103041	33076161	17.9104729	6.847021
322	103684	33386248	17.9443584	6.854124
323	104329	33698267	17.9722008	6.861211
324	104976	34012224	18.0000000	6.868284
325	105625	34328125	18.0277564	6.875343
326	106276	34645976	18.0554701	6.882388
327	106929	34965783	18.0831413	6.889419
328	107584	35287552	18.1107703	6.896435
329	108241	35611289	18.1383571	6.903436
330	108900	35937000	18.1659021	6.910423
331	109561	36264691	18.1934054	6.917396
332	110224	36594368	18.2208672	6.924355
333	110889	36926037	18.2482876	6.931300
334	111556	37259704	18.2756669	6.938232
335	112225	37595375	18.3030052	6.945149
336	112896	37933056	18.3303028	6.952053
337	113569	38272753	18.3575598	6.958943
338	114244	38614472	18.4847763	6.965819
339	114921	38958219	18.4119526	6.972682
340	115600	39304000	18.4390889	6.979532
341	116281	39651821	18.4661853	6.986369
342	116964	40001688	18.4932420	6.993191
343	117649	40353607	18.5202592	7.000000
344	118336	40707584	18.5472370	7.006796
345	119025	41063625	18.5741756	7.013579
346	119716	41421736	18.6010752	7.020349
347	120409	41781923	18.6279360	7.027106
348	121104	42144192	18.6547581	7.033850
349	121801	42508549	18.6815417	7.040581
350	122500	42875000	18.7082869	7.047208

NO.	SQUARE.	CUBE.	SRUARE ROOT.	CUBE ROOT.
351	123201	43243551	18.7349940	7.054003
352	123904	43614208	18.7616630	7.060696
353	124609	43986977	18.7882942	7.067376
354	125316	44361864	18.8148877	7.074043
355	126025	44738875	18.8414437	7.080698
356	126736	45118016	18.8679623	7.087341
357	127449	45499293	18.8944436	7.093970
358	128164	45882712	18.9208879	7.100588
359	128881	46268279	18.9472953	7.107193
360	129600	46656000	18.9736660	7.113786
361	130321	47045881	19.0000000	7.120367
362	131044	47437928	19.0262976	7.126935
363	131769	47832147	19.0525589	7.133492
364	132496	48228544	19.0787840	7.140037
365	133225	48627125	19.1049732	7.146569
366	133956	49027896	19.1311265	7.153090
367	134689	49430863	19.1572441	7.159599
368	135424	49836032	19.1833261	7.166095
369	136161	50243409	19.2093727	7.172580
370	136900	50653000	19.2353841	7.179054
371	137641	51064811	19.2613603	7.185516
372	138384	51478848	19.2873015	7.191966
373	139129	51895117	19.3132079	7.198405
374	139876	52313624	19.3390796	7.204832
375	140625	52734375	19.3649167	7.211247
376	141376	53157376	19.3907194	7.217652
377	142129	53582633	19.4164878	7.224045
378	142884	54010152	19.4422221	7.230427
379	143641	54439939	19.4679223	7.236797
380	144400	54872000	19.4935887	7.243156
381	145161	55306341	19.5192213	7.249504
382	145924	55742968	19.5448203	7.255841
383	146689	56181887	19.5703858	7.262167
384	147456	56623104	19.5959179	7.268482
385	148225	57066625	19.6214169	7.274786
386	148996	57512456	19.6468827	7.281079
387	149769	57960603	19.6723146	7.287362
388	150544	58411072	19.6977156	7.293633
389	151321	58863869	19.7230829	7.299893
390	152100	59319000	19.7484177	7.306143
391	152881	59776471	19.7737199	7.312383
392	153664	60236288	19.7989899	7.318611
393	154449	60698457	19.8242276	7.324829
394	155236	61162984	19.8494332	7.331037
395	156025	61629875	19.8746069	7.337234
396	156816	62099136	19.8997487	7.343420
397	157609	62570773	19 9248588	7.349596
398	158404	63044792	19.9499373	7.355762
399	159201	63521199	19.9749844	7.361917
400	160000	64000000	20.0000000	7.368063

NO.	SQUARE.	CUBE.	SQUARE ROOT.	CUBE ROOT.
401	160801	64481201	20.0249844	7.374198
402	161604	64964808	20.0499377	7.380322
403	162409	65450827	20.0748599	7.386437
404	163216	65939264	20.0997512	7.392542
405	164025	66430125	20.1246118	7.398636
406	164836	66923416	20.1494417	7.404720
407	165649	67419143	20.1742410	7.410794
408	166464	67911312	20.1990099	7.416859
409	167281	68417929	20.2237484	7.422914
410	168100	68921000	20.2484567	7.428958
411	168921	69426531	20.2731349	7.434993
412	169744	69934528	20.2977831	7.441018
413	170569	70444997	20.3224014	7.447033
414	171396	70957944	20.3469899	7.453039
415	172225	71473375	20.3715488	7.459036
416	173056	71991296	20.3960781	7.465022
417	173889	72511713	20.4205779	7.470999
418	174724	73034632	20.4450483	7.476966
419	175561	73560059	20.4694895	7.482924
420	176400	74088000	20.4939015	7.488872
421	177241	74618461	20.5182845	7.494810
422	178084	75151448	20.5426386	7.500740
423	178929	75686967	20.5669638	7.506660
424	179776	76225024	20.5912603	7.512571
425	180625	76765625	20.6155281	7.518473
426	181476	77308776	20.6397674	7.524365
427	182329	77854483	20.6639783	7.530248
428	183184	78402752	20.6881609	7.536121
429	184041	78953589	20.7123152	7.541986
430	184900	79507000	20.7364114	7.547841
431	185761	80062991	20.7605395	7.553688
432	186624	80621568	20.7846097	7.559525
433	187489	81182737	20.8086520	7.565353
434	188356	81746504	20.8326667	7.571173
435	189225	82312875	20.8566536	7.576984
436	190096	82881856	20.8806130	7.582786
437	190969	83453453	20.9045450	7.588579
438	191844	84027672	20.9284495	7.594363
439	192721	84604519	20.9523268	7.600138
440	193600	85184000	20.9761771	7.605905
441	194481	85766121	20.0000000	7.611662
442	195364	86350888	21.0237960	7.617411
443	196249	86938307	21.0475652	7.613151
444	197136	87528384	21.0713075	7.628883
445	198025	88121125	21.0950231	7.634606
446	198916	88716536	21.1187121	7.640321
447	199809	89314623	21.1423745	7.646027
448	200704	89915392	21.1660105	7.651725
449	201601	90518849	21.1896201	7.657414
450	202500	91125000	21.2132034	7.663094

NO.	SQUARE.	CUBE.	SQUARE ROOT.	CUBE
451	203401	91733851	21.2367606	7.66
452	204304	92345408	21.2602916	7.67
453	205209	92959677	21.2837967	7.68
454	206116	93576664	21.3072758	7.68
455	207025	94196375	21.3307290	7.69
456	207936	94818816	21.3541565	7.69
457	208849	95443993	21.3775583	7.70
458	209764	96071912	21.4009346	7.70
459	210681	96702579	21.4242853	7.71
460	211600	97336000	21.4476106	7.71
461	212521	97972181	21.4709106	7.72
462	213444	98611128	21.4941853	7.73
463	214369	99252847	21.5174348	7.73
464	215296	99897344	21.5406592	7.74
465	216225	100544625	21.5638587	7.74
466	217156	101194696	21.5870331	7.75
467	218089	101847563	21.6101828	7.75
468	219024	102503232	21.6333077	7.76
469	219961	103161709	21.6564078	7.76
470	220900	103823000	21.6794834	7.77
471	221841	104487111	21.7025344	7.78
472	222784	105154048	21.7255610	7.78
473	223729	105823817	21.7485632	7.79
474	224676	106496424	21.7715411	7.79
475	225625	107171875	21.7944947	7.80
476	226576	107850176	21.8174242	7.80
477	227529	108531333	21.8403297	7.81
478	228484	109215352	21.8632111	7.81
479	229441	109902239	21.8860686	7.82
480	230400	110592000	21.9089023	7.82
481	231361	111284641	21.9317122	7.83
482	232324	111980168	21.9544984	7.84
483	233289	112678587	21.9772610	7.84
484	234256	113379904	22.0000000	7.85
485	235225	114084125	22.0227155	7.85
486	236196	114791256	22.0454077	7.86
487	237169	115501303	22.0680765	7.86
488	238144	116214272	22.0907220	7.87
489	239121	116930169	22.1133444	7.87
490	240100	117649000	22.1359436	7.88
491	241081	118370771	22.1585198	7.88
492	242064	119095488	22.1810730	7.89
493	243049	119823157	22.2036033	7.89
494	244036	120553784	22.2261408	7.90
495	245025	121287375	22.2485955	7.91
496	246016	122023936	22.2710575	7.91
497	247009	122763473	22.2934968	7.92
498	248004	123505992	22.3159136	7.92
499	249001	124251499	22.3383079	7.93
500	250000	125000000	22.3606798	7.93

NO.	SQUARE.	CUBE.	SQUARE ROOT.	CUBE ROOT.
501	251001	125751501	22.3830293	7.942293
502	252004	126506008	22.4053565	7.947573
503	253009	127263527	22.4276615	7.952847
504	254016	128024064	22.4499443	7.958114
505	255025	128787625	22.4722051	7.963374
506	256036	129554216	22.4944438	7.968627
507	257049	130323843	22.5166605	7.973873
508	258064	131096512	22.5388553	7.979112
509	259081	131872229	22.5610283	7.984344
510	260100	132651000	22.5831796	7.989569
511	261121	133432831	22.6053091	7.994788
512	262144	134217728	22.6274170	8.000000
513	263169	135005697	22.6495033	8.005205
514	264196	135796744	22.6715681	8.010403
515	265225	136590875	22.6936114	8.015595
516	266256	137388096	22.7156334	8.020779
517	267289	138188413	22.7376340	8.025957
518	268324	138991832	22.7596134	8.031129
519	269361	139798359	22.7815715	8.036293
520	270400	140608000	22.8035085	8.041451
521	271441	141420761	22.8254244	8.046603
522	272484	142236648	22.8473193	8.051748
523	273529	143055667	22.8691933	8.056886
524	274576	143877824	22.8910463	8.062018
525	275625	144703125	22.9128785	8.067143
526	276676	145531576	22.9346899	8.072262
527	277729	146363183	22.9564806	8.077374
528	278784	147197952	22.9782506	8.082480
529	279841	148035889	23.0000000	8.087579
530	280900	148877000	23.0217289	8.092672
531	281961	149721291	23.0434372	8.097758
532	283024	150568768	23.0651252	8.102838
533	284089	151419437	23.0867928	8.107912
534	285156	152273304	23.1084400	8.112980
535	286225	153130375	23.1300670	8.118041
536	287296	153990656	23.1516738	8.123096
537	288369	154854153	23.1732605	8.128144
538	289441	155720872	23.1948270	8.133186
539	290521	156590819	23.2163735	8.138223
540	291600	157464000	23.2379001	8.143253
541	292681	158340421	23.2594067	8.148276
542	293764	159220088	23.2808935	8.153293
543	294849	160103007	23.3023604	8.158304
544	295936	160989184	23.3238074	8.163309
545	297025	161878625	23.3452351	8.168308
546	298116	162771336	23.3666429	8.173302
547	299209	163667323	23.3880311	8.178289
548	300304	164566592	23.4093998	8.183269
549	301401	165469149	23.4307490	8.188244
550	302500	166375000	23.4520788	8.193212

NO.	SQUARE.	CUBE.	SQUARE ROOT.	CUBE ROOT.
551	303601	167284151	23.4733892	8.198175
552	304704	168196608	23.4946802	8.203131
553	305809	169112377	23.5159520	8.208082
554	306916	170031464	23.5372046	8.213027
555	308025	170953875	23.5584380	8.217965
556	309136	171879616	23.5796522	8.222898
557	310249	172808693	23.6008474	8.227825
558	311364	173741112	23.6220236	8.232746
559	312481	174676879	23.6431808	8.237661
560	313600	175616000	23.6643191	8.242570
561	314721	176558481	23.6854386	8.247474
562	315844	177504328	23.7065392	8.252371
563	316969	178453547	23.7276210	8.257263
564	318096	179406144	23.7486842	8.262149
565	319225	180362125	23.7697286	8.267029
566	320356	181321496	23.7907545	8.271903
567	321489	182284263	23.8117618	8.276772
568	322624	183250432	23.8327506	8.281635
569	323761	184220009	23.8537209	8.286493
570	324900	185193000	23.8746728	8.291344
571	326041	186169411	23.8956063	8.296190
572	327184	187149248	23.9165215	8.301030
573	328329	188132517	23.9374184	8.305865
574	329476	189119224	23.9582971	8.310694
575	330625	190109375	23.9791576	8.315517
576	331776	191102976	24.0000000	8.320335
577	332929	192100033	24.0208243	8.325147
578	334084	193100552	24.0416306	8.329954
579	335241	194104539	24.0624188	8.334755
580	336400	195112000	24.0831892	8.339551
581	337561	196122941	24.1039416	8.344341
582	338724	197137368	24.1246762	8.349125
583	339889	198155287	24.1453929	8.353904
584	341056	199176704	24.1660919	8.358678
585	342225	200201625	24.1867732	8.363446
586	343396	201230056	24.2074369	8.368209
587	344569	202262003	24.2280829	8.372966
588	345744	203297472	24.2487113	8.377718
589	346921	204336469	24.2693222	8.382465
590	348100	205379000	24.2899156	8.387206
591	349281	206125071	24.3104916	8.391942
592	350464	207474688	24.3310501	8.396673
593	351649	208527857	24.3515913	8.401398
594	352836	209584584	24.3721152	8.406118
595	354025	210644875	24.3926218	8.410832
596	355216	211708736	24.4131112	8.415541
597	356409	212776173	24.4335834	8.420245
598	357604	213847192	24.4540385	8.424944
599	358801	214921799	24.4744765	8.429638
600	360000	216000000	24.4948974	8.434327

NO.	SQUARE.	CUBE.	SQUARE ROOT.	CUBE ROOT.
601	361201	217081807	24.5153013	8.439009
602	362404	218167208	24.5356883	8.443687
603	363609	219256227	24.5560583	8.448360
604	364816	220348864	24.5764115	8.453027
605	366025	221445125	24.5967478	8.457689
606	367236	222545016	24.6170673	8.462347
607	368449	223648543	24.6373700	8.466999
608	369664	224755712	24.6576560	8.471647
609	370881	225866529	24.6779254	8.476289
610	372100	226981000	24.6981781	8.480926
611	373321	228099131	24.7184142	8.485557
612	374544	229220928	24.7386338	8.490184
613	375769	230346397	24.7588368	8.494806
614	376996	231475544	24.7790234	8.499423
615	378225	232608375	24.7991935	8.504034
616	379456	233744896	24.8193473	8.508641
617	380689	234885113	24.8394847	8.513243
618	381924	236029032	24.8596058	8.517840
619	383161	237176659	24.8797106	8.522432
620	384400	238328000	24.8997992	8.527018
621	385641	239483061	24.9198716	8.531600
622	386884	240641848	24.9399278	8.536177
623	388129	241804367	24.9599679	8.540749
624	389376	242970624	24.9799920	8.545317
625	390625	244140625	25.0000000	8.549879
626	391876	245314376	25.0199920	8.554437
627	393129	246491883	25.0399681	8.558990
628	394384	247673152	25.0599282	8.563537
629	395641	248858189	25.0798724	8.568080
630	396900	250047000	25.0998008	8.572618
631	398161	251239591	25.1197134	8.577152
632	399424	252435968	25.1396102	8.581680
633	400689	253636137	25.1594913	8.586204
634	401956	254840104	25.1793566	8.590723
635	403225	256047875	25.1992063	8.595238
636	404496	257259456	25.2190404	8.599747
637	405769	258474853	25.2388589	8.604252
638	407044	259694072	25.2586619	8.608752
639	408321	260917119	25.2784493	8.613248
640	409600	262144000	25.2982213	8.617738
641	410881	263374721	25.3179778	8.622224
642	412164	264609288	25.3377189	8.626706
643	413449	265847707	25.3574447	8.631183
644	414736	267089984	25.3771551	8.635655
645	416025	268336125	25.3968502	8.640122
646	417316	269586136	25.4165301	8.644585
647	418609	270840023	25.4361947	8.649043
648	419904	272097792	25.4558441	8.653497
649	421201	273359449	25.4754784	8.657946
650	422500	274625000	25.4950076	8.662301

NO.	SQUARE.	CUBE.	SQUARE ROOT.	CUBE R
651	423801	275894451	25.5147016	8.666
652	425104	277167808	25.5342907	8.6712
653	426409	278445077	25.5538647	8.6756
654	427716	279726264	25.5734237	8.6801
655	429025	281011375	25.5929678	8.6845
656	430336	282300416	25.6124969	8.6889
657	431649	283593393	25.6320112	8.6933
658	432964	284890312	25.6515107	8.6977
659	434281	286191179	25.6709953	8.7021
660	435600	287496000	25.6904652	8.7065
661	436921	288804781	25.7099203	8.7109
662	438244	290117528	25.7203607	8.715373
663	439569	291434247	25.7487864	8.719759
664	440896	292754944	25.7681975	8.724141
665	442225	294079625	25.7875939	8.728518
666	443556	295408296	25.8069758	8.732891
667	444889	296740963	25.8263431	8.737260
668	446224	298077632	25.8456960	8.741624
669	447561	299418309	25.8650343	8.745984
670	448900	300763000	25.8843582	8.750340
671	450241	302111711	25.9036677	8.754691
672	451584	303464448	25.9229628	8.759038
673	452929	304821217	25.9422435	8.763380
674	454276	306182024	25.9615100	8.767719
675	455625	307546875	25.9807621	8.772053
676	456976	308915776	26.0000000	8.776382
677	458329	310288733	26.0192237	8.780708
678	459684	311665752	26.0384331	8.785029
679	461040	313046839	26.0576284	8.789346
680	462400	314432000	26.0768096	8.793659
681	463761	315821241	26.0959767	8.797967
682	465124	317214568	26.1151297	8.802272
683	466489	318611987	26.1342687	8.806572
684	467856	320013504	26.1533937	8.810868
685	469225	321419125	26.1725047	8.815159
686	470596	322828856	26.1916017	8.819447
687	471969	324242703	26.2106848	8.823730
688	473344	325660672	26.2297541	8.828009
689	474721	327082769	26.2488095	8.832285
690	476100	328509000	26.2678511	8.836556
691	477481	329939371	26.2868789	8.840822
692	478864	331373888	26.3058929	8.845085
693	480249	332812557	26.3248932	8.849344
694	481636	334255384	26.3438797	8.853598
695	483025	335702375	26.3628527	8.857849
696	484416	337153536	26.3818119	8.862095
697	485809	338608873	26.4007576	8.866337
698	487204	340068392	26.4196896	8.870575
699	488601	341532099	26.4386081	8.874809
700	490000	343000000	26.4575131	8.879040

SQUARE.	CUBE.	SQUARE ROOT.	CUBE ROOT.
491401	344472101	26.4764046	8.883266
492804	345948008	26.4952826	8.887488
494209	347428927	26.5141472	8.891706
495616	348913664	26.5329983	8.895920
497025	350402625	26.5518361	8.900130
498436	351895816	26.5706605	8.904336
499849	353393243	26.5894716	8.908538
501264	354894912	26.6082694	8.912736
502681	356400829	26.6270539	8.916931
504100	357911000	26.6458252	8.921121
505521	359425431	26.6645833	8.925307
506944	360944128	26.6833281	8.929490
508369	362467097	26.7020591	8.933668
509796	363994344	26.7207784	8.937844
511225	365525875	26.7394839	8.942014
512656	367061696	26.7581763	8.946180
514089	368601813	26.7768557	8.950343
515524	370146232	26.7955220	8.954502
516961	371694959	26.8141754	8.958658
518400	373248000	26.8328157	8.962809
519841	374805361	26.8514432	8.966957
521284	376367048	26.8700577	8.971100
522729	377933067	26.8886593	8.975240
524176	379503424	26.9072481	8.979376
525625	381078125	26.9258240	8.983508
527076	382657176	26.9443872	8.987637
528529	384240583	26.9629375	8.991762
529984	385828353	26.9814751	8.995883
531441	387420489	27.0000000	9.000000
532900	389017000	27.0185122	9.004113
534361	390617891	27.0370117	9.008222
535824	392223168	27.0554985	9.012328
537289	393832837	27.0739727	9.016430
538756	395446904	27.0924344	9.020529
540225	397065375	27.1108834	9.024623
541696	398688256	27.1293199	9.028714
543169	400315553	27.1477439	9.032802
544644	401917272	27.1661554	9.036885
546121	403583419	27.1845544	9.040965
547600	405224000	27.2029410	9.045041
549081	406869021	27.2213152	9.049114
550564	408518488	27.2396769	9.053183
552049	410172407	27.2580263	9.057248
553536	411830784	27.2763634	9.061309
555025	413493625	27.2946881	9.064367
556516	415160936	27.3130006	9.069422
558009	416832723	27.3313007	9.073472
559504	418508992	27.3495887	9.077519
561001	420189749	27.3678644	9.081563
562500	421875000	27.3861279	9.085603

NO.	SQUARE.	CUBE.	SQUARE ROOT.	CUBE ROOT.
751	564001	423564751	27.4043792	9.089639
752	565504	425259008	27.4226184	9.093672
753	567009	426957777	27.4408455	9.097701
754	568516	428661064	27.4590604	9.101726
755	570025	430368875	27.4772633	9.105748
756	571536	432081216	27.4954542	9.109766
757	573049	433798093	27.5136330	9.113781
758	574564	435519512	27.5317998	9.117793
759	576081	437245479	27.5499546	9.121801
760	577600	438976000	27.5680975	9.125805
761	579121	440711081	27.5862284	9.129806
762	580644	442450728	27.6043475	9.133803
763	582169	444194947	27.6224546	9.137797
764	583696	445943744	27.6405499	9.141788
765	585225	447697125	27.6586334	9.145774
766	586756	449455096	27.6767050	9.149757
767	588289	451217663	27.6947648	9.153737
768	589824	452984832	27.7128129	9.157713
769	591361	454756609	27.7308492	9.161686
770	592900	456533000	27.7488739	9.165656
771	594441	458314011	27.7668868	9.169622
772	595984	460099648	27.7848880	9.173585
773	597529	461889917	27.8028775	9.177544
774	599076	463684824	27.8208555	9.181500
775	600625	465484375	27.8388218	9.185452
776	602176	467288576	27.8567766	9.189401
777	603729	469097433	27.8747197	9.193347
778	605284	470910952	27.8926514	9.197289
779	606841	472729139	27.9105715	9.201228
780	608400	474552000	27.9284801	9.205164
781	609961	476379541	27.9463772	9.209096
782	611524	478211768	27.9642629	9.213025
783	613089	480048687	27.9821372	9.216950
784	614656	481890304	28.0000000	9.220872
785	616225	483736025	28.0178515	9.224791
786	617796	485587656	28.0356915	9.228006
787	619369	487443403	28.0535203	9.232618
788	620944	489303872	28.0713377	9.237527
789	622521	491169069	28.0891438	9.240433
790	624100	493039000	28.1069386	9.244335
791	625681	494913671	28.1247222	9.248234
792	627264	496793088	28.1424946	9.252130
793	628849	498677257	28.1602557	9.256022
794	630436	500566184	28.1780056	9.259911
795	632025	502459875	28.1957444	9.263797
796	633616	504358336	28.2134720	9.267679
797	635209	506261573	28.2311884	9.271559
798	636804	508169592	28.2488938	9.275435
799	638401	510082399	28.2665881	9.279308
800	640000	512000000	28.2842712	9.283177

SQUARE.	CUBE.	SQUARE ROOT.	CUBE ROOT.
641601	513922401	28.3019434	9.287044
643204	515849608	28.3196045	9.290907
644809	517781627	28.3372546	9.294767
646416	519718464	28.3548938	9.298623
648025	521660125	28.3725219	9.302477
649636	523606616	28.3901391	9.306327
651249	525557943	28.4077454	9.310175
652864	527514112	28.4253408	9.314019
654481	529475129	28.4429253	9.317859
656100	531441000	28.4604989	9.321697
657721	533411731	28.4780617	9.325532
659344	535387328	28.4956137	9.329363
660969	537366797	28.5131549	9.333191
662596	539353144	28.5306852	9.337016
664225	541343375	28.5482048	9.340838
665856	543338496	28.5657137	9.344657
667489	545338513	28.5832119	9.348473
669124	547343432	28.6006993	9.352285
670761	549353259	28.6181760	9.356095
672400	551368000	28.6356421	9.359901
674041	553387661	28.6530976	9.363704
675684	555412248	28.6705424	9.367505
677329	557441767	28.6879766	9.371302
678976	559476224	28.7054002	9.375096
680625	561515625	28.7228132	9.378887
682276	563559976	28.7402157	9.382675
683929	565609283	28.7576077	9.386460
685584	567663552	28.7749891	9.390241
687241	569722789	28.7923601	9.394020
688900	571787000	28.8097206	9.397796
690561	573856191	28.8270706	9.401569
692224	575930368	28.8444102	9.405338
693889	578009537	28.8617394	9.409105
695556	580093704	28.8790582	9.412869
697225	582182875	28.8963666	9.416630
698896	584277056	28.9136646	9.420387
700569	586376253	28.9309523	9.424141
702244	588480472	28.9482297	9.427893
703921	590589719	28.9654967	9.431642
705600	592704000	28.9827535	9.435388
707281	594823321	29.0000000	9.439130
708964	596947688	29.0172363	9.442870
710649	599077107	29.0344623	9.446607
712336	601211584	29.0516781	9.450341
714025	603351125	29.0688837	9.454071
715716	605495736	29.0860791	9.457799
717409	607645423	29.1032644	9.461524
719104	609800192	29.1204396	9.465247
720801	611960049	29.1376046	9.468966
722500	614125000	29.1547505	9.472682

NO.	SQUARE.	CUBE.	SQUARE ROOT.	CUBE ROOT.
851	724201	616295051	29.1719043	9.476395
852	725904	618470208	29.1890390	9.480106
853	727609	620650477	29.2061637	9.483813
854	729316	622835864	29.2232784	9.487518
855	731025	625026375	29.2403830	9.491219
856	732736	627222016	29.2574777	9.494918
857	734449	629422793	29.2745623	9.498614
858	736164	631628712	29.2916370	9.502307
859	737881	633839779	29.3087018	9.505998
860	739600	636056000	29.3257566	9.509685
861	741321	638277381	29.3428015	9.513369
862	743044	640503928	29.3598365	9.517051
863	744769	642735647	29.3768616	9.520730
864	746496	644972544	29.3938769	9.524406
865	748225	647214625	29.4108823	9.528079
866	749956	649461896	29.4278779	9.531749
867	751689	651714363	29.4448637	9.535417
868	753424	653972032	29.4618397	9.539081
869	755161	656234909	29.4788059	9.542748
870	756900	658503000	29.4957624	9.546402
871	758641	660776311	29.5127091	9.550058
872	760384	663054848	29.5296461	9.553712
873	762129	665338617	29.5465734	9.557363
874	763876	667627624	29.5634910	9.561010
875	765625	669921875	29.5803989	9.564655
876	767376	672221376	29.5972972	9.568297
877	769129	674526133	29.6141858	9.571937
878	770884	676836152	29.6310648	9.575574
879	772641	679151439	29.6479325	9.579208
880	774400	681472000	29.6647939	9.582839
881	776161	683797841	29.6816442	9.586468
882	777924	686128968	29.6984848	9.590093
883	779689	688465387	29.7153159	9.593716
884	781456	690807104	29.7321375	9.597337
885	783225	693154125	29.7489496	9.600954
886	784996	695506456	29.7657521	9.604569
887	786769	698764103	29.7825452	9.608181
888	788444	700227072	29.7993289	9.611791
889	790321	702595369	29.8161030	9.615397
890	792100	704969000	29.8328678	9.619001
891	793881	707347971	29.8496231	9.622603
892	795664	709732288	29.8663690	9.626201
893	797449	712121957	29.8831056	9.629797
894	799236	714516984	29.8998328	9.633390
895	801025	716917375	29.9165506	9.636981
896	802816	719323136	29.9332591	9.640569
897	804609	721734273	29.9499583	9.644154
898	806404	724150792	29.9666481	9.647736
899	808201	726572699	29.9833287	9.651316
900	810000	729000000	30.0000000	9.654893

NO.	SQUARE.	CUBE.	SQUARE ROOT.	CUBE ROOT.
901	811801	731432701	30.0166620	9.658468
902	813604	733870808	30.0333148	9.662040
903	815409	736314327	30.0499584	9.665609
904	817216	738763264	30.0665928	9.669176
905	819025	741217625	30.0832179	9.672740
906	820836	743677416	30.0998339	9.676301
907	822649	746142643	30.1164407	9.679860
908	824464	748613312	30.1330383	9.6 3416
909	826281	751089429	30.1496269	9.686970
910	828100	753571000	30.1632063	9.690521
911	829921	756058031	30.1827765	9.694069
912	831744	758550528	30.1993377	9.697615
913	833569	761048497	30.2158899	9.701158
914	835396	763551944	30.2324329	9.704698
915	837225	766060875	30.2489669	9.708236
916	839056	768575296	30.2654919	9.711772
917	840889	771095213	30.2820079	9.715305
918	842724	773620632	30.2985148	9.718835
919	844561	776151559	30.3150128	9.722363
920	846400	778688000	30.3315018	9.725888
921	848241	781229961	30.3479818	9.729410
922	850084	783777448	30.3644529	9.732930
923	851929	786330467	30.3809151	9.736448
924	853776	788889024	30.3973683	9.739963
925	855625	791453125	30.4138127	9.743475
926	857476	794022776	30.4302481	9.746985
927	859329	796597983	30.4466747	9.750493
928	861184	799178752	30.4630924	9.753998
929	863041	801765089	30.4795013	9.757500
930	864900	804357000	30.4959014	9.761000
931	866761	806954491	30.5122926	9.764497
932	868624	809557568	30.5286750	9.767992
933	870489	812166237	30.5450487	9.771484
934	872356	814780504	30.5614136	9.774974
935	874225	817400375	30.5777697	9.778461
936	876096	820025856	30.5941171	9.782946
937	877969	822656953	30.6104557	9.785428
938	879844	825293672	30.6267857	9.788908
939	881721	827936019	30.6431069	9.792386
940	883600	830584000	30.6594194	9.795861
941	885481	833237621	30.6757233	9.799333
942	887364	835896888	30.6920185	9.802803
943	889249	838561807	30.7083051	9.806271
944	891136	841232884	30.7245830	9.809736
945	893025	843908625	30.7408523	9.813198
946	894916	846590536	30.7571130	9.816659
947	896809	849278123	30.7733651	9.820117
948	898704	851971392	30.7896086	9.823572
949	800601	954670349	30.8058436	9.827025
950	802500	957375000	30.8220700	9.830475

NO.	SQUARE.	CUBE.	SQUARE ROOT.	CUBE
951	904401	860085351	30.8382879	9.8
952	906304	862801408	30.8544972	9.8
953	908209	865523177	30.8706981	9.8
954	910116	868250664	30.8868904	9.8
955	912025	870983875	30.9030743	9.8
956	913936	873722816	30.9192497	9.8
957	915849	876467493	30.9354166	9.8
958	917764	879217912	30.9515751	9.8
959	919681	881974079	30.9677251	9.8
960	921600	884736000	30.9838668	9.8
961	923521	887503681	31.0000000	9.8
962	925444	890277128	31.0161248	9.8
963	927369	893056347	31.0322413	9.8
964	929296	895841344	31.0483494	9.8
965	931225	893632125	31.0644491	9.8
966	933156	901428696	31.0805405	9.8
967	935089	904231063	31.0966236	9.8
968	937024	907039232	31.1126984	9.8
969	938961	909853209	31.1287648	9.8
970	940900	912673000	31.1448230	9.8
971	942841	915498611	31.1608729	9.9
972	944784	918330048	31.1769145	9.9
973	946729	921167317	31.1929479	9.9
974	948676	924010424	31.2089731	9.9
975	950625	926859375	31.2249900	9.9
976	952576	929714176	31.2409987	9.9
977	954529	932574833	31.2569992	9.9
978	956484	935441352	31.2729915	9.9
979	958441	938313739	31.2889757	9.9
980	960400	941192001	31.3049517	9.9
981	962361	944076141	31.3209195	9.9
982	964324	946966168	31.3368792	9.9
983	966289	949862087	31.3528308	9.9
984	968256	952763904	31.3687743	9.9
985	970225	955671625	31.3847097	9.9
986	972196	958585256	31.4006369	9.9
987	974169	961504803	31.4165561	9.9
988	976144	964430272	31.4324673	9.9
989	978121	967361669	31.4483704	9.9
990	980100	970299000	31.4642654	9.9
991	982081	973242271	31.4801525	9.9
992	984064	976191488	31.4960315	9.9
993	986049	979146657	31.5119025	9.9
994	988036	982107784	31.5277655	9.9
995	990025	985074875	31.5436206	9.9
996	992016	988047936	31.5594677	9.9
997	994009	991026973	31.5753068	9.9
998	996004	994011992	31.5911380	9.9
999	998001	997002999	31.6069613	9.9

(STEREOTYPED.)

HIGHER GEOMETRY

AND

TRIGONOMETRY:

BEING THE

THIRD PART OF A SERIES

ON

ELEMENTARY AND HIGHER

GEOMETRY, TRIGONOMETRY, AND MENSURATION,

CONTAINING MANY VALUABLE DISCOVERIES AND IMPROVEMENTS IN MATHEMATICAL
SCIENCE, ESPECIALLY IN RELATION TO THE QUADRATURE OF THE CIRCLE,
AND SOME OTHER CURVES, AS WELL AS THE CUBATURE OF CERTAIN
CURVILINEAR SOLIDS; DESIGNED AS A TEXT-BOOK FOR COLLEGIATE
AND ACADEMIC INSTRUCTION, AND AS A PRACTICAL
COMPENDIUM OF MENSURATION.

BY NATHAN SCHOLFIELD.

LONDON:
WILEY AND PUTNAM, 6, WATERLOO PLACE.
1845.

PREFACE.

s part of the series consists of spherical geometry, taken
from Brewster's translation of Legendre's work. Ana-
plane and spherical trigonometry, based on the subject,
nd in Rutherford's edition of Hutton's Mathematics, being
ally abridged from the larger works of Cagnoli, and
; but, in this work, much improved and enlarged. To
are added many practical exercises on the subject, by
f application. In this treatise will be found many cu-
and highly useful problems in trigonometrical surveying,
pographical operations, not before published. The pro-
s of the circle are introduced advantageously into trigo-
rical problems—hence we are enabled, by geometrical
uction, and trigonometrical analysis, to determine many
vise extremely difficult problems, in a manner at once
, elegant, and satisfactory. The application of alge-
geometry, is discussed in such manner as to combine
inciples of the two sciences. The properties of the *pa-*
c, *elliptical*, and *hyperbolic curves*, being such as are
d by the sections of a cone, and hence are usually de-
ated *conic sections*, are also discussed. This subject is,
some alterations and additions, taken from Rutherford's
of Hutton's Mathematics. It is the design of the au-
o preserve an unbroken connection from pure elemen-
o the higher Geometry and mensuration; and with this
in view the present volume, being the third part of the
, is prepared.

The well established reputation and the high respectability of the authors from whom our selections have been made, renders it unnecessary for us to discuss their merits in order to secure a favorable reception of this. It will only be necessary for us, in the following pages, to preserve the same degree of accuracy and perspicuity in our digressions as characterise those works, and we shall have nothing to fear from the criticisms of scientific amateurs and mathematicians.

CONTENTS.

SPHERICAL GEOMETRY.

PAGE.
nd General Propositions, - - - - - - - 9

ANALYTICAL PLANE TRIGONOMETRY.

efinitions and Illustration of Principles, - - - 27
ieneral Formulæ, - - - - - - - 45
Formulæ for the Solution of Triangles, - - - 66
Construction of Trigonometrical Tables, - - - 73
ogarithms, - - - - - - - - - 77
Solution of Right Angled Triangles, - - - 93
-Solution of Oblique Angled Triangles, - - - 98
—On the use of Subsidiary Angles, - - - - 107
On the Solution of Geometrical Problems by Trigonometry, - - - - - - - . - 110
roblems in Trigonometrical Surveying, &c. - - 113

SPHERICAL TRIGONOMETRY.

eneral Principles and Illustrations, - - - - 131
olution of Right Angled Spherical Triangles, - - 144
Solution of Oblique Angled Spherical Triangles, - 150
On the use of Subsidiary Angles, - - - - 154

APPLICATION OF ALGEBRA TO GEOMETRY.

PA
Construction of Algebraical Quantities, - - - - - -
Geometrical Questions, the modes of forming Equations therefrom, and their Solutions, - - - - - - - -
Determination of Algebraic Expressions for Surfaces and Solids,

CONIC SECTIONS.

The Parabola and its Properties, - - - - - -
The Ellipse and its Properties, - - - - - - - -
The Hyperbola and its Properties, - - - - - -

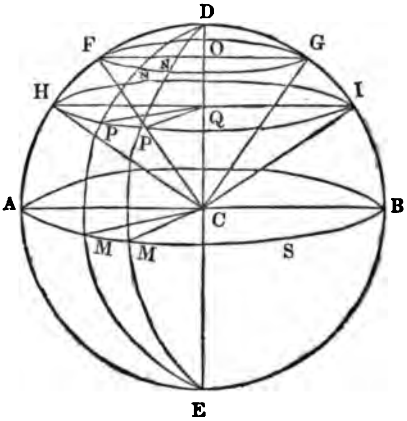
SPHERICAL GEOMETRY.

SPHERICAL GEOMETRY.

DEFINITIONS.

l. The *sphere* is a solid terminated by a curve surface, all
: points of which are equally distant from a point within,
led the *centre.*

Fhe sphere may be con-
ved to be generated by
: revolution of a semi-
:le DAE about its di-
eter DE; for the surface
cribed in this move-
nt, by the curve DAE,
l have all its points
ally distant from its
tre C.



. The *radius of a sphere* is a straight line, drawn from the
tre to any point of the surface; the *diameter*, or *axis*, is a
passing through this centre, and terminated on both sides
he surface.
ll the radii of a sphere are equal; all the diameters are
l, and each double of the radius.
It will be shown (Prop. I.) that every section of the
re, made by a plane, is a circle. This granted, a *great*
e is a section which passes through the centre; a *small*
e, one which does not pass through the centre.
A *plane* is *tangent* to a sphere, when their surfaces have
ne point in common.

5. The *pole of a circle* of a sphere is a point in the surf equally distant from all the points in the circumference of circle.

6. A *spherical triangle* is a portion of the surface of a sph bounded by three arcs of great circles.

Those arcs, named the *sides* of the triangle, are alw supposed to be each less than a semi-circumference. The gles which their planes form with each other, are the an of the triangle.

7. A spherical triangle takes the name of *right-ang* *isosceles*, *equilateral*, in the same cases as a rectilineal trian

8. A *spherical polygon* is a portion of the surface sphere, terminated by several arcs of great circles.

9. A *lune* is that portion of the surface of a sphere w is included between two great semicircles, meeting in a c mon diameter.

10. A *spherical wedge*, or *ungula*, is that portion of the s sphere which is included between the same great semi-cir and has the lune for its base.

11. A *spherical pyramid* is a portion of the solid sphere cluded between the planes of a solid angle, whose verte the centre. The *base* of the pyramid is the spherical poly intercepted by the same planes.

12. A *zone* is the portion of the surface of the sphere cluded between two parallel planes, which form its *bases*. of those planes may be tangent to the sphere; in which c the zone has only a single base.

13. A *spherical segment* is the portion of the solid sp included between two parallel planes which form its base

One of these planes may be tangent to the sphere; in w case, the segment has only a single base.

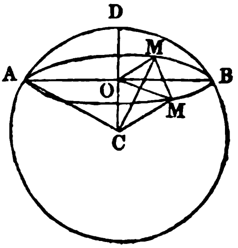
14. The *altitude of a zone* or *of a segment* is the dista between the two parallel planes, which form the bases of zone or segment.

15. Whilst the semicircle DAE (Def. 1.) revolving ro its diameter DE, describes the sphere, any circular secto DCF or FCH, describes a solid, which is named a *spher* *sector*.

16. The symbal ∵ which occurs in this volume, is use denote *because;* when applied in algebraic notation.

PROPOSITION I. THEOREM.

Every section of a sphere, made by a plane, is a circle.



Let AMB be a section, made by a plane, in the sphere, whose centre is C. From the point C, draw CO perpendicular to the plane AMB; and different lines CM, CM, to different points of the curve AMB, which terminates the section.

The oblique lines CM, CM, CA, are equal, being radii of the sphere; hence (Prop. VI. B. I. *El. S. Geom.*) they are equally distant from the perpendicular CO; therefore all the lines OM, MO, OB, are equal. Consequently, the section AMB is a circle, whose centre is O.

Cor. 1. If the section passes through the centre of the sphere, its radius will be the radius of the sphere; hence, all great circles are equal.

Cor. 2. Two great circles always bisect each other; for their common intersection, passing through the centre, is a diameter.

Cor. 3. Every great circle divides the sphere and its surface into two equal parts; for, if the two hemispheres were separated, and afterwards placed on the common base, with their convexities turned the same way, the two surfaces would exactly coincide, no point of the one being nearer the centre than any point of the other.

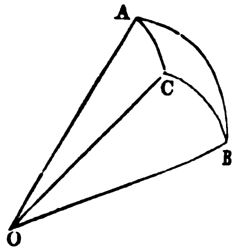
Cor. 4. The centre of a small circle, and that of the sphere, are in the same straight line, perpendicular to the plane of the small circle.

Cor. 5. Small circles are the less the further they lie from the centre of the sphere; for, the greater CO is, the less is the chord AB, the diameter of the small circle AMB.

Cor. 6. An arc of a great circle may always be made to pass through any two given points of the surface of the sphere; for the two given points, and the centre of the sphere, make three points, which determine the position of a plane. But if the two given points were at the extremities of a diameter, these two points and the centre would then lie in one straight line, and an infinite number of great circles might be made to pass through the two given points.

PROPOSITION II. THEOREM.

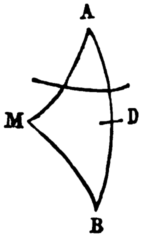
In every spherical triangle, any side is less than the sum of the other two.



Let O be the centre of the sphere, and ACB the triangle: draw the radii OA, OB, OC. Imagine the planes AOB, AOC, COB, to be drawn; those planes will form a solid angle at the centre O; and the angles AOB, AOC, COB, will be measured by AB, AC, BC, the sides of the spherical triangle. But each of the three plane angles forming a solid angle is less than the sum of the other two (Prop. XXI., B. I. *El. S. Geom.*): hence any side of the triangle ABC is less than the sum of the other two.

PROPOSITION III. THEOREM.

The shortest distance from one point to another, on the surface of a sphere, is the arc of the great circle which joins the two given points.



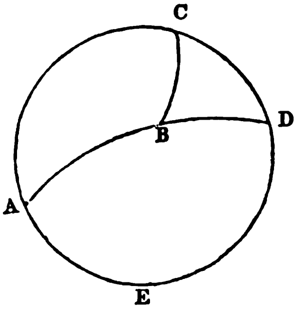
Let ADB be the arc of the great circle which joins the points A and B; and without this line, if possible, let M be a point of the shortest path between A and B. Through the point M, draw MA, MB, arcs of great circles; and take BD=MB.

By the last Proposition, the arc ADB is shorter than AM+MB; take BD=BM respectively from both; there will remain AD<AM. Now, the distance of B from M, whether it be the same with the arc BM, or with any other line, is equal to the distance of B from D; for, by making the plane of the great circle BM to revolve about the diameter which passes through B, the point M may be brought into the position of the point D; and the shortest line between M and B, whatever it may be, will then be identical with that between D and B: hence the two paths from A to B, one passing through M, the other through D, have an equal part in each, the part from M to B equal to the part from D to B. The first part is the shorter by hypothesis; hence the distance

from A to M must be shorter than the distance from A to D, which is absurd; the arc AM being proved greater than AD. Hence no point of the shortest line from A to B can lie out of he arc ADB; hence this arc is itself the shortest distance beween its two extremities.

PROPOSITION IV. THEOREM.

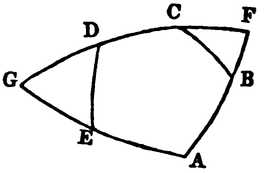
he sum of the three sides of a spherical triangle is less than the circumference of a great circle.



Let ABC be any spherical tringle; produce the sides AB, .C, till they meet again in D. 'he arcs ABD, ACD, will be emicircumferences, since (Prop. Cor. 2.) two great circles alays bisect each other. But in e triangle BCD, we have (Prop. I.) the side $BC<BD+CD$; add $B+AC$ to both; we shall have $B+AC+BC<ABD+ACD$:—hat is to say, less than a circumerence.

PROPOSITION V. THEOREM.

he sum of all the sides of any spherical polygon is less than the circumference of a great circle.

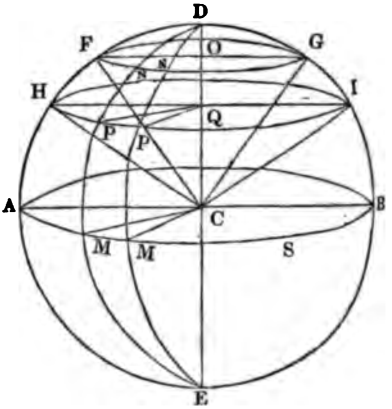


Take the pentagon ABCDE, for xample. Produce the sides AB, C, till they meet in F; then, since C is less than $BF+CF$, the perieter of the pentagon ABCDE will less than that of the quadrilateral EDF. Again produce the sides E, FD, till they meet in G; we all have $ED<EG+DG$; hence the perimeter of the quadrieral AEDF is less than that of the triangle AFG; which t is itself less than the circumference of a great circle: nce, for a still stronger reason, the perimeter of the polygon CDE is less than this same circumference.

Scholium. This proposition is fundamentally the same as op. XXII. B. I. *El. S. Geom.*; for O being the centre of the sphere, a solid angle may be conceived as formed at O, by the plane angles AOB, BOC, COD, &c.; and the sum of these angles must be less than four right angles; which is exactly the proposition we have been engaged with. The demonstration here given is different from that of Prop. XXII. B. I. *El. S. Geom.*; both, however, suppose that the polygon ABCDE is convex, or that no side produced will cut the figure.

PROPOSITION VI. THEOREM.

The poles of a great circle of the sphere, are the extremities of that diameter of the sphere which is perpendicular to this circle; and these extremities are also the poles of all small circles parallel to it.



Let ED be perpendicular to the great circle AMB; then will E and D be its poles; as also the poles of the parallel small circles HPP, FNG.

For, DC being perpendicular to the plane AMB, is perpendicular to all the straight lines CA, CM, CB, &c., drawn through its foot in this plane; hence all the arcs DA, DM, DB, &c., are quarters of the circumference. So likewise are all the arcs EA, EM, EB, &c.; hence the points D and E are each equally distant from all the points of the circumference AMB; hence (Def. 5.) they are the poles of that circumference.

Again, the radius DC, perpendicular to the plane AMB, is perpendicular to its parallel FNG; hence (Prop. I. *Cor.* 4.) it passes through O the centre of the circle FNG; hence, if the oblique lines DF, DN, DG be drawn, these oblique lines will diverge equally from the perpendicular DO, and will themselves be equal. But, the chords being equal, the arcs are equal; hence the point D is the pole of the small circle FNG; and, for like reasons, the point E is the other pole.

Cor. 1. Every arc DM, drawn from a point in the arc of a great circle AMB to its pole, is a quarter of the circumfernce, which, for the sake of brevity, is usually named a *quadans*, or quadrant: and this quadrant at the same time makes right angle with the arc AM. For (Prop. XVII. B. I. *El.* l. *Geom.*) the line DC being perpendicular to the plane AMC, very plane DMC passing through the line DC, is perpendicuar to the plane AMC; hence the angle of these planes, or Def. 6.) the angle AMD, is a right angle.

Cor. 2. To find the pole of a given arc AM, draw the inlefinite arc MD perpendicular to AM; take MD equal to a uadrant; the point D will be one of the poles of the arc AM: r thus, at the two points A and M, draw the arcs AD and MD erpendicular to AM; their point of intersection, D, will be he pole required.

Cor. 3. Conversely, if the distance of the point D from ach of the points A and M, is equal to a quadrant, the point D will be the pole of the arc AM, and also the angles DAM, AMD, will be right.

For, let C be the centre of the sphere, and draw the radii CA, CD, CM. Since the angles ACD, MCD are right, the line CD is perpendicular to the two straight lines CA, CM; hence it is perpendicular to their plane (Prop. V. B. I. *El. S. Geom.*); hence the point D is the pole of the arc AM; and consequently the angles DAM, AMD are right.

Scholium. The properties of these poles enable us to describe arcs of a circle on the surface of a sphere, with the same facility as on a plane surface. It is evident, for instance, that by turning the arc DF, or any other line extending to the same distance, round the point D, the extremity F will describe the small circle FNG; and, by turning the quadrant DFA round the point D, its extremity A will describe the arc f the great circle AM.

If the arc AM were required to be produced, and nothing were given but the points A and M through which it was to ass, we should first have to determine the pole D, by the inersection of two arcs described from the points A and M as entres, with a distance equal to a quadrant. The pole D eing found, we might describe the arc AM and its prolongaion, from D as a centre, and with the same distance as before.

In fine, if it is required from a given point P to let fall a erpendicular on the given arc AM, produce this arc to S, ill the distance PS be equal to a quadrant; then, from the ole S, and with the same distance, describe the arc PM, which will be the perpendicular required.

PROPOSITION VII. THEOREM.

Every plane perpendicular to a radius at its extremity is tangent to the sphere.

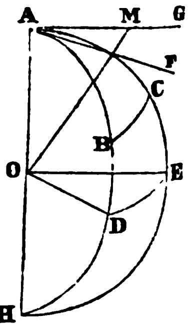
Let FAG (see the next diagram) be a plane perpendicular to the radius OA, at its extremity A. Any point M in this plane being assumed, and OM, AM being joined, the angle OAM will be right, and hence the distance OM will be greater than OA. Hence the point M lies without the sphere; and as the same can be shown for every other point of the plane FAG, this plane can have no point but A common to it and the surface of the sphere; hence (Def. 4.) it is tangent.

Scholium. In the same way, it may be shown that two spheres have but one point in common, and therefore touch each other, when the distance between their centres is equal to the sum, or the difference of their radii; in which case, the centres and the point of contact lie in the same straight line.

PROPOSITION VIII. THEOREM.

The angle formed by two arcs of great circles, is equal to the angle formed by the tangents of these two arcs at their points of intersection, and is measured by the arc described from this point of intersection, as a pole, and limited by the sides, produced if necessary.

Let the angle BAC be formed by the two arcs AB, AC; then it will be equal to the angle FAG formed by the tangents AF, AG, and be measured by the arc DE, described about A as a pole.



For the tangent AF, drawn in the plane of the arc AB, is perpendicular to the radius AO; and the tangent AG, drawn in the plane of the arc AC, is perpendicular to the same radius AO. Hence, (Prop. XX. B. I. *El. S. Geom.*) the angle FAG is equal to the angle contained by the planes OAB, OAC; which is that of the arcs AB, AC, and is named BAC.

In like manner, if the arcs AD and AE are both quadrants, the line OD, OE will be perpendicular to OA, and the angle DOE will still be equal to the angle of the planes AOD, AOE: hence the arc DE is the measure of the angle contained by these planes, or of the angle CAB.

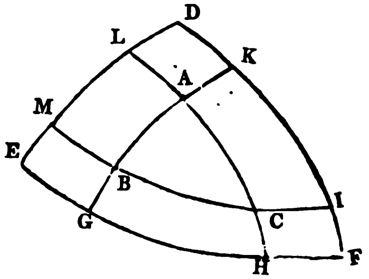
Cor. The angles of spherical triangles may be compared gether, by means of the arcs of great circles described from ir vertices as poles and included between their sides : hence s easy to make an angle of this kind equal to a given angle.

Scholium. Vertical angles, such as ACO and BCN (see diaam to Prop. XXI.) are equal ; for either of them is still the gle formed by the two planes ACB, OCN.

It is further evident, that, in the intersection of two arcs CB, OCN, the two adjacent angles ACO, OCB taken togeer, are equal to two right angles.

PROPOSITION IX. THEOREM.

from the vertices of the three angles of a spherical triangle, as poles, three arcs be described forming a second triangle, the vertices of the angles of this second triangle will be respectively poles of the sides of the first.



From the vertices A, B, C, poles, let the arcs EF, FD, D be described, forming on e surface of the sphere, the angle DFE ; then will the ints D, E, and F be reectively poles of the sides C, AC, AB.

For, the point A being the le of the arc EF, the disnce AE is a quadrant ; the int C being the pole of the arc DE, the distance CE is likeise a quadrant : hence the point E is removed the length of quadrant from each of the points A and C ; hence (Prop VI. r. 3.) is the pole of the arc AC. It might be shown, by the me method, that D is the pole of the arc BC, and F that of e arc AB.

Cor. Hence the triangle ABC may be described by means of EF, as DEF is described by means of ABC.

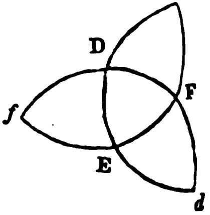
PROPOSITION X. THEOREM.

he same supposition continuing as in the last proposition, each angle in the one of the triangles, will be measured by the semicircumference minus *the side lying opposite to it in the other triangle.*

Produce the sides (see the preceding diagram) AB, AC, if necessary, till they meet EF, in G and H. The point A being the pole of the arc GH, the angle A will be measured by that arc. But the arc EH is a quadrant, and likewise GF, E being the pole of AH, and F of AG; hence EH+GF is equal to the semicircumference. Now, EH+GF is the same as EF+GH; hence the arc GH, which measures the angle A, is equal to a semicircumference *minus* the side EF. In like manner, the angle B will be measured by ½ *circ.*—DF: the angle C, by ½ *circ.*—DE.

And this property must be reciprocal in the two triangles, since each of them is described in a similar manner by means of the other. Thus we shall find the angles D, E, F, of the triangle DEF to be measured respectively by ½ *circ.*—BC, ½ *circ.*—AC. ½ *circ.*—AB. Thus the angle D, for example, is measured by the arc MI; but MI+BC=MC+BI=½ *circ.*; hence the arc MI, the measure of D, is equal to ½ *circ.*—BC; and so of all the rest.

Scholium. It must further be observed, that besides the triangle DEF, three others might be formed by the intersection of the three arcs DE, EF, DF. But the proposition immediately before us is applicable only to the central triangle, which is distinguished from the other three by the circumstance (see diagram to Prop. IX.) that the two angles A and D lie on the same side of BC, the two B and E on the same side of AC, and the two C and F on the same side of AB.

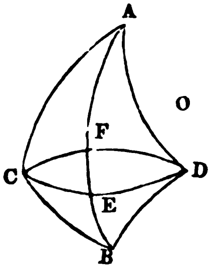


Various names have been given to the triangles ABC, DEF; we shall call them *polar triangles.*

PROPOSITION XI. THEOREM.

If around the vertices of the two angles of a given spherical triangle, as poles, the circumference of two circles be described which shall pass through the third angle of the triangle; if then, through the other point in which those circumferences intersect, and the two first angles of the triangle, the arcs of great circles be drawn, the triangle thus formed will have all its parts equal to those of the first triangle.

Let ABC be the given triangle, CED, DFC the arc described about A and B as poles; then will the triangle ADB have all its parts equal to those of ABC.



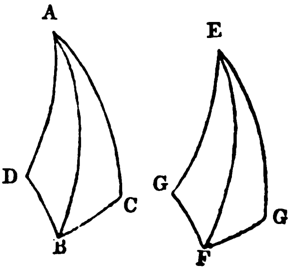
For, by construction, the side AD=AC, DB=BC, and AB is common: hence hose two triangles have their sides equal, each to each. We are now to show, that he angles opposite these equal sides are also equal.

If the centre of the sphere is supposed o be at O, a solid angle may be conceived as formed at O by the three plane angles AOB, AOC, BOC; likewise another olid angle may be conceived as formed by the three plane angles AOB, AOD, BOD. And because the sides of the triangle ABC are equal to those of the triangle ADB, the plane angles forming the one of these solid angles, must be equal to he plane angles forming the other, each to each. But in this case we have shown (Prop. XXIII. *El. S. Geom.*) that the planes, in which the equal angles lie, are equally inclined to each other; hence, all the angles of the spherical triangle DAB are respectively equal to those of the triangle CAB, namely, DAB=BAC, DBA=ABC, and ADB=ACB; hence, the sides and the angles of the triangle ADB are equal to the sides and the angles of the triangle ACB.

Scholium. The equality of those triangles is not, however, an absolute equality, or one of superposition; for it would be mpossible to apply them to each other exactly, unless they were isosceles. The equality meant here is what we have already named an equality by *symmetry*; therefore, we shall call the triangles ACB, ADB, *symmetrical triangles.*

PROPOSITION XII. THEOREM.

Two triangles on the same sphere, or on equal spheres, are equal in all their parts, when they have each an equal angle included between equal sides.



Suppose the side AB=EF, the ide AC=EG, and the angle BAC =FEG; the triangle EFG may be placed on the triangle ABC, or on ABD symmetrical with ABC, just as two rectilineal triangles are placed upon each other, when they have an equal angle included between equal sides. Hence all the parts of the triangle EFG will be equal to all the parts of the triangle ABC; that is, besides the three parts equal by hypothesis, we shall have the side BC=FG, the angle ABC=EFG, and the angle ACB=EGF.

PROPOSITION XIII. THEOREM.

Two triangles on the same sphere, or on equal spheres, are equal in all their parts, when two angles and the included side of the one are respectively equal to two angles and the included side of the other.

For, one of those triangles, or the triangle symmetrical with it, may be placed on the other, as is done in the corresponding case of rectilineal triangles, (Prop. IX. B. II. *El. Geom.*)

PROPOSITION XIV. THEOREM.

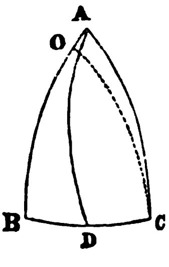
If two triangles on the same sphere, or on equal spheres have all their sides respectively equal, their angles will likewise be all respectively equal, the equal angles lying opposite the equal sides.

This truth is evident from Proposition XI, where it is shown that, with three given sides AB, AC, BC, (see the diagram,) there can only be two triangles ACB, ABD, differing as to the position of their parts, and equal as to the magnitude of those parts. Hence, those two triangles, having all their sides respectively equal in both, must either be absolutely equal, or at least *symmetrically* so; in both of which cases their corresponding angles must be equal, and lie opposite to equal sides.

PROPOSITION XV. THEOREM.

In every isosceles spherical triangle, the angles opposite the equal sides are equal; and conversely, if two angles of a spherical triangle are equal, the triangle is isosceles.

First. Suppose the side AB=AC; we shall have the angle C=B. For, if the arc be drawn from the vertex A to the middle point D of the base, the two triangles ABD, ACD will have all the sides of the one respectively equal to the corresponding sides of the other, namely, AD common, BD=DC, and AB=AC: hence, by the last Proposition, their angles will be equal; therefore B=C.



Secondly. Suppose the angle B=C; we

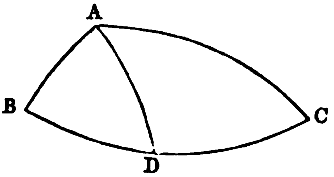
all have the side AC=AB. For, if not, let AB be the greater f the two; take BO=AC, and join OC. The two sides BO, C are equal to the two AC, BC; the angle OBC, contained y the first two is equal to ACB contained by the second two. ence (Prop. XII.) the two triangles BOC, ACB have all their ther parts equal; hence the angle OCB=ABC: but by hyothesis, the angle ABC=ACB; hence we have OCB=ACB, hich is absurd; hence it is absurd to suppose AB different om AC; hence the sides AB, AC, opposite to the equal anles B and C, are equal.

Scholium. The same demonstration proves the angle BAD =DAC, and the angle BDA=ADC. Hence the two last are ght angles; hence *the arc drawn from the vertex of an isosles spherical triangle to the middle of the base, is at right anles to the base, and bisects the vertical angle.*

PROPOSITION XVI. THEOREM.

any spherical triangle, the greater side is opposite the greater angle; and conversely, the greater angle is opposite the greater side.

Let the angle A be greater than the angle B, then will BC greater than AC; and conversely, if BC is greater than C, then will the angle A be greater than B.



First. Suppose the angle >B; make the angle BAD= : then (Prop. XV.) we shall ive AD=DB; but AD+DC greater than AC; hence, itting DB in place of AD, we all have DB+DC, or BC>AC.

Secondly. If we suppose BC>AC, the angle BAC will be eater than ABC. For, if BAC were equal to ABC, we should ive BC=AC; if BAC were less than ABC, we should then, has just been shown, find BC<AC. Both these conclusions e false: hence the angle BAC is greater than ABC.

PROPOSITION XVII. THEOREM.

If two triangles on the same sphere, or on equal spheres, are mutually equiangular, they will also be mutually equilateral.

Let A and B be the two given triangles; P and Q their polar triangles. Since the angles are equal in the triangles A and B, the sides will be equal in their polar triangles P and Q, (Prop. X. :) but since the triangles P and Q are mutually equilateral, they must also (Prop. XIV.) be mutually equiangular; and, lastly, the angles being equal in the triangles P and Q, it follows (Prop. X.) that the sides are equal in their polar triangles A and B. Hence the mutually equiangular triangles A and B are at the same time mutually equilateral.

Scholium. This proposition is not applicable to rectilineal triangles; in which equality among the angles indicates only proportionality among the sides. Nor is it difficult to account for the difference observable, in this respect, between spherical and rectilineal triangles. In the proposition now before us, as well as in Propositions XII, XIII, XIV, which treat of the comparison of triangles, it is expressly required that the arcs be traced on the same sphere, or on equal spheres. Now similar arcs are to each other as their radii; hence, on equal spheres, two triangles cannot be similar without being equal. Therefore it is not strange that equality among the angles should produce equality among the sides.

The case would be different, if the triangles were drawn upon unequal spheres; there, the angles being equal, the triangles would be similar, and the homologous sides would be to each other as the radii of their spheres.

PROPOSITION XVIII. THEOREM.

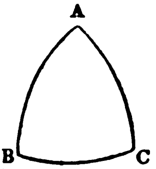
The sum of all the angles in any spherical triangle is less than six right angles, and greater than two.

For, in the first place, every angle of a spherical triangle is less than two right angles (see the following Scholium): hence the sum of all the three is less than six right angles.

Secondly, the measure of each angle of a spherical triangle (Prop. X.) is equal to the semicircumference *minus* the corresponding side of the polar triangle; hence the sum of all the three, is measured by three semicircumferences *minus* the sum of all the sides of the polar triangle. Now (Prop. IV.), this latter sum is less than a circumference ; therefore, taking it away from three semicircumferences, the remainder will be greater than one semicircumference, which is the measure of two right angles ; hence, in the second place, the sum of all the angles in a spherical triangle is greater than two right angles.

Cor. 1. The sum of all the angles of a spherical triangle is not constant, like that of all the angles of a rectilineal triangle ; it varies between two right angles and six, without ever arriving at either of these limits. Two given angles therefore do not serve to determine the third.

Cor. 2. A spherical triangle may have two, or even three angles, right, two or three obtuse.



If the triangle ABC is *bi-rectangular*, in other words, has two right angles B and C, the vertex A will (Prop. X.) be the pole of the base BC ; and the sides AB, AC will be quadrants.

If the angle A is also right, the triangle ABC will be *tri-rectangular;* its angles will all be right, and its sides quadrants. The tri-rectangular triangle is contained eight times in the surface of the sphere.

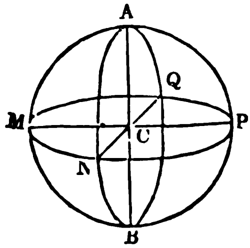
Scholium. In all the preceding observations, we have supposed, in conformity with (Def. 6,) that our spherical triangles have always each of their sides less than a semicircumference; from which it follows that any one of their angles is always less than two right angles. For (see diagram to Prop. IV.) if the side AB is less than a semicircumference, and AC is so likewise, both those arcs will require to be produced before they can meet in D. Now the two angles ABC, CBD taken together, are equal to two right angles ; hence the angle ABC itself, is less than two right angles.

We may observe, however, that some spherical triangles do exist, in which certain of the sides are greater than a semicircumference, and certain of the angles greater than two right angles. Thus, if the side AC is produced so as to form a whole circumference ACE, the part which remains after substracting the triangle ABC from the hemisphere, is a new triangle also designated by ABC, and having AB, BC, AEDC for its sides. Here, it is plain, the side AEDC is greater than the semicircumference AED ; and, at the same time, the angle B opposite to it exceeds two right angles, by the quantity CBD.

The triangles whose sides and angles are so large, have been excluded from our Definition ; but the only reason was, that the solution of them, or the determination of their parts, is always reducible to the solution of such triangles as are comprehended by the Definition. Indeed, it is evident enough, that if the sides and angles of the triangle ABC are known, it will be easy to discover the angles and sides of the triangle which bears the same name, and is the difference between a hemisphere and the former triangle.

PROPOSITION XIX. THEOREM.

The surface of a lune is to the surface of the sphere, as the angle of this lune, is to four right angles, or as the arc which measures that angle, is to the circumference.



Let AMNB be a lune; then will its surface be to the surface of the sphere as the angle NCM to four right angles, or as the arc NM to the circumference of a great circle.

Suppose, in the first place, the arc MN to be the circumference MNPQ as some one rational number is to another, as 5 to 48, for example. The circumference MNPQ being divided into 48 equal parts, MN will contain 5 of them; and if the pole A were joined with the several points of division, by as many quadrants, we should in the hemisphere AMNPQ have 48 triangles, all equal, because all their parts are equal. Hence the whole sphere must contain 96 of those partial triangles, the lune AMBNA will contain 10 of them; hence the lune is to the sphere as 10 is to 96, or as 5 to 48, in other words, as the arc MN is to the circumference.

If the arc MN is not commensurable with the circumference, we may still show, by a mode of reasoning frequently exemplified already, that in this case also, the lune is to the sphere as MN is to the circumference.

Cor. 1. Two lunes are to each other as their respective angles.

Cor. 2. It was shown above (Prop. XVIII. Cor. 2.) that the whole surface of the sphere is equal to eight tri-rectangular triangles; hence, if the area for one such triangle is taken for unity, the surface of the sphere will be represented by 8. This granted, the surface of the lune, whose angle is A, will be expressed by 2A (the angle A being always estimated from the ght angle assumed as unity :) since 2A : 8 : : A : 4. Thus e have here two different unities; one for angles, being the ght angle; the other for surfaces being the tri-rectangular herical triangle, or the triangle whose angles are all right, d whose sides are quadrants.

Or if the area of one such triangle is represented by T, the rface of the whole sphere will be expressed by 8T, and the rface of the lune whose angle is A, will be expressed by $A \times T$. for

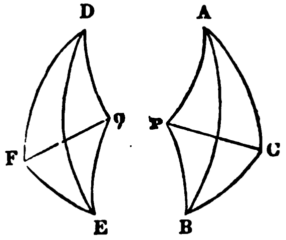
$$4 : A :: 8T : 2A \times T$$

which expression, A represents such a part of unity, as the ngle of the lune is of one right angle.

Scholium. The spherical ungula, bounded by the planes AMB, NB, is to the whole solid sphere, as the angle A is to four ght angles. For, the lunes being equal, the spherical ungulas ill also be equal; hence two spherical ungulas are to each ther, as the angles formed by the planes which bound them.

PROPOSITION XX. THEOREM.

Two symmetrical spherical triangles are equal in surface.



Let ABC, DEF be two symmetical triangles, that is to say, two iangles having their sides AB= E, AC=DF, CB=EF, and yet inpable of coinciding with each her: we are to show that the surce ABC is equal to the surface EF.

Let P be the pole of the small cire passing through the three points A, B, C; from this point raw (Prop. VI.) the equal arcs, PA, PB, PC; at the point F, ake the angle DFQ=ACP, the arc FQ=CP; and join DQ, Q.

The sides DF, FQ are equal to the sides AC, CP; the ane DFQ=ACP: hence (Prop. XII.) the two triangles DFQ, CP are equal in all their parts; hence the side DQ=AP, and e angle DQF=APC.

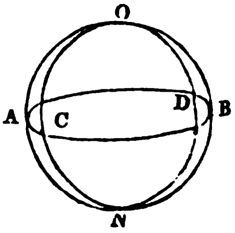
In the proposed triangles DFE, ABC, the angles DFE, ACB pposite to the equal sides DE, AB, being equal (Prop. XIII.) the angles DFQ, ACP, which are equal by construction, be ken away from them, there will remain the angle QFE, ual to PCB. Also the sides QF, FE are equal to the sides C, CB; hence the two triangles FQE, CPB are equal in all their parts; hence the side QE=PB, and the angle FQE =CPB.

Now, the triangles DFQ, ACP, which have their sides respectively equal, are at the same time isosceles, and capable of coinciding, when applied to each other; for having placed PA on its equal QF, the side PC will fall on its equal QD, and thus the two triangles will exactly coincide; hence they are equal; and the surface DQF=APC. For a like reason, the surface FQE=CPB, and the surface DQE=APB; hence we have DQF+FQE—DQE=APC+CPB—APB, or DFE= ABC; hence the two symmetrical triangles ABC, DEF are equal in surface.

Scholium. The poles P and Q might lie within the triangles ABC, DEF: in which case it would be requisite to add the three triangles DQF, FQE, DQE together, in order to make up the triangle DEF; and in like manner to add the three triangles APC, CPB, APB together, in order to make up the triangle ABC: in all other respects, the demonstration and the result would still be the same.

PROPOSITION XXI. THEOREM.

If the circumferences of two great circles intersect each other on the surface of a hemisphere, the sum of the opposite triangles thus formed, is equivalent to the surface of a lune whose angle is equal to the angle formed by the circles.



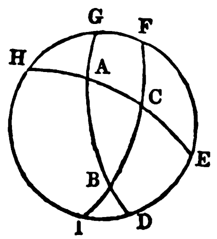
Let the circumferences AOB, COD, intersect on the hemisphere OACBD; then will the opposite triangles AOC, BOD be equal to the lune whose angle is BOD.

For, producing the arcs OB, OD on the other hemisphere, till they meet in N, the arc OBN will be a semi-circumference, and AOB one also; and taking OB from both, we shall have BN=AO. For a like reason, we have DN=CO, and BD=AC. Hence the two triangles AOC, BDN have their three sides respectively equal; besides, they are so placed as to be symmetrical; hence (Prop. XIX. Sch.) they are equal in surface, and the sum of the triangles AOC, BOD is equivalent to the lune OBNDO, whose angle is BOD.

Scholium. It is likewise evident that the two spherical pyramids, which have the triangles AOC, BOD for bases, are together equivalent to the spherical ungula whose angle is BOD.

PROPOSITION XXII. THEOREM.

The surface of a spherical triangle is measured by the excess of the sum of its three angles above two right angles, multiplied by the tri-rectangular triangle.



Let ABC be the proposed triangle: produce its sides till they meet the great circle DEFG, drawn at pleasure without the triangle. By the last Theorem, the two triangles ADE, AGH, are together equivalent to the lune whose angle is A, and which is measured by 2A.T (Prop. XIX. Cor. 2.) Hence we have ADE+AGH=2A.T; and, for a like reason, BGF+BID=2B.T, and CIH+CFE=2C.T. But the sum of these six triangles exceeds the hemisphere by twice the triangle ABC, and the hemisphere is represented by 4T; therefore, twice the triangle ABC is equal to 2A.T+2B.T+2C.T—4T; and consequently, once ABC=(A+B+C—2)T; hence every spherical triangle is measured by the sum of all its angles *minus* two right angles, multiplied by the tri-rectangular triangle.

Cor. 1. However many right angles there may be in the sum of the three angles minus two right angles, just so many tri-rectangular triangles, or eighths of the sphere, will the proposed triangle contain. If the angles, for example, are each equal to $\frac{4}{3}$ of a right angle, the three angles will amount to four right angles, and the sum of the angles minus two right angles will be represented by 4—2, or 2; therefore the surface of the triangle will be equal to two tri-rectangular triangles, or to the fourth part of the whole surface of the sphere.

Scholium. While the spherical triangle ABC is compared with the tri-rectangular triangle, the spherical pyramid, which has ABC for its base, is compared with the tri-rectangular pyramid, and a similar proportion is found to subsist between hem. The solid angle at the vertex of the pyramid, is in like nanner compared with the solid angle at the vertex of the tri-ectangular pyramid. These comparisons are founded on the coincidence of the corresponding parts. If the bases of the pyramids coincide, the pyramids themselves will evidently coincide, and likewise the solid angles at their vertices. From this, some consequences are deduced.

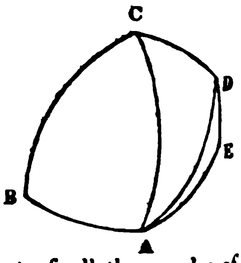
First. Two triangular spherical pyramids are to each other as their bases; and, since a polygonal pyramid may always be divided into a certain number of triangular ones, it follows that any two spherical pyramids are to each other as the polygons which form their bases.

Second. The solid angles at the vertices of these pyramids are also as their bases: hence, for comparing any two solid angles, we have merely to place their vertices at the centres of two equal spheres, and the solid angles will be to each other as the spherical polygons intercepted between their planes or faces.

The vertical angle of the tri-rectangular pyramid is formed by three planes, at right angles to each other. This angle, which may be called a *right solid angle,* will serve as a very natural unit of measure for all other solid angles. If, for example, the area of the triangle is ¾ of the tri-rectangular triangle, then the corresponding solid angle will also be ¾ of the right solid angle.

PROPOSITION XXIII. THEOREM.

The surface of a spherical polygon is measured by the sum of all its angles, minus *two right angles multiplied by the number of sides in the polygon* less *two, into the tri-rectangular triangle.*



From one of the vertices A, let diagonals AC, AD, be drawn to all the other vertices; the polygon ABCDE will be divided into as many triangles, *minus* two, as it has sides. But the surface of each triangle is measured by the sum of all its angles *minus* two right angles, into the tri-rectangular triangle; and the sum of the angles in all the triangles is evidently the same as that of all the angles of the polygon: hence, the surface of the polygon is equal to the sum of all its angles, diminished by twice as many right angles as it has sides, less two, into the tri-rectangular triangle.

Scholium. Let s be the sum of all the angles in a spherical polygon, n the number of its sides, and T the tri-rectangular triangle; the right angle being taken for unity, the surface of the polygon will be measured by

$$\big(s-2\,(n-2,)\big)\ T, \text{ or } (s-2\,n+4)\ \ T$$

ANALYTICAL PLANE TRIGONOMETRY.

CHAPTER I.

ANE TRIGONOMETRY is the science which treats of the re-
ıs of the sides and angles of plane triangles.

every triangle there are six parts: three sides and three
ːs; which have such relations to each other that the value
ıe depends on the value of the others; and if a sufficient
ɔer of these are known the others may thereby be deter-
d.

ıe sides of triangles consist of absolute magnitude, but the
ːs are only the relations of those sides to each other in
ion or direction, without regard to their magnitudes.

ıgles have no absolute measure in terms of the sides; but
ıevertheless, susceptible of measure; for if two lines meet
other the space included between them within a given
nce from their point of contact is proportional to their
ɪal inclination, and hence (Prop. XVIII. Cor. B. III. *El.*
ɪ.) the arc of the circumference of a circle intercepted by
lines drawn from its centre, may be regarded as the mea-
of the angle or inclination of those lines, and therefore the
ɔf the circumference may be regarded as the measure of
ılar magnitude.

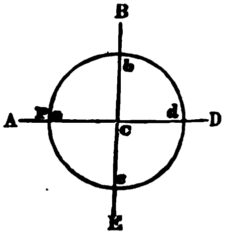
ɔr this purpose the circumference of the circle is supposed
divided into 360 equal parts, called degrees, and each of
degrees is divided into 60 equal parts called minutes, and
minute into 60 equal parts called seconds; and so on, to
s, fourths, &c.

hese divisions are designated by the following characters,
′ ‴ &c. Thus the expression 30° 20′ 12″ 22‴, repre-
ɪ an arc or an angle of 30 degrees 20 minutes 12 seconds
hirds.

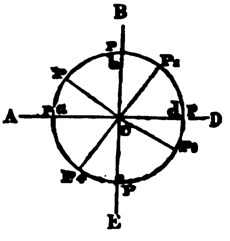
he circumference of any circle may in this manner be ap-
l as the measure of angles, without regard to its magni-
or the length of its radius; hence a degree is not a mag-

nitude of any definite length, but is a certain portion of the whole circumference of any circle, for it is evident that the 360th part of the circumference of a large circle is greater than the same part of a smaller one, but the number of degrees in the small circumference is the same as in the large one.

The fourth part of the circumference of a circle is called a quadrant and contains 90 degrees: hence 90 degrees is the measure of the right angle.



Thus, if we draw two straight lines AD, BE, so as to cross each other at right angles, and from their point of intersection, C, we discribe a circle with any radius so as to cut those lines in any points, as *a*, *b*, *d*, *e*, the circumference of the circle will thus be divided into four equal arcs, *ab*, *bd*, *de*, *ea*, each of which measures or subtends a right angle at the centre C, of the circle.



If a line CP be made to revolve round a fixed point C as the centre of a circle, and so as to pass successively through every point of the circumference, commencing in the point *a*, then, while it is in the position *Ca*, or while it coincides with the line *Ca*, those two lines form but one, and intercept no arc on the circumference of the circle, and hence form no angle with each other; but when the line C*a*P comes into the position CP, it forms with AC an acute angle at C, which is measured by the arc *a*P, and when it comes into the position C*b*P, it then forms a right angle ACP with the line AC, which angle is measured by the quadrant *ab*. Now let it come into the position CP_2, and the angle which it forms with CA, will be measured by the arc aP_2, which is greater than a right angle, and hence is an obtuse angle.

Let it now come into the position C*d*P; it then coincides with the right line C*d*, which is a portion of the line AC produced, since the line CP, in this position, coincides with the line AD, it can be said to form with it no angle; yet the space passed over by the line CP, from the position C*a*P, is equal to two quadrants, or two right angles equal to 180 degrees. and for trigonometrical investigation the lines C*b*P and CA are said to subtend the angle measured by the arc ***abd***.

After passing the point *d*, and coming into the position CP_3, it forms with AC, and on the upper side of it the angle P_3CA

easured by the arc $ae\mathrm{P}_3$, but having passed over the arc $d\mathrm{P}_3$, is said to contain, with the line CA, the angle ACP_3 on e upper side of those lines measured by the arc $abd\mathrm{P}_3$, eater than two right angles. When it comes in the posin $\mathrm{C}e\mathrm{P}$ it is said to subtend, with the line AC from the same le of it, the angle measured by the arc *abde*, or three quadnts, equal to three right angles.

When in the position CP_4, it is said to contain with CA, and the same side of it, an angle greater than three right angles.

Finally, when the line CP has completed an entire revolun, having returned to its original position, CA, it will have rmed an angle with it equal to four right angles.

If the line CP continues to revolve, it is manifest that the gle will increase, and may with this view form with CA, gles greater than four, than five, or than any given number right angles.

ab is called the first quadrant of the circle, *bd* the second, the third, and *ea* the fourth quadrant.

It must be borne in mind, that the line CP cannot, geomeically, be said to contain with another line, AC, an angle eater, nor quite equal to, two right angles, but in view of its pposed motion round one of its extremities, C, as a centre, is said to contain, with the line AC, all the angular space rough which it has passed in its revolution.

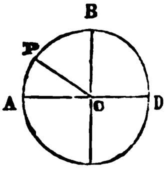
Thus, let the line CP have performed one complete revolun, from the position $\mathrm{C}a\mathrm{P}$ to the same position again, the gle which it forms with the line CA, though absolutely noing, is in view of its supposed motion measured by the quadnts $ab+bd+de+ea$, each of which quadrants are readily regnized as being contained by their several lines of division, hen by removing those lines of division of the circumference, ose several angles are all converted into one containing the hole circumference; hence, in view of this motion or relan of the two lines, they are said to contain an angle meared by the whole circumference.

DEFINITIONS AND ILLUSTRATIONS.

The following symbols are sometimes used.

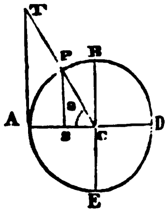
1. The complement of an arc or of an angle, is what remains after taking that arc or that angle from 90 degrees. Thus, if θ be any arc or angle, the complement is $90° - \theta$

2. Supplement of an arc or an angle, is what remains after taking that arc or angle from two right angles, or 180 degrees. Thus, if θ be any arc or angle, the supplement of θ is $180° - \theta$.



If AP be any arc, and ACP be any angle, θ measured by that arc, then the complement of the angle θ is the angle PCB measured by the arc PB, and the supplement of the angle θ is the angle PCD measured by the arc PBD, and if BCP is any angle θ measured by the arc PB, then PCA is the complement of θ, and if PCD is any angle θ, then will PCA be its supplement.

3. To represent the ratios of the sides and angles of triangles, right lines are drawn in and about a circle called sines, tangents, secants, &c.



Draw two right lines AD,BE cutting each other at right angles in the point C, with the centre C and any distance as radius, describe a circle cutting the lines in the points A, B, D, E.

Draw the radius CP forming with CA any angle ACP=θ. From P draw PS perpendicular on CA. From A draw AT a tangent to the circumference at A. Produce CP to meet AT in T.

4. Then the ratio of PS to the radius CA of the circle, is called the *sine* of the angle PCA.

Or, $$\frac{PS}{CA} = \sin. \theta.$$

5. The ratio of AT to the radius CA of the circle, is called the *tangent* of the angle TCA or PCA.

Hence, $$\frac{AT}{CA} = \tan. \theta.$$

6. And the ratio of CT to the radius is the *secant* of the angle PCA.

Or, $$\frac{CT}{CA} = \sec. \theta.$$

7. The ratio of AS to the radius of the circle, is called the *sed sine* of the angle PCA.

Or, $$\frac{AS}{CA}=\text{v. sin. }\theta.$$

3. The sine of the complement of an angle, is called the sine nplement, or *cosine* of that angle.

Thus, sin. (90°—θ)=cos. θ, hence cos. (90°—θ)=sin. θ.

9. The tangent of the complement of any given angle, is led the *cotangent* of that angle.

Thus, tan. (90°—θ)=cot. θ, hence cot. (90°—θ)=tan. θ

10. The secant of the complement of any given angle, is lled the *cosecant* of that angle.

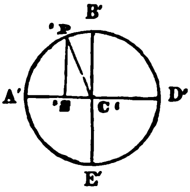
Or, sec. (90°—θ)=cosec. θ, hence cosec. (90°—θ)=sec. θ.

11. The versed sine of the complement of any angle, is call- the *co-versed sine* of that angle.

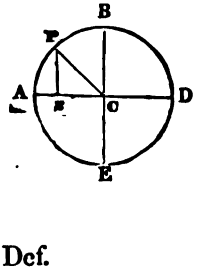
Or, v. sin. (90°—θ)=co-v. sin. θ,
and hence co-v. sin. (90—θ)=v. sin. θ.

In order to show that the ratio of CS to the radius of the rcle in the last figure, is the cosine of the angle PCA; that the sine of its complement,

Or that $$\frac{CS}{CA}=\cos.\ \theta,$$



Draw a circle A'B'D'E' equal to the circle BDE, and from C' the centre, draw C'P', aking with C'A' the angle P'C'A' equal to e angle PCB; that is, to the compliment ' PCA, or to (90°—θ).

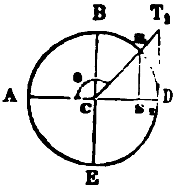


Then, since CP is equal to C'P', and the igles at S and S' are right angles, the angle PS equal to the angle P'C'S' the two triangles PCS, P'C'S' are equal in every respect; 'S=C'S', CS=P'S'.

Therefore, $$\frac{CS}{CA}=\frac{P'S'}{CA}$$
$$=\text{sin. } P'C'A' \text{ by Def.}$$
$$=\text{sin. } (90^\circ-\theta) \text{ by construct.}$$
$$=\text{cos. } \theta \text{ by Def.}$$

We have hitherto considered an angle PCA less than a right ngle, but the same definitions are applied, whatever may be he magnitude of the angle.

Thus, for example, let us take an angle P_2CA situated in the second quadrant, *that is*, an angle greater than one right angle, and less than two.



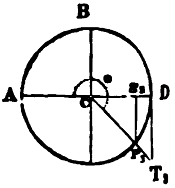
From P_2 let fall P_2S_2 perpendicular on AD, from D draw DT_2 a tangent to the circle at D. meeting CP_2 produced in T_2; then, as before,

$$\frac{P_2S_2}{CA}=\text{sin. } P_2CA$$
$$\frac{CS_2}{CA}=\text{cos. } P_2CA$$
$$\frac{DT_2}{CA}=\text{tan. } P_2CA$$
$$\frac{CT_2}{CA}=\text{sec. } P_2CA$$
$$\frac{AS_2}{CA}=\text{v. sin. } P_2CA.$$

Again, let the angle in question be situated in the third quadrant, that is, let it be an angle greater than two, and less than three right angles.

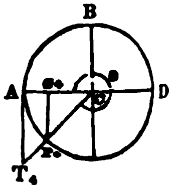
Making a construction analogous to that in the two former cases, we shall have

$$\frac{P_3S_3}{CA}=\text{sin. } P_3CA$$
$$\frac{CS_3}{CA}=\text{cos. } P_3CA$$
$$\frac{DT_3}{CA}=\text{tan. } P_3CA$$
$$\frac{CT_3}{CA}=\text{sec. } P_3AC$$
$$\frac{AS_3}{CA}=\text{v. sin. } P_3CA.$$



Lastly, let the angle be situated in the fourth quadrant; that is, let it be an angle greater than three, and less than four right angles, then as before,

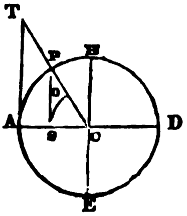
$$\frac{P_4S_4}{CA}=\text{sin. } P_4CA$$
$$\frac{CS_4}{CA}=\text{cos. } P_4CA$$
$$\frac{AT_4}{CA}=\text{tan. } P_4CA$$



$$\frac{CT_4}{CA}=\text{sec. } P_4CA$$

$$\frac{AS_4}{CA}=\text{v. sin. } P_4CA.$$

We shall now proceed to establish some important general relations, between the trigonometrical quantities which are immediately deducible from the above definitions, and from the principles of Geometry.



Resuming the figure of Def. (3):

Since CSP is a right-angled triangle, and CP the hypothenuse,

$$PS^2+CS^2=CP^2$$

Dividing by CP²,

$$\frac{PS^2}{CP^2}+\frac{CS^2}{CP^2}=1$$

that is, $\sin.^2\theta+\cos.^2\theta=1$ - - - - - - - - - - - - - (1)

The triangles PSC, TAC, are equiangular and similar; hence,

$$\frac{PS}{CS}=\frac{AT}{CA}$$

Therefore $$\frac{\frac{PS}{CA}}{\frac{CS}{CA}}=\frac{AT}{CA}$$

that is, $$\frac{\sin.\ \theta}{\cos.\ \theta}=\tan.\ \theta \quad \text{- - - - - - - - - -} \quad (2)$$

In last case, for θ substitute (90°—θ); then

$$\frac{\sin.\ (90^\circ-\theta)}{\cos.\ (90^\circ-\theta)}=\tan.\ (90^\circ-\theta)$$

Or, $$\frac{\cos.\ \theta}{\sin.\ \theta}=\cot.\ \theta \quad \text{- - - - - - - - - - -} \quad (3)$$

From (2) and (3) we have

$$\frac{\sin.\ \theta}{\cos.\ \theta}=\tan.\ \theta, \text{ and } \frac{\cos.\ \theta}{\sin.\ \theta}=\cot.\ \theta.$$

Hence, $\tan.\ \theta=\frac{1}{\cot.\ \theta}$, or, $\tan.\ \theta\ \cot.\ \theta=1$ - - - - - - - (4)

By similar triangles CTA, CPS.

$$\frac{CT}{CA}=\frac{CP}{CS}$$

$$=\frac{1}{\frac{CS}{CP}}$$

Or, $$\text{sec. } \theta = \frac{1}{\text{cos. } \theta}, \text{ or, sec. } \theta \text{ cos. } \theta = 1 \quad - \quad - \quad (5)$$

By Definition,

$$\begin{aligned} \text{cosec. } \theta &= \text{sec. } (90^\circ - \theta) \\ &= \frac{1}{\text{cos. } (90^\circ - \theta)} \text{ by the last case,} \\ &= \frac{1}{\text{sin. } \theta}, \text{ or, cosec. } \theta \text{ sin. } \theta = 1 \quad - \quad - \quad (6) \end{aligned}$$

Since CAT is a right angled triangle, and CT the hypothenuse

$$CA^2 + AT^2 = CT^2$$

Dividing by CA^2,

$$1 + \frac{AT^2}{CA^2} = \frac{CT^2}{CA^2}$$

that is, $$1 + \text{tan.}^2\, \theta = \text{sec.}^2\, \theta \quad - \; - \; - \; - \; - \; - \; - \; - \; - \; - \quad (7)$$

By (3) we have $$\text{cot. } \theta = \frac{\text{cos. } \theta}{\text{sin. } \theta}$$

Therefore, $$\text{cot.}^2\, \theta = \frac{\text{cos.}^2\, \theta}{\text{sin.}^2\, \theta}$$

Adding 1 to each side of the equation,

$$\begin{aligned} 1 + \text{cot.}^2\, \theta &= 1 + \frac{\text{cos.}^2\, \theta}{\text{sin.}^2\, \theta} \\ &= \frac{\text{sin.}^2\, \theta + \text{cos.}^2\, \theta}{\text{sin.}^2\, \theta} \\ &= \frac{1}{\text{sin.}^2\, \theta} \text{ by (1)} \\ &= \text{cosec.}^2\, \theta \text{ by (6)} \quad - \; - \; - \; - \quad (8) \end{aligned}$$

By Definition, $$\begin{aligned} \text{versin. } \theta &= \frac{SA}{CA} \\ &= \frac{CA - CS}{CA} \\ &= 1 - \frac{CS}{CA} \\ &= 1 - \text{cos. } \theta \quad - \; - \; - \; - \; - \; - \; - \; - \; - \quad (9) \end{aligned}$$

By Definition,

$$\begin{aligned} \text{coversin. } \theta &= \text{versin. } (90^\circ - \theta) \\ &= 1 - \text{cos.} (90^\circ - \theta,) \text{ by the last case.} \\ &= 1 - \text{sin. } \theta \quad - \; - \; - \; - \; - \; - \; - \quad (10) \end{aligned}$$

The above results, which are of the highest importance in all trigonometrical investigations, are collected and arranged in the following table, which ought to be committed to memory:—

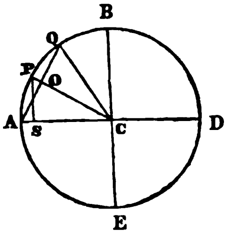
TABLE I.

1. $\text{Sin.}^2\,\theta+\text{cos.}^2\,\theta=1$
2. $\dfrac{\text{sin. }\theta}{\text{cos. }\theta}=\text{tan. }\theta$
3. $\dfrac{\text{cos. }\theta}{\text{sin. }\theta}=\text{cot. }\theta$
4. $\text{tan. }\theta\ \text{cot. }\theta=1$
5. $\text{sec. }\theta\ \text{cos. }\theta=1$
6. $\text{cosec. }\theta\ \text{sin. }\theta=1$
7. $1+\text{tan.}^2\,\theta=\text{sec.}^2\,\theta$
8. $1+\text{cot.}^2\,\theta=\text{cosec.}^2\,\theta$
9. $\text{v. sin. }\theta=1-\text{cos. }\theta$
10. $\text{coversin. }\theta=1-\text{sin. }\theta.$

12. The *chord* of an arc is the ratio of the straight line joining the two extremities of the arc to the radius of the circle.

PROPOSITION.

The chord of any arc is equal to twice the sine of half the arc.



Take any arc AQ, subtending at the centre of the circle the angle $ACQ=\theta$.

Draw the straight line CP bisecting the angle ACQ.

Join A, Q; from P let fall PS perpendicular on CA.

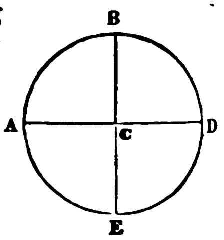
Since CP bisects ACQ, the vertical angle of the isosceles triangle ACQ, it bisects the base AQ at right angles.

Therefore, $AO=OQ$, and the angles at O are right angles.

Again, since the triangles AOC, PSC, have the angles CSP, COA, right angles, and the angle PCS common to the two triangles, and also the side CP of the one equal the side CA of the other, these triangles are in every respect equal.

$$\therefore\ PS=AO=OQ$$
$$\therefore\ AQ=2PS$$
$$\therefore\ \frac{AQ}{CA}=2\frac{PS}{CA}$$
$$\text{or, chord }\theta=2\text{ sin. }PCA$$
$$=2\text{ sin. }\frac{\theta}{2}$$

We shall now proceed to explain the principle by which the *signs* of the trigonometrical quantities are regulated —



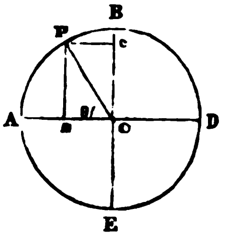
All lines measured from the point C along CA, that is, *to the left*, are considered positive, or have the sine +.

All lines measured from the point C along CD, that is, in the opposite direction *to the right* are considered negative, or have the sign —.

All lines measured from the point C along CB, that is, *upwards*, are considered positive or have sign +.

All lines measured from the point C along CE, that is, in the opposite direction *downwards*, are considered negative, or have the sign —.

Let us determine according to this principle, the signs of the sines and cosines of angles in the different quadrants.



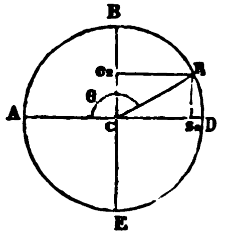
In the first quadrant, sin. $\theta=\frac{PS}{CA}$

$$\cos.\ \theta=\frac{CS.}{CA}$$

Here PS=C*c* is reckoned from C along CB upwards, and is therefore positive.

CS is reckoned from C along CA, to the left and is therefore positive.

In the first quadrant, therefore the *sine and cosin are both positive.*



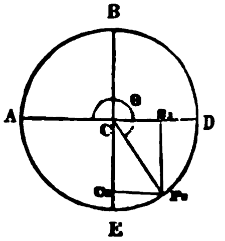
In the second quadrant, sin. $\theta=\frac{P_2S_2}{CA}$

$$\cos.\theta=\frac{CS_2.}{CA}$$

Here $P_2S_2=Cc_2$ is reckoned from C along CB upwards, and is therefore positive.

CS_2 is reckoned from C along CD to the right and is therefore negative.

In the second quadrant, therefore, *the sine is positive and the cosine negative.*



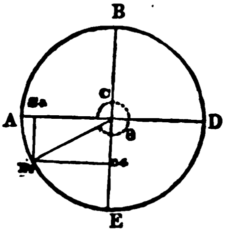
In the third quadrant, sin. $\theta=\frac{P_3S_3}{CA}$

$$\cos.\ \theta=\frac{CS_3}{CA}$$

Here $P_3S_3=Cc_3$ is reckoned from C along CE, downwards, and is therefore negative.

CS_3 is reckoned from C along CD, *to the right*, and is therefore negative.

In the third quadrant, therefore, *the sine*
:d cosine are both negative.



In the fourth quadrant, $\sin.\,\theta = \dfrac{P_4S_4}{CA}$

$$\cos.\,\theta = \frac{CS_4}{CA}$$

ere $P_4S_4 = Cc_4$ is reckoned from C along
E downwards and is therefore negative.
S_4 is reckoned from C along CA, to the
t and is therefore positive.

In the fourth quadrant, therefore, *the sine is negative, and*
e cosine positive.

Hence we conclude, that *the sine is positive in the first and*
cond quadrants, and negative in the third and fourth ; and the
sine is positive in the first and fourth, and negative in the
cond and third, or in other words:

The sine of an angle less than 180° *is positive and the sine*
an angle greater than 180° *and less than* 360° *is negative.*

The cosine of an angle less than 90° *is positive, the cosine of*
angle greater than 90°, *and less than* 270°, *is negative, and*
e cosine of an angle greater than 270°, *and less than* 360,° *is*
ositive.

The signs of the sine and the cosine being determined, the
gns of all the other trigonometrical quantities may be at once
tablished by referring to the relations in Table 1.

Thus, for the tangent,

$$\tan.\,\theta = \frac{\sin.\,\theta}{\cos.\,\theta}$$

Hence, it appears that when the sine and cosine have the
me sign the tangent will be positive, and when they have
ifferent signs it will be negative.

Therefore, *the tangent is positive in the first and third quad-*
nts and negative in the second and fourth.

he same holds good for the cotangent; for

$$\cot.\,\theta = \frac{\cos.\,\theta}{\sin.\,\theta}$$

gain, since

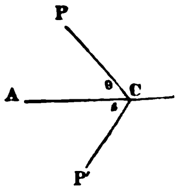
$$\sec.\,\theta = \frac{1}{\cos.\,\theta}$$

e *sign of the secant is always the same with that of the co-*
ne ; and, since

$$\text{cosec.}\,\theta = \frac{1}{\sin.\,\theta}$$

like manner, *the sign of the cosecant is always the same with*
at of the sine.

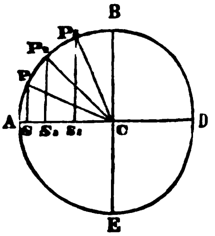
The versed sine is always positive, being reckoned from A
lways in the same direction.



It is sometimes convenient to give different signs to angles themselves. We have hitherto supposed angles of different magnitudes to be generated by the revolution of the moveable radius CP round C in a direction from left to right; and the angles so formed have been considered positive, or affected with the sign +. If we now suppose the angle $\beta=$ to be generated by the revolution of the radius CP′ in the opposite direction, we may, upon a principle analogous to th former, consider the angle β as negative, and affect it with th sign —.

We shall now determine the variations in the magnitude o the sine and cosine for angles of different magnitudes.

In the first quadrant:



Let CP, CP_2, CP_3, ... be different positions of the revolving radius in the first quadrant; and from P, P_2, P_3, ... draw PS, P_2S_2, P_3S_3, perpendiculars on CA.

It is manifest, that as the angle increases the sine increases; for

$$\frac{P_2S_2}{CA} > \frac{PS}{CA} \text{ and } \frac{P_3S_3}{CA} > \frac{P_2S_2}{CA}$$

When the angle becomes very small, PS becomes very small also; and when the revolving radius coincides with CA, that is, when the angle becomes 0, then PS disappears altogether, and is $=0$.

Hence since, generally, $\sin.\ \theta=\frac{PS}{CA}$ and since, when $\theta=0$, $PS=0$;

$$\therefore \sin.\ 0=\frac{0}{CA}$$
$$=0$$

On the other hand, when the angle becomes equal to 90°, PS coincides with CB, and is equal to it.

Hence since, generally, $\sin.\ \theta=\frac{PS}{CA}$, and since, when $\theta=90°$, $PS=CB$;

$$\therefore \sin.\ 90°=\frac{CB}{CA}$$
$$=1;\ \therefore CB=CA.$$

Again, it is manifest, that as the angle increases the cosine diminishes; for

$$\frac{CS}{CA} > \frac{CS_2}{CA} \text{ and } \frac{CS_2}{CA} > \frac{CS_3}{CA}$$

When the angle is very small, CS is very nearly equal to CA; and when the revolving radius coincides with CA, that is, when the angle is 0, then CS coincides with CA and is equal to it.

Hence since, generally, cos. $\theta=\frac{CS}{CA}$, and since, when $\theta=0$, CS=CA;

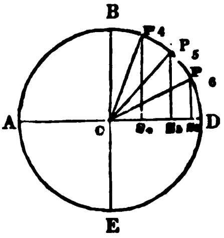
$$\therefore \text{ cos. } 0=\frac{CA}{CA}=1$$

On the other hand, as the angle increases, CS diminishes, and when the angle becomes equal to 90°, CS disappears altogether, and is =0.

Hence since, generally, cos. $\theta=\frac{CS}{CA}$, and since, when $\theta=90°$, CS =0;

$$\therefore \text{ cos. } 90°=\frac{0}{CA}=0$$

Let us now take different positions of the revolving radius *in the second quadrant.*



It is manifest, that as the angle increases the sine diminishes; for

$$\frac{P_4S_4}{CA}>\frac{P_5S_5}{CA}, \text{ and } \frac{P_5S_5}{CA}>\frac{P_6S_6}{CA}$$

As the angle goes on increasing, PS goes on diminishing; and when CP coincides with CD, that is, when the angle becomes equal to 180°, PS disappears altogether, and is equal 0.

Hence since, generally, sin. $\theta=\frac{PS}{CA}$, and since, when $\theta=180°$, PS=0;

$$\therefore \text{ sin. } 180°=0.$$

On the other hand, as the angle increases the cosine increases; for

$$\frac{CS_4}{CA}<\frac{CS_5}{CA} \text{ and } \frac{CS_5}{CA}<\frac{CS_6}{CA}$$

and when the revolving radius coincides with CD and the angle becomes 180°, CP coincides with CD and is equal to it.

Hence since, generally, cos. $\theta=\frac{CS}{CA}$ and since, when $\theta=180°$, CS=CD;

$$\therefore \text{ cos. } 180°=\frac{CD}{CA}=-1;$$

$$\therefore CD=CA.$$

The negative sign here is employed, because the cosine is reckoned *to the right* along CD.

Reasoning in the same manner for the third and fourth quadrants, we shall find,

$$\sin. 270° = -1$$
$$\cos. 270° = 0$$
$$\sin. 360° = 0$$
$$\cos. 360° = 1.$$

Thus, it appears,

That as the angle increases in the first quadrant, from 0 up to 90°,
The sine, being positive, increases from 0 up to 1,
The cosine being positive decreases from 1 down to 0.

That, as the angle increases in the second quadrant, from 90° up to 180°,
The sine, being positive, decreases from 1 down to 0,
The cosine, being negative, passes from 0 to —1.

That, as the angle increases in the third quadrant, from 180° up to 270°,
The sine, being negative, passes from 0 to —1,
The cosine, being negative, passes from —1 to 0.

That, as the angle increases in the fourth quadrant from 270° up to 360°,
The sine, being negative, passes from —1 to 0,
The cosine, being positive, increases from 0 up to 1.

The variations in the magnitude of the sines and cosines, being those of the other trigonometrical quantities may be determined by the means of the relations in Table I.

Thus, since, $\tan. \theta = \frac{\sin. \theta}{\cos. \theta}$

$$\tan. 0 = \frac{\sin. 0}{\cos. 0} = \frac{0}{1} = 0$$

$$\tan. 90° = \frac{\sin. 90°}{\cos. 90°} = \frac{1}{0} = \textit{ad infinitum}, \text{ or } \infty.$$

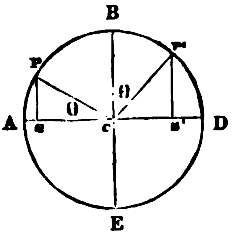
The truth of this last relation may be readily illustrated, by referring to the geometrical construction; when it will be seen that for the angle of 90° AT becomes parallel to CP; and therefore, the point T, in which the two lines meet, is at an infinite distance.

So, also, $\cot. 0 = \infty$

$$\cot. 90° = 0$$

and so for all the rest.

We shall next proceed to point out some important general relations, which exist between the trigonometrical functions of angles less than 90° and those of angles greater than 90°



Draw CP, making with CA any angle PCA which we may call θ; let fall PS perpendicular from P on CA. Draw CP′, making with BC the angle BCP′$=$PCA$=\theta$; and from P′ let fall P′S′ perpendicular on CD.

Then the angle P′CA$=90°+\theta$.

The two triangles PCS, P′CS′, have the side PC of the one equal to the side P′C of the other, also the angles at S and S′ right angles, and the angle CPS of the one equal to the angle P′CS′ of the other; therefore the two triangles are in every respect equal; and

$$PS=CS', \quad CS=P'S'.$$

Therefore, $$\frac{P'S'}{CA}=\frac{CS}{CA}$$

Or, $$\text{sin. } P'CA=\text{cos. } PCA,$$

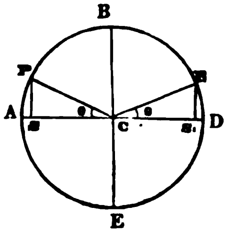
that is, $$\text{sin. }(90°+\theta)=\text{cos. }\theta.$$

Again,

$$\frac{CS'}{CA}=\frac{PS}{CA}$$

Or, $$-\text{cos. } P'CA=\text{sin. } PCA,$$

that is $$\text{cos. }(90°+\theta)=-\text{sin. }\theta.$$



As before, draw CP, making any angle θ with CA, and draw CP′, making with CD the angle P′CD, equal to θ.

Then the angle P′CA$=180°-\theta$.

The two triangles PCS, P′CS′ are manifestly in all respects equal; and

$$PS=P'S' \quad CS=CS'$$

Therefore, $$\frac{PS}{CA}=\frac{P'S'}{CA}$$

that is, $\text{sin. }\theta=\text{sin. }(180°-\theta)$

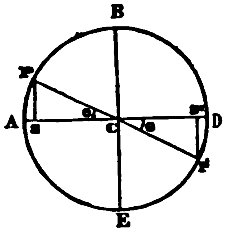
an important proposition which enunciated in words, is, *the sine of an angle is equal to the sine of its supplement.*

Again,

$$\frac{CS}{CA}=\frac{CS'}{CA}$$

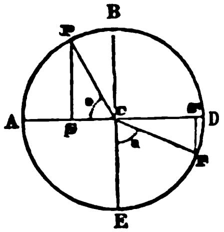
$$\text{cos. }\theta=-\text{cos. }(180°-\theta),$$

that is, *the cosine of an angle, and the cosine of its supplement are equal in absolute magnitude, but have opposite signs.*



If, as in the annexed figure, we draw CP′, making with CD an angle DCP′ equal to the angle θ, we shall find in like manner,

$$\sin.\ (180^\circ+\theta)=-\sin.\ \theta$$
$$\cos.\ (180^\circ+\theta)=\cos.\ (180^\circ-\theta)$$
$$=-\cos.\ \theta.$$



If we draw CP′, making with CE an angle ECP′$=\theta$, then

$$\sin.\ (270^\circ-\theta)=-\cos.\ \theta$$
$$\cos.\ (270^\circ-\theta)=-\sin.\ \theta$$

as is evident from Def. 8, and the rule for signs; and, in like manner, we may proceed for angles in the fourth quadrant. These relations being established between the sines and cosines, the corresponding relations between other trigonometrical functions may be deduced immediately from Table 1.

Thus,

$$\tan.\ (90^\circ+\theta)=\frac{\sin.\ (90^\circ+\theta)}{\cos.\ (90^\circ+\theta)}$$
$$=\frac{\cos.\ \theta}{-\sin.\ \theta}$$
$$=-\cot.\ \theta$$
$$\tan.\ (180^\circ-\theta)=\frac{\sin.\ (180^\circ-\theta)}{\cos.\ (180^\circ-\theta)}$$
$$=\frac{\sin.\ \theta}{-\cos.\ \theta}$$
$$=-\tan.\ \theta$$

and so for all the rest

The student may exercise himself by verifying such of the results in the following table as have not been formally demonstrated.

TABLE II.

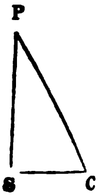
	=0	*sin. (180°+θ)	=— sin. θ
	=1	*cos. (180°+θ)	=— cos. θ
	=0	tan. (180°+θ)	=tan. θ
	= ∞	cot. (180°+θ)	=cot. θ
	=1	sec. (180°+θ)	=— sec. θ
)	= ∞	cosec. (180°+θ)	=— cosec. θ
°—θ)	=cos. θ	sin. (270°—θ)	=— cos. θ
)°—θ)	=sin. θ	cos. (270°—θ)	=— sin. θ
°—θ)	=cot. θ	tan. (270°—θ)	=cot. θ
°—θ)	=tan. θ	cot. (270°—θ)	=tan. θ
°—θ)	=cosec. θ	sec. (270°—θ)	=— cosec. θ
(90—θ)	=sec. θ	cosec. (270°—θ)	=— sec. θ
	=1	sin. 270°	=—1
°	=0	cos. 270°	=0
°	= ∞	tan. 270°	= ∞
°	=0	cot. 270°	=0
°	= ∞	sec. 270°	= ∞
90°	=1	cosec. 270°	=—1
°+θ)	=cos. θ	sin. (270°+θ)	=— cos. θ
)°+θ)	= — sin. θ	cos. (270°+θ)	=sin. θ
)°+θ)	= — cot. θ	tan. (270°+θ)	=— cot. θ
)°+θ)	= — tan. θ	cot. (270°+θ)	=— tan. θ
)°+θ)	= — cosec. θ	sec. (270°+θ)	=cosec. θ
(90°+θ)	=sec. θ	cosec. (270°+θ)	=— sec. θ
0°—θ)	=sin. θ	sin. (360°—θ)	=— sin. θ
0°—θ)	= — cos. θ	cos. (360°—θ)	=cos. θ.
30°—θ)	= — tan. θ	tan. (360°—θ)	=— tan. θ
30°—θ)	= — cot. θ	cot. (360°—θ)	=— cot. θ
30°—θ)	= — sec. θ	sec. (360°—θ)	=sec. θ
(180°—θ)	=cosec.	cosec. (360—θ)	=—cosec. θ
0°	=0	sin. 360°	=0
0°	= —1	cos. 360°	=1
0°	=0	tan. 360°	=0
0°	= — ∞	cot. 360°	= ∞
0°	= —1	sec. 360°	=1
180°	=0	cosec. 360°	= ∞

results in the above table which are most frequently
e marked with an asterisk, and ought to be committed
ory.

iave in the preceding pages confined ourselves to the
ration of angles not greater than 360°, but the student
l no difficulty in applying the above principles to an-
any magnitude whatsoever.

We shall conclude this introductory chapter, by demonstrating two propositions which are of the highest importance in our subsequent investigations. The first is,

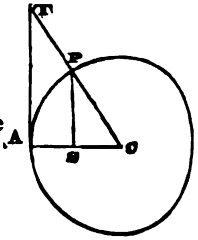
In any right-angled triangle, the ratio which the side opposite to one of the acute angles bears to the hypothenuse, is the sine of that angle; the ratio which the side adjacent to one of the acute angles bears to the hypothenuse, is the cosine of that angle; and the ratio which the side opposite to one of the acute angles bears to the side adjacent to that angle, is the tangent of that angle.



Let CSP be a plane triangle right-angled at S.

Then, $\frac{PS}{CP}=\sin. C, \frac{CS}{CP}=\cos. C, \frac{PS}{CS}=\tan. C,$

or $\frac{CS}{CP}=\sin. P. \frac{SP}{CP}=\cos. P. \frac{SC}{SP}=\tan. P.$



From C as a centre with the radius CP, describe a circle.

Produce CS, to meet the circumference in A.

From A draw AT a tangent to the circle at A.

Produce CP to meet AT in T.

Then, from Definitions (2) (3) (4),

$\frac{PS}{CP}=\sin. C, \frac{CS}{CP}=\cos. C, \frac{AT}{CP}=\tan. C,$

for $CP=AC$.

But the triangles TAC, PSC, are similar;

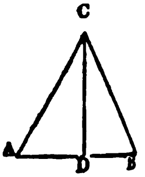
Therefore, $\frac{AT}{CP}=\frac{PS}{CS}=\tan. C,$

Cor.

$$PS=CP \sin. C=CP \cos. P$$
$$CS=CP \cos. C=CP \sin. P$$
$$PS=CS \tan. C=CS \cot. P$$

The second proposition is,

In any plane triangle, the ratio of any two of the sides is equal to the ratio of the sines of the angles opposite to them.



Let ABC be a plane triangle; it is required to prove that

$$\frac{CB}{CA}=\frac{\sin. A}{\sin. B}, \frac{CB}{BA}=\frac{\sin. A}{\sin. C}, \frac{CA}{BA}=\frac{\sin. B}{\sin. C}$$

From C let fall the perpendicular CD on AB.

Then, since CDB is a plane triangle right-angled at D, by last proposition

$$\text{CD}=\text{CB sin. B} \quad - - - - - - - - - - - - \quad (1)$$

Again, since CDA is a plane triangle right-angle at A,

$$\text{CD}=\text{CA sin. A} \quad - - - - - - - - - - \quad (2)$$

Equating these two values of CD,

$$\text{CB sin B}=\text{CA sin. A};$$

Therefore, $\frac{\text{CB}}{\text{CA}}=\frac{\text{sin. A.}}{\text{sin. B.}}$

In like manner, by dropping perpendiculars from B and A upon the side AC, CB we can prove,

$$\frac{\text{CB}}{\text{BA}}=\frac{\text{sin. A,}}{\text{sin. C,}} \quad \frac{\text{CA}}{\text{BA}}=\frac{\text{sin. B,}}{\text{sin. C.}}$$

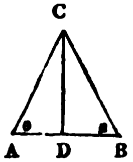
In treating of plane triangles, it is convenient to designate the three angles by the capital letters A, B, C, and the sides opposite to these angles by the corresponding small letters a, b, c. According to this notion, the last proposition will be

$$\frac{a}{b}=\frac{\text{sin. A}}{\text{sin. B}}, \quad \frac{a}{c}=\frac{\text{sin. A}}{\text{sin. C}}, \quad \frac{b}{c}=\frac{\text{sin. B}}{\text{sin, } c},$$

CHAPTER II.

GENERAL FORMULÆ.

Given the sines and cosines of two angles, to find the sine of their sum.



Let ABC be a plane triangle; from C let fall CD perpendicular on AB,

Let angle $\text{CAB}=\theta$,

and angle $\text{CBA}=\beta$,

$$\text{Then, AB} = \text{BD}+\text{DA}$$
$$= \text{BC cos. } \beta + \text{AC cos. } \theta,$$

because BDC and ADC are right-angled triangles,

Dividing each member of the equation by AB,

$$1 = \frac{\text{BC}}{\text{AB}} \text{cos. } \beta + \frac{\text{AC}}{\text{AB}} \text{cos. } \theta$$

$$= \frac{\text{sin. } \theta}{\text{sin.C}} \text{cos } \beta + \frac{\text{sin.} \beta}{\text{sin.C}} \text{cos. } \theta, \text{ by last Prop. in Chap. I.}$$

$$\therefore \text{sin. C} = \text{sin. } \theta \text{ cos. } \beta + \text{sin. } \beta \text{ cos. } \theta.$$

But, since ABC is a plane triangle, $\theta+\beta+\text{C} = 180°$

$$\therefore \text{C} = 180° - (\theta+\beta)$$
$$\text{sin. C} = \text{sin. } \{180° - (\theta+\beta)\}$$

$= \text{sin } (\theta+\beta,)$ because $180° - (\theta+\beta)$ is the supplement of $(\theta+\beta.)$

Hence, $\sin.(\beta+\theta) = \sin.\theta \cos\beta + \sin.\beta \cos.\theta$ - - - - (a)
That is, *the sine of the sum of two arcs or angles is equal the sine of the first multiplied by the cosine of the second, plus the sine of the second multiplied by the cosine of the first.*

This expression, from its great importance, is called the *fundamental formula* of Plane Trigonometry, and nearly the whole science may be derived from it.

Given the sines and cosines of two angles, to find the sine of their difference.

By formula (a).

$$\sin.(\theta+\beta) = \sin.\theta \cos.\beta + \sin.\beta \cos.\theta.$$

For θ substitute $180^\circ-\theta$, the above will become

$$\sin.\{180^\circ-(\theta-\beta)\} = \sin.(180^\circ-\theta)\cos.\beta + \sin.\beta \cos.(180^\circ-\theta)$$

But $\sin.\{180^\circ-(\theta-\beta)\} = \sin.(\theta-\beta)$ ∵ $180^\circ-(\theta-\beta)$ is the supplement of $(\theta-\beta.)$

And, $\sin.(180^\circ-\theta) = \sin.\theta,$

And, $\cos.(180^\circ-\theta) = -\cos.\theta$

Substitute, therefore, these values in the above expression, it becomes

$$\sin.(\theta-\beta) = \sin.\theta \cos.\beta - \sin.\beta \cos.\theta \quad \text{- - - -} \quad (b)$$

That is, *the sine of the difference of two arcs or angles, is equal the sine of the first × cosine of the second, — the sine of the second × cosine of the first.*

Given the sines and cosines of two angles, to find the cosine of their sum.

By formula (a)

$$\sin.(\theta+\beta) = \sin.\theta \cos.\beta + \sin.\beta \cos.\theta.$$

For θ substitute $90^\circ+\theta$, the above will become

$$\sin.\{90^\circ+(\theta+\beta)\} = \sin.(90^\circ+\theta)\cos.\beta + \sin.\beta \cos.(90^\circ+\theta)$$

But, $\sin.\{90^\circ+(\theta+\beta)\} = \cos.(\theta+\beta)$ by Table II.
And, $\sin.(90^\circ+\theta) = \cos.\theta.$
And, $\cos.(90^\circ+\theta) = -\sin.\theta.$

Substituting, therefore, these values in the above expression, it becomes, $\cos.(\theta+\beta) = \cos.\theta \cos.\beta - \sin.\theta \sin.\beta$ - - (c)

That is, *the cosine of the sum of two arcs or angles, is equal to the cosine of the first multiplied by the cosine of the second, minus the sine of the first multiplied by the sine of the second.*

Given the sines and cosines of two angles, to find the cosine of their difference.

By formula (a:)

$$\sin.\ (\theta+\beta) \quad = \sin.\ \theta\ \cos.\ \beta + \sin.\ \beta\ \cos.\ \theta,$$

For θ substitute $90°-\theta$, the above will become

$$\sin.\ \{90°-(\theta-\beta)\} = \sin.\ (90°-\theta)\ \cos.\ \beta + \sin.\ \beta\ \cos.\ (90°-\theta),$$

But, $\sin.\ \{90°-(\theta-\beta)\} = \cos.\ (\theta-\beta)$, By Table II.

$\sin.\ (90°-\theta) \quad = \cos.\ \theta$ ....

$\cos.\ (90°-\theta) \quad = \sin.\ \theta$ ....

Substituting, therefore, these values in the above expression, it becomes

$$\cos.\ (\theta-\beta) \quad = \cos.\ \theta\ \cos.\ \beta + \sin.\ \theta\ \sin.\ \beta \quad - \ - \quad (d)$$

That is, *the cosine of the difference of two arcs or angles, is equal to the cosine of the first multiplied by the cosine of the second, plus sine of the first into the sine of the second.*

Given the tangents of two angles, to find the tangent of their sum.

By Table I.:

$$\tan.\ (\theta+\beta) \quad = \frac{\sin.\ (\theta+\beta)}{\cos.\ (\theta+\beta)}$$

$$= \frac{\sin.\ \theta\ \cos.\ \beta + \sin.\ \beta\ \cos.\ \theta}{\cos.\ \theta\ \cos.\ \beta - \sin.\ \theta\ \sin.\ \beta} \text{ by } (a) \text{ and } (c)$$

Dividing both numerator and denominator of fraction by $\cos.\ \theta\ \cos\ \beta$:

$$= \frac{\dfrac{\sin.\ \theta\ \cos.\ \beta}{\cos.\ \theta\ \cos.\ \beta} + \dfrac{\sin.\ \beta\ \cos.\ \theta}{\cos.\ \beta\ \cos.\ \theta}}{1 - \dfrac{\sin.\ \theta\ \sin.\ \beta}{\cos.\ \theta\ \cos.\ \beta}}$$

Simplifying,

$$= \frac{\tan.\ \theta + \tan.\ \beta}{1 - \tan.\ \theta\ \tan.\ \beta} \quad - \ - \ - \ - \ - \ - \ - \ - \quad (e)$$

That is, *the tangent of the sum of two arcs or angles, is equal to the sum of the tangents of the two arcs, divided by* 1 *minus the product of the two tangents.*

Given the tangents of two angles, to find the tangent of their difference.

By Table I.:

$$\tan.\ (\theta-\beta) = \frac{\sin.\ (\theta-\beta)}{\cos.\ (\theta-\beta)}$$

$$= \frac{\sin.\ \theta\ \cos.\ \beta - \sin.\ \beta\ \cos.\ \theta}{\cos.\ \theta\ \cos.\ \beta + \sin.\ \theta\ \sin.\ \beta.} \text{ by } (b) \text{ and } (d)$$

Dividing both numerator and denominator by cos. θ cos. β:

$$=\frac{\frac{\sin.\ \theta\ \cos.\ \beta}{\cos.\ \theta\ \cos.\ \beta}-\frac{\sin.\ \beta\ \cos.\ \theta}{\cos.\ \theta\ \cos.\ \beta}}{1+\frac{\sin.\ \theta\ \sin.\ \beta}{\cos.\ \theta\ \cos.\ \beta}}$$

Simplifying, $$=\frac{\tan.\ \theta-\tan.\ \beta}{1+\tan.\ \theta\ \tan.\ \beta}\ \text{- - - - - - -}\ (f)$$

Hence, *the tangent of the difference of two arcs or angles, is equal to the difference of the tangents of the two arcs, divided by* 1 *plus the product of the two tangents.*

The student will have no difficulty in deducing the following:

$$\cot.\ (\theta+\beta)=\frac{\cot.\ \theta\ \cot.\ \beta-1}{\cot.\ \beta+\cot.\ \theta}$$

$$\cot.\ (\theta-\beta)=\frac{\cot.\ \theta\ \cot.\ \beta+1}{\cot.\ \beta-\cot.\ \theta}$$

$$\sec.\ (\theta+\beta)=\frac{\sec.\ \theta\ \sec.\ \beta\ \text{cosec.}\ \theta\ \text{cosec.}\ \beta}{\text{cosec.}\ \theta\ \text{cosec.}\ \beta-\sec.\ \theta\ \sec.\ \beta}$$

$$\sec.\ (\theta-\beta)=\frac{\sec.\ \theta\ \sec.\ \beta\ \text{cosec.}\ \theta\ \text{cosec.}\ \beta}{\text{cosec.}\ \theta\ \text{cosec.}\ \beta+\sec.\ \theta\ \sec.\ \beta}$$

$$\text{cosec.}\ (\theta+\beta)=\frac{\sec.\ \theta\ \sec.\ \beta\ \text{cosec.}\ \theta\ \text{cosec.}\ \beta}{\sec.\ \theta\ \text{cosec.}\ \beta+\sec.\ \beta\ \text{cosec.}\ \theta}$$

$$\text{cosec.}\ (\theta-\beta)=\frac{\sec.\ \theta\ \sec.\ \beta\ \text{cosec.}\ \theta\ \text{cosec.}\ \beta}{\sec.\ \theta\ \text{cosec.}\ \beta-\sec.\ \beta\ \text{cosec.}\ \theta}$$

To determine the sine of twice a given angle.

By formula (*a*):

$$\sin.\ (\theta+\beta)=\sin.\ \theta\ \cos.\ \beta+\sin.\ \beta\ \cos.\ \theta.$$

Let $\theta=\beta$, then the above becomes

$$\sin.\ 2\ \theta=\sin.\ \theta\ \cos.\ \theta+\sin.\ \theta\ \cos.\ \theta$$

$$=2\ \sin.\ \theta\ \cos.\ \theta\ \text{- - - - - - -}\ (g1)$$

That is, *the sine of twice a given angle, is equal to twice the sine of the given angle multiplied by its cosine.*

In the last formula, for θ substitute $\frac{\theta}{2}$; then,

$$\sin.\ 2\times\frac{\theta}{2}=2\ \sin.\ \frac{\theta}{2}\ \cos.\ \frac{\theta}{2},$$

$$\text{Or, }\sin.\ \theta=2\ \sin.\ \frac{\theta}{2}\ \cos.\ \frac{\theta}{2}\ \text{- - - - - - -}\ (g2)$$

To determine the cosine of twice a given angle.

By formula (*c*):

$$\cos.(\theta+\beta)=\cos.\theta\cos.\beta-\sin.\theta\sin.\beta$$

Let $\theta=\beta$ then the above becomes

$$\cos.2\theta=\cos.^2\theta-\sin.^2\theta \quad \text{- - - - - - -} \quad (h1)$$

By table I. $\sin.^2\theta=1-\cos.^2\theta$; substituting this for $\sin.^2\theta$:

$$\cos.2\theta=2\cos.^2\theta-1 \quad \text{- - - - - - - - -} \quad (h2)$$

Again, since $\cos.^2\theta=1-\sin.^2\theta$, substitute this for $\cos.^2\theta$:

$$\cos.2\theta=1-2\sin.^2\theta \quad \text{- - - - - - - - -} \quad (h3)$$

Hence, *the cosine of twice a given arc or angle, is equal to* 1 *minus twice the square of the sine of the given angle.*

To determine the tangent of twice a given angle.

By formula (*e*):

$$\tan.(\theta+\beta)=\frac{\tan.\theta+\tan.\beta}{1-\tan.\theta\tan.\beta}$$

Let $\theta=\beta$, the above becomes

$$\tan.2\theta=\frac{2\tan.\theta}{1-\tan.^2\theta} \quad \text{- - - - - - - - - - -} \quad (i)$$

The tangent of twice a given arc, is equal to twice the tangent of the given arc, divided by 1 *minus the square of the tangent.*

The student will easily deduce the following:

$$\cot.2\theta=\frac{\cot.^2\theta-1}{2\cot.\theta}=\frac{\cot.\theta-\tan.\theta}{2}$$

$$\sec.2\theta=\frac{\sec.^2\theta\,\text{cosec.}^2\theta}{\text{cosec.}^2\theta-\sec^2\theta}$$

$$\text{cosec.}2\theta=\frac{\sec.^2\theta\,\text{cosec.}^2\theta}{2\sec.\theta\,\text{cosec.}\theta}=\frac{\sec.\theta\,\text{cosec.}\theta}{2}$$

To determine the sine of half a given angle.

By formula (*h3*):

$$\cos.2\theta=1-2\sin.^2\theta$$

For θ substitute $\frac{\theta}{2}$; the above becomes,.

$$\cos.2\frac{\theta}{2}=1-2\sin.^2\frac{\theta}{2}$$

Or, $\quad \cos.\theta=1-2\sin.^2\frac{\theta}{2}$

$$\therefore 2\sin.^2\frac{\theta}{2}=1-\cos.\theta$$

$$\sin.\frac{\theta}{2}=\sqrt{\frac{1-\cos.\theta}{2}} \quad \text{- - - - - - -} \quad (j)$$

That is, *the sine of half a given angle or arc is equal to the square root of* 1 *minus the cosine of the arc divided by two.*

To determine the cosine of half a given angle.

By formal ($h2$):

$$\cos.\,2\theta = 2\cos.^2\theta - 1$$

For θ substitute $\frac{\theta}{2}$; the above becomes,

$$\cos.\,2\frac{\theta}{2} = 2\cos.^2\frac{\theta}{2} - 1$$

Or,

$$\cos.\,\theta = 2\cos.^2\frac{\theta}{2} - 1$$

$$\therefore \quad 2\cos.^2\frac{\theta}{2} = 1 + \cos.\,\theta$$

$$\cos.\,\frac{\theta}{2} = \sqrt{\frac{1+\cos.\,\theta}{2}} \quad \ldots\ldots \quad (k)$$

To determine the tangent of half a given angle.

Divide formula (j) by (k):

$$\frac{\sin.\,\frac{\theta}{2}}{\cos.\,\frac{\theta}{2}} = \sqrt{\frac{1-\cos.\,\theta}{1+\cos.\,\theta}}$$

Or,

$$\tan.\,\frac{\theta}{2} = \sqrt{\frac{1-\cos.\,\theta}{1+\cos.\,\theta}} \quad \ldots\ldots \quad (l1)$$

Multiply both numerator and denominator by $\sqrt{1-\cos.\,\theta}$; the above becomes,

$$\tan.\,\frac{\theta}{2} = \frac{1-\cos.\,\theta}{\sin.\,\theta} \quad \ldots\ldots \quad (l2)$$

Multiply both numerator and denominator of (l 1) by $\sqrt{1+\cos.\,\theta}$; we have,

$$\tan.\,\frac{\theta}{2} = \frac{\sin.\,\theta}{1+\cos.\,\theta} \quad \ldots\ldots \quad (l3)$$

The student will easily deduce the following:

$$\cot.\,\frac{\theta}{2} = \sqrt{\frac{1+\cos.\,\theta}{1-\cos.\,\theta}}$$

$$= \frac{1+\cos.\,\theta}{\sin.\,\theta}$$

$$= \frac{\sin.\,\theta.}{1-\cos.\,\theta}$$

$$\text{sec.}\,\frac{\theta}{2} = \sqrt{\frac{2\,\text{sec.}\,\theta}{\text{sec.}\,\theta+1}}$$

$$\text{cosec.}\,\frac{\theta}{2} = \sqrt{\frac{2\,\text{sec.}\,\theta}{\text{sec.}\,\theta-1}}$$

To determine the sine of $(n+1)\,\theta$, in terms of $n\,\theta$, $(n-1)\,\theta$ and θ.

By formula (*a*) and (*b*):

$$\text{sin.}\,(\beta+\theta)=\text{sin.}\,\beta\,\text{cos.}\,\theta+\text{sin.}\,\theta\,\text{cos.}\,\beta$$
$$\text{sin.}\,(\beta-\theta)=\text{sin.}\,\beta\,\text{cos.}\,\theta-\text{sin.}\,\theta\,\text{cos.}\,\beta$$

Add these two equations,

$$\text{sin.}\,(\beta+\theta)+\text{sin.}\,(\beta-\theta)=2\,\text{sin.}\,\beta\,\text{cos.}\,\theta$$

Subtract $\text{sin.}\,(\beta-\theta)$ from each member,

$$\text{sin.}\,(\beta+\theta)=2\,\text{sin.}\,\beta\,\text{cos.}\,\theta-\text{sin.}\,(\beta-\theta)$$

Let $\beta=n\,\theta$, the above becomes

$$\text{sin.}\,(n+1)\theta=2\,\text{sin.}\,n\,\theta\,\text{cos.}\,\theta-\text{sin.}\,(n-1)\,\theta\,..\,(m)$$

In the above formula, let $n=1$; $\therefore n+1=2,\ n-1=0$

$$\therefore\ \text{sin.}\,2\,\theta=2\,\text{sin.}\,\theta\,\text{cos.}\,\theta-\text{sin.}\,0$$
$$=2\,\text{sin.}\,\theta\,\text{cos.}\,\theta,$$ the same result as in (*g*).

Let $n=2$; $\therefore n+1=3,\ n-1=1$;

$$\therefore\ \text{sin.}\,3\,\theta=2\,\text{sin.}\,2\,\theta\,\text{cos.}\,\theta-\text{sin.}\,\theta$$
$$=2\times2\,\text{sin.}\,\theta\,\text{cos.}\,\theta\times\text{cos.}\,\theta-\text{sin.}\,\theta$$
$$=4\,\text{sin.}\,\theta\,\text{cos.}^2\,\theta-\text{sin.}\,\theta$$
$$=4\,\text{sin.}\,\theta\,(1-\text{sin.}^2\,\theta)-\text{sin.}\,\theta$$
$$=3\,\text{sin.}\,\theta-4\,\text{sin.}^3\,\theta\ \text{------}\ (n)$$

Let $n=3$; $\therefore n\times1=4,\ n-1=2$;

$\therefore$ By formula (*m*):

$$\text{sin.}\,4\,\theta=2\times\text{sin.}\,3\,\theta\times\text{cos.}\,\theta-\text{sin.}\,2\,\theta$$
$$=2\,(3\,\text{sin.}\,\theta-4\,\text{sin.}^3\,\theta)\,\text{cos.}\,\theta-2\,\text{sin.}\,\theta\,\text{cos.}\,\theta$$
$$=(8\,\text{cos.}^3\,\theta-4\,\text{cos.}\,\theta)\,\text{sin.}\,\theta$$

It is manifest that, by continuing the same process, we may find in succession, $\text{sin.}\,5\,\theta$, $\text{sin.}\,6\,\theta$, - - - &c.

To determine the cosine of $(n+1)\,\theta$, in terms of $n\,\theta$, $(n-1)\,\theta$, and θ.

By formula (*c*) and (*d*):

$$\text{cos.}\,(\beta+\theta)=\text{cos.}\,\beta\,\text{cos.}\,\theta-\text{sin.}\,\beta\,\text{sin.}\,\theta$$
$$\text{cos.}\,(\beta-\theta)=\text{cos.}\,\beta\,\text{cos.}\,\theta+\text{sin.}\,\beta\,\text{sin.}\,\theta$$

Add these two equations,

$$\text{cos.}\,(\beta+\theta)+\text{cos.}\,(\beta-\theta)=2\,\text{cos.}\,\beta\,\text{cos.}\,\theta$$

Subtract $\text{cos.}\,(\beta-\theta)$ from each member,

$$\text{cos.}\ (\beta+\theta)=2\ \text{cos.}\ \beta\ \text{cos.}\ \theta-\text{cos.}\ (\beta-\theta)$$

Let $\beta=n\ \theta$, the above becomes

$$\text{cos.}\ (n+1)\ \theta=2\ \text{cos.}\ n\ \theta\ \text{cos.}\ \theta-\text{cos.}\ (n-1)\ \theta\ \cdot\ \cdot\ (o)$$

In the above formula, let $n=1$; $\therefore\ n+1=2,\ n-1=0$;

$$\begin{aligned}\text{Then,}\quad \text{cos.}\ 2\ \theta&=2\ \text{cos.}\ \theta\ \text{cos.}\ \theta-\text{cos}\ 0\\&=2\ \text{cos.}^2\ \theta-1,\ \text{the same result as in } (h2).\end{aligned}$$

Let $n=2$, $\therefore\ n+1=3,\ n-1=1$;

$$\begin{aligned}\therefore\quad \text{cos.}\ 3\ \theta&=2\ \text{cos.}\ 2\ \theta\ \text{cos.}\ \theta-\text{cos.}\ \theta\\&=2\ (2\ \text{cos.}^2\ \theta-1)\ \text{cos.}\ \theta-\text{cos.}\ \theta\\&=4\ \text{cos.}^3\ \theta-3\ \text{cos.}\ \theta\ \ -\ -\ -\ -\ \ (p)\end{aligned}$$

Let $n=3$; $\therefore\ n+1=4,\ n-1=2$;

$$\begin{aligned}\therefore\quad \text{cos.}\ 4\ \theta&=2\ \text{cos.}\ 3\ \theta\ \text{cos.}\ \theta-\text{cos.}\ 2\ \theta\\&=2\ (4\ \text{cos.}^3\ \theta-3\ \text{cos.}\ \theta)\ \text{cos.}\ \theta-(2\ \text{cos.}^2\ \theta-1)\\&=8\ \text{cos.}^4\ \theta-8\ \text{cos.}^2\ \theta+1\end{aligned}$$

It is manifest that, by continuing the same process, we may find, in succession, cos. 5 θ, cos. 6 θ, - - - &c.

By adding and subtracting (a) and (b), and by adding and subtracting (c) and (d), we obtain the following formulæ, which are of considerable utility.

$$\left.\begin{aligned}\text{sin.}\ (\theta+\beta)+\text{sin.}\ (\theta-\beta)&=\ \ 2\ \text{sin.}\ \theta\ \text{cos.}\ \beta\\\text{sin.}\ (\theta+\beta)-\text{sin.}\ (\theta-\beta)&=\ \ 2\ \text{sin.}\ \beta\ \text{cos.}\ \theta\\\text{cos.}\ (\theta+\beta)+\text{cos.}\ (\theta-\beta)&=\ \ 2\ \text{cos.}\ \theta\ \text{cos.}\ \beta\\\text{cos.}\ (\theta+\beta)-\text{cos.}\ (\theta-\beta)&=-2\ \text{sin.}\ \theta\ \text{sin.}\ \beta\end{aligned}\right\}\ -\ -\ -\ -\ (q)$$

Any angle θ may, by a simple artifice, be put under the form,

$$\theta=\frac{\theta+\beta}{2}+\frac{\theta-\beta}{2}$$

And, in like manner,

$$\beta=\frac{\theta+\beta}{2}-\frac{\theta-\beta}{2}$$

$$\therefore\ \text{sin.}\ \theta=\text{sin.}\left\{\frac{\theta+\beta}{2}+\frac{\theta-\beta}{2}\right\}$$

$$=\text{sin.}\ \frac{\theta+\beta}{2}\ \text{cos.}\ \frac{\theta-\beta}{2}+\text{sin.}\ \frac{\theta-\beta}{2}\ \text{cos.}\ \frac{\theta+\beta}{2}\ -\ -\ -\ (1)$$

$$\text{sin.}\ \beta=\text{sin.}\left\{\frac{\theta+\beta}{2}-\frac{\theta-\beta}{2}\right\}$$

$$=\text{sin.}\ \frac{\theta+\beta}{2}\ \text{cos.}\ \frac{\theta-\beta}{2}-\text{sin.}\ \frac{\theta-\beta}{2}\ \text{cos.}\ \frac{\theta+\beta}{2}\ -\ -\ -\ (2)$$

$$\text{cos.}\ \theta=\text{cos.}\left\{\frac{\theta+\beta}{2}+\frac{\theta-\beta}{2}\right\}$$

$$=\text{cos.}\ \frac{\theta+\beta}{2}\ \text{cos.}\ \frac{\theta-\beta}{2}-\text{sin.}\ \frac{\theta+\beta}{2}\ \text{sin.}\ \frac{\theta-\beta}{2}\ -\ -\ -\ (3)$$

$$\text{cos.}\,\beta=\text{cos.}\left\{\frac{\theta+\beta}{2}-\frac{\theta-\beta}{2}\right\}$$

$$=\text{cos.}\,\frac{\theta+\beta}{2}\,\text{cos.}\,\frac{\theta-\beta}{2}+\text{sin.}\,\frac{\theta+\beta}{2}\,\text{sin.}\,\frac{\theta-\beta}{2}\quad\text{- - -}\quad(4)$$

Add together (1) and (2):

$$\text{sin.}\,\theta+\text{sin.}\,\beta=2\,\text{sin.}\,\frac{\theta+\beta}{2}\,\text{cos.}\,\frac{\theta-\beta}{2}\quad\text{- - - - - - -}\quad(r)$$

Subtract (2) from (1),

$$\text{sin.}\,\theta-\text{sin.}\,\beta=2\,\text{sin.}\,\frac{\theta-\beta}{2}\,\text{cos.}\,\frac{\theta+\beta}{2}\quad\text{- - - - - - -}\quad(s)$$

Add together (3) and (4),

$$\text{cos.}\,\theta+\text{cos.}\,\beta=2\,\text{cos.}\,\frac{\theta+\beta}{2}\,\text{cos.}\,\frac{\theta-\beta}{2}\quad\text{- - - - - - -}\quad(t)$$

Subtract (4) from (3),

$$\text{cos.}\,\theta-\text{cos.}\,\beta=-2\,\text{sin.}\,\frac{\theta+\beta}{2}\,\text{sin.}\,\frac{\theta-\beta}{2}\quad\text{- - - - - -}\quad(v)$$

These formulæ, which are of the greatest importance, might have been immediately deduced from the group (q), by changing $\theta+\beta$ into θ, $\theta-\beta$ into β, θ into $\frac{\theta+\beta}{2}$, β into $\frac{\theta-\beta}{2}$.

Divide (r) by (s):

$$\frac{\text{sin.}\,\theta+\text{sin.}\,\beta}{\text{sin.}\,\theta-\text{sin.}\,\beta}=\frac{2\,\text{sin.}\,\frac{\theta+\beta}{2}\,\text{cos.}\,\frac{\theta-\beta}{2}}{2\,\text{sin.}\,\frac{\theta-\beta}{2}\,\text{cos.}\,\frac{\theta+\beta}{2}}$$

$$=\frac{\text{tan.}\,\frac{\theta+\beta}{2}}{\text{tan.}\,\frac{\theta-\beta}{2}}=\frac{\text{tan.}\,\frac{1}{2}\,(\theta+\beta)}{\text{tan.}\,\frac{1}{2}\,(\theta-\beta)}\quad\text{- - - -}\quad(w)$$

Multiply (a) by (b); then,

$$\text{sin.}\,(\theta+\beta)\,\text{sin.}\,(\theta-\beta)=\text{sin.}^2\,\theta\,\text{cos.}^2\,\beta-\text{sin.}^2\,\beta\,\text{cos.}^2\,\theta$$

$$=\text{sin.}^2\,\theta-\text{sin.}^2\,\beta\quad\text{- - - - - -}\quad(x)$$

Multiply (c) by (d); then,

$$\text{cos.}\,(\theta+\beta)\,\text{cos.}\,(\theta-\beta)=\text{cos.}^2\,\theta\,\text{cos.}^2\,\beta-\text{sin.}^2\,\theta\,\text{sin.}^2\,\beta$$

$$=\text{cos.}^2\,\theta-\text{sin.}^2\,\beta\quad\text{- - - - -}\quad(y)$$

We will now investigate a few properties where *more* than two arcs or angles are concerned, and which may be of use in the subsequent part of this work.

Let θ, β, γ, be any three arcs or angles.

Then,

$$\sin.\ (\beta+\gamma)=\frac{\sin.\ \theta\ \sin.\ \gamma+\sin.\ \beta\ \sin.\ (\theta+\beta+\gamma)}{\sin.\ (\theta+\beta)}$$

For by formula (*a*)

$\sin.\ (\theta+\beta+\gamma)=\sin.\ \theta\ \cos.\ (\beta+\gamma)+\cos.\ \theta\ \sin.\ (\beta+\gamma)$ which [putting $\cos.\ \beta\ \cos.\ \gamma - \sin.\ \beta\ \sin.\ \gamma$, for $\cos.\ (\beta+\gamma)$], is $=\sin.\ \theta\ \cos.\ \beta\ \cos.\ \gamma - \sin.\ \theta\ \sin.\ \beta\ \sin.\ \gamma+\cos.\ \theta\ \sin.\ (\beta+\gamma)$; and, multiplying by $\sin.\ \beta$, and adding $\sin.\ \theta\ \sin\ \gamma$, there results $\sin.\ \theta\ \sin.\ \gamma+\sin.\ \beta\ \sin.\ (\theta+\beta+\gamma)=\sin.\ \theta\ \cos.\ \beta\ \cos.\ \gamma\ \sin.\ \beta+\sin.\ \theta\ \sin.\ \gamma\ \cos.^2\beta+\cos.\ \theta\ \sin.\ \beta\ \sin.\ (\beta+\gamma)=\sin.\ \theta\ \cos.\ \beta\ (\sin.\ \beta\ \cos.\ \gamma+\cos.\ \beta\ \sin.\ \gamma)+\cos.\ \theta\ \sin.\ \beta\ \sin.\ (\beta+\gamma)=(\sin.\ \theta\ \cos.\ \beta+\cos.\ \theta\ \sin.\ \beta)\ \sin.\ (\beta+\gamma)=\sin.\ (\theta+\beta)\ \sin.\ (\beta+\gamma)$.

Hence, dividing by $\sin.\ (\theta+\beta)$, we have,

$$\sin.\ (\beta+\gamma)=\frac{\sin.\ \theta\ \sin.\ \gamma+\sin.\ \beta\ \sin.(\theta+\beta+\gamma)}{\sin.\ (\theta+\beta)}$$

In a similar manner it may be shown, that

$$\sin.\ (\beta-\gamma)=\frac{\sin.\ \theta\ \sin.\ \gamma-\sin.\ \beta\ \sin.\ (\theta-\beta+\gamma)}{\sin.\ (\theta-\beta)}$$

If θ, β, γ, δ, represents any four arcs or angles, then writing $\gamma+\delta$ for γ in the preceding investigation, there will result

$$\sin.\ (\beta+\gamma+\delta)=\frac{\sin.\ \theta\ \sin.\ (\gamma+\delta)+\sin.\ \beta\ \sin.\ (\theta+\beta+\gamma+\delta)}{\sin.\ (\theta+\beta)}$$

A like process for five arcs or angles will give

$$\sin.\ (\beta+\gamma+\delta+\zeta)=\frac{\sin.\theta\ \sin.(\gamma+\delta+\zeta)+\sin.\beta\sin.(\theta+\beta+\gamma+\delta+\zeta)}{\sin.\ (\theta+\beta)}$$

And for any number θ, β, γ, &c. . . . λ

$$\sin.(\beta+\gamma+..\lambda)=\frac{\sin.\theta\ \sin.(\gamma+\delta+...\lambda)+\sin.\beta\ \sin.(\theta+\beta+\gamma+...\lambda)}{\sin.\ (\theta+\beta)}$$

Taking again the three θ, β, γ, we have

$$\sin.\ (\beta-\gamma)=\sin.\ \beta\ \cos.\ \gamma-\sin.\ \gamma\ \cos.\ \beta$$
$$\sin.\ (\gamma-\theta)=\sin.\ \gamma\ \cos.\ \theta-\sin.\ \theta\ \cos.\ \gamma$$
$$\sin.\ (\theta-\beta)=\sin.\ \theta\ \cos.\ \beta-\sin.\ \beta\ \cos.\ \theta$$

Multiplying the first of these equations by $\sin.\ \theta$, second by $\sin.\ \beta$, third by $\sin.\ \gamma$; then adding together the equations thus transformed ; there will result,

$$\sin.\ \theta\ \sin\ (\beta-\gamma)+\sin.\ \beta\ \sin.\ (\gamma-\theta)+\sin.\ \gamma\ \sin.\ (\theta-\beta)=0$$
$$\sin.\ \theta\ \sin.\ (\beta-\gamma)+\cos.\ \beta\ \sin.\ (\gamma-\theta)+\cos.\ \gamma\ \sin.\ (\theta-\beta)=0$$

These two equations resulting from any three angles whatever may evidently be applied to the three angles of any triangles.

Let the series of arcs or angles $\theta, \beta, \gamma, \delta$ - - - - - - λ, be contemplated, then we have formula (x)

$$\begin{aligned}
&\sin.\ (\theta+\beta)\ \sin.\ (\theta-\beta)=\sin.^2\ \theta-\sin.^2\ \beta\\
&\sin.\ (\beta+\gamma)\ \sin.\ (\beta-\gamma)=\sin.^2\ \beta-\sin.^2\ \gamma.\\
&\sin.\ (\gamma+\delta)\ \sin.\ (\gamma-\delta)=\sin.^2\ \gamma-\sin.^2\ \delta\\
&\qquad\qquad \&c.\ldots\ldots = \&c.\ldots\ldots\\
&\sin.\ (\lambda+\theta)\ \sin.\ (\lambda-\theta)=\sin.^2\ \lambda-\sin.^2\ \theta
\end{aligned}$$

Adding these equations together, we have
$\sin.\ (\theta+\beta)\ \sin.\ (\theta-\beta)+\sin.\ (\beta+\gamma)\ \sin.\ (\beta-\gamma)+\sin.\ (\gamma+\delta)$ $\sin.\ (\gamma-\delta)+\ldots\ldots\ldots\sin.\ (\lambda+\theta)\ \sin.\ (\lambda-\theta)=0$

Proceeding in a similar manner with the $\sin.\ (\theta-\beta)$, $\cos.\ (\theta+\beta)$, $\sin.\ (\beta-\gamma)$, $\cos.\ (\beta+\gamma)$, &c., there will at length be obtained $\cos.(\theta+\beta)\ \sin.(\theta-\beta)+\cos.(\beta+\gamma)\ \sin.(\beta-\gamma)+\ldots$ $\cos.\ (\lambda+\theta)\ \sin.\ (\lambda-\theta)=0$

If the arcs θ, β, γ - - - - λ form an arithmetical progression of which the first term is 0 the ratio ϱ and the last term λ, any number n of circumferences, then will $\beta-\theta=\varrho$, $\gamma-\beta=\varrho$, &c., $\theta+\beta=\varrho$, $\beta+\gamma=3\varrho$, &c.; dividing the whole by the $\sin.\ \varrho$, the preceding equations will become

$$\left.\begin{aligned}
&\sin.\ \varrho+\sin.\ 3\varrho+\sin.\ 5\varrho+\&c.=0\\
&\cos.\ \varrho+\cos.\ 3\varrho+\cos.\ 5\varrho+\&c.=0
\end{aligned}\right\} \text{- - - - -}\ (z)$$

If ξ were equal 2ϱ, these equations would become

$$\begin{aligned}
&\sin.\ \varrho+\sin.\ (\varrho+\xi)+\sin.\ (\varrho+2\xi)+\sin.\ (\varrho+3\xi)+\&c.=0\\
&\cos.\ \varrho+\cos.\ (\varrho+\xi)+\cos.\ (\varrho+2\xi)+\cos.\ (\varrho+3\xi)+\&c.=0
\end{aligned}$$

The last equations, however, only show the sums of the sines and cosines of arcs or angles in arithmetical progression when the common difference is to the first term in the ratio of 2 to 1. To find a general expression for an infinite series of this kind, let

$S+\sin.\ \theta+\sin.\ (\theta+\beta)+\sin.\ (\theta+2\beta)+\sin.\ (\theta+3\beta)+$ - - - - &c.
Then since this series is a recurring series whose scale of relations is $2\cos.\ \beta-1$, it will arise from the development of a fraction whose denominator $1-2\chi\cos.\ \beta+\chi^2$ making $\chi=1$.

Now this fraction will be,

$$=\frac{\sin.\ \theta+\chi(\sin.(\theta+\beta)-2\sin.\ \theta\ \cos.\ \beta}{1-2\chi\cos.\ \beta+\chi^2}$$

Therefore, when $\chi=1$, we have,

$$S=\frac{\sin.\ \theta+\sin.\ (\theta+\beta)-2\sin.\ \theta\cos.\ \beta}{2-2\cos.\ \beta};$$

And this because, $2\sin.\ \theta\ \cos.\ \beta=\sin.\ (\theta+\beta)+\sin.\ (\theta-\beta)$

$$=\frac{\sin.\ \theta-\sin.\ (\theta-\beta)}{2\ (1-\cos.\ \beta)}\ \text{by formula } (q).$$

Now putting θ' for $(\theta+\beta)$ and β' for $(\theta-\beta)$ we have from formula (s):

$$\sin.\,\theta' - \sin.\,\beta' = 2\cos.\tfrac{1}{2}\,(\theta'+\beta')\,\sin.\tfrac{1}{2}\,(\theta'-\beta')$$

Hence, it follows that,

$$\sin.\,\theta - \sin.\,(\theta-\beta) = 2\cos.\,(\theta-\tfrac{1}{2}\beta)\,\sin.\tfrac{1}{2}\beta$$

Besides which we have,

$$1-\cos.\,\beta = 2\sin.^2\tfrac{1}{2}\beta$$

Consequently the preceding expression becomes,

$S = \sin.\,\theta + \sin.\,(\theta+\beta) + \sin.\,(\theta+2\beta) + \sin.\,(\theta+3\beta)$ &c. *ad infinitum*,

$$= \frac{\cos.\,(\theta-\frac{1}{2}\beta)}{2\sin.\frac{1}{2}\beta} \quad - - - - - - - - \quad (z2)$$

To find the sum of $n+1$ terms of this series, we have simply to consider that the sum of the terms past the $(n+1)$th, that is the sum of

$\sin.\,(\theta+(n+1)\beta) + \sin.\,(\theta+(n+2)\beta) + \sin.(\theta+(n+3)\beta) +$ &c. *ad infinitum*, is by the preceeding theorem,

$$= \frac{\cos.\,(\theta+(n+\frac{1}{2})\beta)}{2\sin.\frac{1}{2}\beta}$$

Deducting this from the former expression, there will remain

$$\sin.\,\theta + \sin.\,(\theta+\beta) + \sin.\,(\theta+2\beta) + \sin.\,(\theta+3\beta) + - - - - - - - - \sin.\,(\theta+n\beta) = \frac{\cos.\,(\theta-\frac{1}{2}\beta) - \cos.\,(\theta+(n+\frac{1}{2})\beta)}{2\sin.\,\frac{1}{2}\beta}$$

$$= \frac{\sin.\,(\theta+\frac{1}{2}\,n\beta)\,\sin.\frac{1}{2}\,(n+1)\beta}{\sin.\,\frac{1}{2}\beta} \quad - \quad (z3)$$

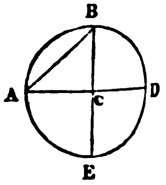
By like means it may be found, that the sum of the cosines of arcs or angles in arithmetical progression, is $\cos.\,\theta + \cos.\,(\theta+\beta) + \cos.\,(\theta+2\beta) + \cos.\,(\theta+3\beta) +$ &c. *ad infinitum*,

$$= -\frac{\sin.(\theta+\frac{1}{2}\beta)}{2\sin.\,\frac{1}{2}\beta} \quad - - - \quad (z4)$$

Also,

$$\cos.\,\theta + \cos.\,(\theta+\beta) + \cos.\,(\theta+2\beta) + \cos.\,(\theta+3\beta) + - - - - - - - (\cos.\,\theta+n\beta) = \frac{\cos.\,(\theta+\frac{1}{2}\beta(\sin.\frac{1}{2}\,(n+1)\beta}{\sin.\frac{1}{2}\,\beta} \quad - - \quad (z5)$$

To find the numerical value of the sine, cosine, &c., of 45°.



In the circle ABD, draw CA, CB, radii at right angles; join AB.

Then by Definition (12)

$$\text{Chord ABC } (90^\circ) = \frac{AB}{AC}$$

$$\text{Chord}^2\ 90^\circ = \frac{AB^2}{AC^2}$$

$$= \frac{AC^2+BC^2}{AC^2}$$

$$=\frac{2\,AC^2}{AC^2} \quad \because BC=AC$$

$$= \quad 2 \;-\;-\;-\;-\;-\;-\;-\;-\;-\;-\; (1)$$

Now, the chord of an arc is equal to twice the sine of half the arc; therefore,

$$2 \text{ sin. } 45°=\text{chord } 90°$$

$$4 \text{ sin.}^2\, 45°=\text{chord}^2\, 90°$$

$$=2, \text{ by Equation (1)};$$

$$\therefore \quad \text{sin. } 45°= \frac{1}{\sqrt{2}}$$

Again, by table I.:

$$\text{sin.}^2\, \theta+\text{cos.}^2\, \theta=1$$

$$\therefore \quad \text{cos.}^2\, 45°=1-\text{sin.}^2\, 45°$$

$$=1-\frac{1}{2}$$

$$=\frac{1}{2}$$

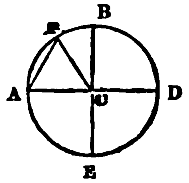
$$\therefore \quad \text{cos. } 45°= \frac{1}{\sqrt{2}}= \text{sin. } 45°.$$

Also,

$$\text{tan. } 45°=\frac{\text{sin. } 45°}{\text{cos. } 45°}$$

$$=1=\text{cot. } 45°.$$

To find the numerical value of the sine, cosine, &c., of 30°.



In the circle ABD, draw CP, making with CA the angle ACP=60°; join A, P.

Now,

$$2 \text{ sin. } 30°=\text{chord } 60°$$

$$=\frac{AP}{AC}$$

$$=\frac{AC}{AC} \quad \because AP=AC, \because \text{ the triangle APC is equiangular, and therefore equilateral.}$$

$$= 1$$

$$\therefore \quad \text{sin. } 30°=\frac{1}{2}$$

Again,

$$\text{cos. } 30°=\sqrt{1-\text{sin.}^2\, 30°}$$

$$=\sqrt{1-\tfrac{1}{4}}=\frac{\sqrt{3}}{2}=\frac{1}{2}\sqrt{3}$$

Also,

$$\begin{aligned}\text{tan. } 30^\circ &= \frac{\text{sin. } 30^\circ}{\text{cos. } 30^\circ}\\ &= \frac{1}{\sqrt{3}}\end{aligned}$$

$$\text{cot. } 30^\circ = \frac{1}{\text{tan. } 30^\circ} = \sqrt{3}$$

To find the numerical value of the sine, cosine, &c., of 60°.

$$\begin{aligned}\text{sin. } 60^\circ &= \text{cos. } (90^\circ - 60^\circ)\\ &= \text{cos. } 30^\circ\\ &= \frac{\sqrt{3}}{2}, \text{ by last art.}\end{aligned}$$

Again,

$$\begin{aligned}\text{cos. } 60^\circ &= \text{sin. } (90^\circ - 60^\circ)\\ &= \text{sin. } 30^\circ\\ &= \frac{1}{2}\end{aligned}$$

Also,

$$\text{tan. } 60^\circ = \sqrt{3}$$

$$\text{cot. } 60^\circ = \frac{1}{\sqrt{3}}.$$

It is required to find the sum of all the natural sines to every minute in the quadrant, radius $=1$. In this problem, the actual addition of all the terms would be a very tiresome labor, but the solution by means of formula (z3), is rendered very easy.

Applying that formula we have sin. $(\theta + \frac{1}{2}n\beta) =$ sin. 45°, sin. $\frac{1}{2}(n+1)\beta =$ sin. 45°, 0′, 30″ and sin. $\frac{1}{2}\beta =$ sin. 30′,
$\frac{\text{sin. } 45^\circ \text{ sin. } 45^\circ\ 0'\ 30''}{\text{sin. } 39''} = 3438.2467465$, the same sum required.

Let it be required to find the sum of the sines to every minute of the arc of 60°.

Here the numerical expression in the equation would become
$\frac{\text{sin. } 30^\circ \times \text{sin. } 30^\circ\ 0'\ 30''}{\text{sin. } 30''} = .5 \times .500126 \div .00014545953$
$= 1719.123373.25$ equal the sum of all the natural sines to every minute of the arc of 60°.

It may be useful to exhibit the most useful results in this chapter, in the following table.

TABLE III.

(1.) $\sin.\ (\theta\pm\beta) = \sin.\ \theta\ \cos.\ \beta \pm \sin.\ \beta\ \cos.\ \theta$

(2.) $\cos.\ (\theta\pm\beta) = \cos.\ \theta\ \cos.\ \beta \mp \sin.\ \theta\ \sin.\ \beta$

(3.) $\tan.\ (\theta\pm\beta) \quad \dfrac{\tan.\ \theta\pm\tan.\ \beta}{1\mp\tan.\ \theta\ \tan.\ \beta}$

(4.) $\sin.\ 2\ \theta = 2\ \sin.\ \theta\ \cos.\ \theta$

(5.) $\cos.\ 2 = \cos.^2\theta - \sin.^2\theta = 2\ \cos.^2\ \theta - 1 = 1 - 2\ \sin.^2\ \theta$

(6.) $\tan.\ 2\ \theta = \dfrac{2\ \tan.\ \theta}{1-\tan.^2\ \theta}$

(7.) $\sin.\ \dfrac{\theta}{2} = \sqrt{\dfrac{1-\cos.\ \theta}{2}}$

(8.) $\cos.\ \dfrac{\theta}{2} = \sqrt{\dfrac{1+\cos.\ \theta}{2}}$

(9.) $\tan.\ \dfrac{\theta}{2} = \sqrt{\dfrac{1-\cos.\ \theta}{1+\cos.\ \theta}} = \dfrac{1-\cos.\ \theta}{\sin.\ \theta} = \dfrac{\sin.\ \theta}{1+\cos.\ \theta}$

(10.) $\sin.\ \theta = 2\ \sin.\ \dfrac{\theta}{2}\ \cos.\ \dfrac{\theta}{2}$

(11.) $\sin.\ 3\ \theta = 3\ \sin.\ \theta - 4\ \sin.^3\ \theta$

(12.) $\cos.\ 3\ \theta = 4\ \cos.^3\ \theta - 3\ \cos.\ \theta$

(13.) $\sin.\ (n+1)\ \theta = 2\ \sin.\ n\theta\ \cos.\ \theta - \sin.\ (n-1)\ \theta$

(14.) $\cos.\ (n+1)\ \theta = 2\ \cos.\ n\theta\ \cos.\ \theta - \cos.\ (n-1)\ \theta$

(15.) $\sin.\ \theta+\sin.\ \beta = 2\ \sin.\ \dfrac{\theta+\beta}{2}\ \cos.\ \dfrac{\theta-\beta}{2}$

(16.) $\sin.\ \theta-\sin.\ \beta = 2\ \sin.\ \dfrac{\theta-\beta}{2}\ \cos.\ \dfrac{\theta+\beta}{2}$

(17.) $\cos.\ \theta+\cos.\ \beta = 2\ \cos.\ \dfrac{\theta+\beta}{2}\ \cos.\ \dfrac{\theta-\beta}{2}$

(18) $\cos.\ \theta-\cos.\ \beta = -2\ \sin.\ \dfrac{\theta+\beta}{2}\ \sin.\ \dfrac{\theta-\beta}{2}$

(19.) $\dfrac{\sin.\ \theta+\sin.\ \beta}{\sin.\ \theta-\sin.\ \beta} = \dfrac{\tan.\ \dfrac{\theta+\beta}{2}}{\tan.\ \dfrac{\theta-\beta}{2}}$

(20.) $\sin.\ (\theta+\beta)+\sin.\ (\theta-\beta) = 2\ \sin.\ \theta\ \cos.\ \beta$

(21.) $\sin.\ (\theta+\beta)-\sin.\ (\theta-\beta) = 2\ \sin.\ \beta\ \cos.\ \theta$

(22.) $\cos.\ (\theta+\beta)+\cos.\ (\theta-\beta) = 2\ \cos.\ \theta\ \cos.\ \beta$

(23.) $\cos.(\theta+\beta)-\cos.(\theta-\beta)=-2\sin.\theta\sin.\beta$

(24.) $\sin.(\theta+\beta)\cos.(\theta-\beta)=\sin.^2\theta-\sin.^2\beta=\cos.^2\beta-\cos.^2\theta$

(25.) $\cos.(\theta+\beta)\cos.(\theta-\beta)=\cos.^2\theta-\sin.^2\beta=\cos.^2\theta+\cos.^2\beta-1$

(26.) $\sin.\theta+\sin.(\theta+\beta)+\sin.(\theta+2\beta)+\sin.(\theta+3\beta)+\cdots$

$$\cdots\sin.(\theta+n\beta)=\frac{\sin.(\theta+\frac{1}{2}n\beta)\sin.\frac{1}{2}(n+1)\beta}{\sin.\frac{1}{2}\beta}$$

(27.) $\cos.\theta+\cos.(\theta+\beta)+\cos.(\theta+2\beta)+\cos.(\theta+3\beta)+\cdots$

$$\cdots\cos.(\theta+n\beta)=\frac{\cos.(\theta+\frac{1}{2}\beta)\sin.\frac{1}{2}(n+1)\beta}{\sin.\frac{1}{2}\beta}$$

(28.) $\sin.45^\circ=\cos.45^\circ=\frac{1}{\sqrt{2}}$

(29.) $\tan.45^\circ=\cot.45^\circ=1$

(30.) $\sin.30^\circ=\cos.60^\circ=\frac{1}{2}$

(31.) $\cos.30^\circ=\cos.60^\circ=\frac{\sqrt{3}}{2}$

(32.) $\tan.30^\circ=\cot.60^\circ=\frac{1}{\sqrt{3}}$

(33.) $\cot.30^\circ=\tan.60^\circ=\sqrt{3}$

The formulæ of Trigonometry may be multiplied to almost any extent, and the same quantity may be expressed in a vast number of different ways. An intimate acquaintance with those given in the above table is *essential* to the progress of the student.

The following, although of less frequent occurrence, may occasionally be found useful, and can be readily deduced from the above.

(34.) $$\left\{\begin{array}{l}\sin.(45^\circ\pm\theta)\\ \cos.(45^\circ\mp\theta)\end{array}\right\}=\frac{\cos.\theta\pm\sin.\theta}{\sqrt{2}}$$

(35.) $$\tan.(45^\circ\pm\theta)=\frac{1\pm\tan.\theta}{1\mp\tan.\theta}$$

(36.) $$\tan.^2\left(45^\circ\pm\frac{\theta}{2}\right)=\frac{1\pm\sin.\theta}{1\mp\sin.\theta}$$

(37.) $$\tan.\left(45^\circ\pm\frac{\theta}{2}\right)=\frac{1\pm\sin.\theta}{\cos.\theta}=\frac{\cos.}{1+\sin.\theta}$$

(38.) $$\frac{\sin.(\theta+\beta)}{\sin.(\theta-\beta)}=\frac{\tan.\theta+\tan.\beta}{\tan.\theta-\tan.\beta}=\frac{\cot.\beta\mp\cot.\theta}{\cot.\beta-\cot.\theta}$$

39.) $\dfrac{\cos.\ (\theta+\beta)}{\cos.\ (\theta-\beta)} = \dfrac{\cot.\ \beta-\tan.\ \theta}{\cot.\ \beta+\tan.\ \theta} = \dfrac{\cot.\ \theta-\tan.\ \beta}{\cot.\ \theta+\tan.\ \beta}$

40.) $\dfrac{\sin.\ \theta+\sin.\ \beta}{\cos.\ \theta+\cos.\ \beta} = \tan.\ \dfrac{\theta+\beta}{2}$

41.) $\dfrac{\sin.\ \theta+\sin.\ \beta}{\cot.\ \theta-\cos.\ \beta} = -\cot.\ \dfrac{\theta-\beta}{2}$

42.) $\dfrac{\sin.\ \theta-\sin.\ \beta}{\cos.\ \theta+\cos.\ \beta} = \tan.\ \dfrac{\theta-\beta}{2}$

43.) $\dfrac{\sin.\ \theta-\sin.\ \beta}{\cos.\ \theta-\cos.\ \beta} = -\cot.\ \dfrac{\theta+\beta}{2}$

44.) $\dfrac{\cos.\ \theta+\cos.\ \beta}{\cos.\ \theta-\cos.\ \beta} = -\cot.\ \dfrac{\theta+\beta}{2}\cot.\ \dfrac{\theta-\beta}{2}$

(45.) $\tan.\ \theta+\tan.\ \beta = \dfrac{\sin.\ (\theta+\beta)}{\cos.\ \theta\ \cos.\ \beta}$

(46.) $\cot.\ \theta+\cot.\ \beta = \dfrac{\sin.\ (\theta+\beta)}{\sin.\ \theta\ \sin.\ \beta}$

(47.) $\tan.\ \theta-\tan.\ \beta = \dfrac{\sin.\ (\theta-\beta)}{\cos.\ \theta\ \cos.\ \beta}$

(48.) $\cot.\ \theta-\cot.\ \beta = -\dfrac{\sin.\ (\theta-\beta)}{\sin.\ \theta\ \sin.\ \beta}$

(49.) $\tan.^2\ \theta-\tan.^2\ \beta = \dfrac{\sin.\ (\theta+\beta)\ \sin.\ (\theta-\beta)}{\cos.^2\ \theta\ \cos.^2\ \beta}$

(50.) $\cot.^2\ \theta-\cot.^2\ \beta = -\dfrac{\sin.\ (\theta+\beta)\ \sin.\ (\theta-\beta)}{\sin.^2\ \theta\ \sin.^2\ \beta}$

In order to become familiar with the various combinations, and dexterous in the application of these expressions, the student will do well to exercise himself by verifying the following values of Sin. θ, Cos θ, Tan. θ, which are extracted from the large work of Cagnoli.

TABLE OF THE MOST USEFUL ANALYTICAL VALUES OF SIN. θ, COS. θ, TAN. θ.

VALUES OF SIN. θ.	VALUES OF COS. θ.
1. $\cos.\,\theta\,\tan.\,\theta$	16. $\dfrac{\sin.\,\theta}{\tan.\,\theta}$
2. $\dfrac{\cos.\,\theta}{\cot.\,\theta}$	17. $\sin.\,\theta\,\cot.\,\theta$
3. $\sqrt{1-\cos.^2\theta}$	18. $\sqrt{1-\sin.^2\theta}$
4. $\dfrac{1}{\sqrt{1+\cot.^2\theta}}$	19. $\dfrac{1}{\sqrt{1+\tan.^2\theta}}$
5. $\dfrac{\tan.\,\theta}{\sqrt{1+\tan.^2\theta}}$	20. $\dfrac{\cot.\,\theta}{\sqrt{1+\cot.^2\theta}}$
6. $2\sin.\,\frac{\theta}{2}\cos.\frac{\theta}{2}$	21. $\cos.^2\frac{\theta}{2}-\sin.^2\frac{\theta}{2}$
7. $\dfrac{\sqrt{1-\cos.\,2\theta}}{2}$	22. $1-2\sin.^2\frac{\theta}{2}$
8. $\dfrac{2\tan.\frac{\theta}{2}}{1+\tan.^2\frac{\theta}{2}}$	23. $2\cos.^2\frac{\theta}{2}-1$
9. $\dfrac{2}{\cot.\,\frac{\theta}{2}+\tan.\frac{\theta}{2}}$	24. $\sqrt{\dfrac{1+\cos.\,2\theta}{2}}$
10. $\dfrac{\sin.\,(30^\circ+\theta)-(\sin.\,30^\circ-\theta)}{\sqrt{3}}$	25. $\dfrac{1-\tan.^2\frac{\theta}{2}}{1+\tan.^2\frac{\theta}{2}}$
11. $2\sin.^2(45^\circ+\frac{\theta}{2})-1$	26. $\dfrac{\cot.\,\frac{\theta}{2}-\tan.\,\frac{\theta}{2}}{\cot.\,\frac{\theta}{2}+\tan.\,\frac{\theta}{2}}$
12. $1-2\sin.^2(45^\circ-\frac{\theta}{2})$	27. $\dfrac{1}{1+\tan.\,\theta\,\tan.\frac{\theta}{2}}$
13. $\dfrac{1-\tan.^2(45^\circ-\frac{\theta}{2})}{1+\tan.^2(45^\circ-\frac{\theta}{2})}$	28. $\dfrac{2}{\tan.(45^\circ+\frac{\theta}{2})+\cot.(45^\circ+\frac{\theta}{2})}$
14. $\dfrac{\tan.(45^\circ+\frac{\theta}{2})-\tan.(45^\circ-\frac{\theta}{2})}{\tan.(45^\circ+\frac{\theta}{2})+\tan.(45^\circ-\frac{\theta}{2})}$	29. $2\cos.(45^\circ+\frac{\theta}{2})\cos.(45^\circ-\frac{\theta}{2})$
15. $\sin.\,(60^\circ+\theta)-\sin.\,(60^\circ-\theta)$	30. $\cos.\,(60^\circ+\theta)+\cos.\,(60^\circ-\theta)$

VALUES OF TAN. θ.

$\frac{\text{in. }\theta}{\text{os. }\theta}$	37. $\frac{2\cot.\frac{\theta}{2}}{\cot.^2\frac{\theta}{2}-1}$
$\frac{1}{\text{ot. }\theta}$	38. $\frac{2}{\cot.\frac{\theta}{2}-\tan.\frac{\theta}{2}}$
$\sqrt{\frac{1}{\cos.^2\theta}-1}$	39. $\cot.\theta-2\cot.2$
$\frac{\sin.\theta}{\sqrt{1-\sin.^2\theta}}$	40. $\frac{1-\cos.2\theta}{\sin.2\theta}$
$\frac{\sqrt{1-\cos.^2\theta}}{\cos.\theta}$	41. $\frac{\sin.2\theta}{1+\cos.2\theta}$
$\frac{2\tan.\frac{\theta}{2}}{1-\tan.^2\frac{\theta}{2}}$	42. $\sqrt{\frac{1-\cos.2\theta}{1+\cos.2\theta}}$.
	43. $\frac{\tan.(45^\circ+\frac{\theta}{2})-\tan.(45^\circ-\frac{\theta}{2})}{2}$

om certain properties of the circles to be discussed in an-
: volume, other important trigonometrical formulæ, may be
ced, furnishing us with more expeditious means of deter-
ng, numerically, the values of some of the trigonometri-
ines, and ratios, all of which will occur in their order.

o develop sin. x *and cos.* x *in a series ascending by the powers of* x.

he series for sin. x must vanish when $x=0$, and therefore
erm in the series can be independent of x, nor can the
powers of x occur in the series; for if we suppose

$$\sin. x = a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+ \ldots .$$

en $\sin.(-x)=-a_1x+a_2x^2-a_3x^3+a_4x^4-a_5x^5+ \ldots$

ut $\sin.(-x)=-\sin. x$

$$=-a_1x-a_2x^2-a_3x^3-a_4x^4-a_5x^5- \ldots$$

$\therefore\ a_2=-a_2,\ a_4=-a_4, \ldots$; hence $a_2=0,\ a_4=0 \ldots$

$$\therefore \sin. x= a_1x+a_3x^3+a_5x^5+a_7x^7+ \ldots \quad (1)$$

Again, the series for cos. x must $= 1$ when $x = 0$, and therefore the series must contain a term independent of x, and it must be 1; also the series can contain no odd powers of x, for if we suppose

$$\cos. x = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots$$

$$\text{then } \cos. (-x) = 1 - a_1x + a_2x^2 - a_3x^3 + a_4x^4 - \ldots\ldots$$

$$\text{but } \cos. (-x) = \cos. x$$

$$= 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots$$

$$\therefore\ a_1 = -a_1,\ a_3 = -a_3,\ \ldots\ \therefore a_1 = 0,\ a_3 = 0\ \ldots$$

$$\therefore \cos. x = 1 + a_2x^2 + a_4x^4 + a_6x^6 + \quad - - - - - \quad (2)$$

$$\text{Hence } \cos. x + \sin. x = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 \quad - - (3)$$

$$\cos. x - \sin. x = 1 - a_1x + a_2x^2 - a_3x^3 + a_4x^4 - a_5x^5 + \quad - - (4)$$

Now in equation (3) write $x + h$ for x, and we have

$$\cos.(x+h) + \sin(x+h) = 1 + a_1(x+h) + a_2(x+h)^2 + a_3(x+h)^3 + \quad (5)$$

$$\begin{aligned}
\text{but } \cos. (x+h) + \sin. (x+h) &= \cos. x \cos. h - \sin. x \sin. h \\
&\quad + \sin. x \cos. h + \cos. x \sin. h \\
&= \cos. h\,(\cos. x + \sin. x) + \sin. h\,(\cos. x - \sin. x) \\
&= (1 + a_2h^2 + a_4h^4 + \ldots).(1 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots) \\
&\quad + (a_1h + a_3h^3 + a_5h^5 + \ldots)(1 - a_1x + a_2x^2 - a_3x^3 + \ldots)
\end{aligned}$$

$$\left.\begin{array}{r}
= 1 + a_1x + a_2x^2 + a_3x^3 \quad + \ldots \\
+ a_1h - a_1{}^2xh + a_1a_2x^2h + \ldots \\
+ a_2h^2 + a_1a_2xh^2 \quad + \ldots \\
+ a_3h^3 \quad + \ldots \\
+ \ldots
\end{array}\right\} \quad - - - \quad (6)$$

Comparing equations (5) and (6) we have

$$\left.\begin{array}{r}
1 + a_1x + a_2x^2 \quad + a_3x^3 \quad + \\
+ a_1h + 2a_2xh + 3a_3x^2h + \\
+ a_2h^2 + 3a_3xh^2 + \\
+ a_3h^3 + \\
+
\end{array}\right\} = \left\{\begin{array}{r}
1 + a_1x + a_2x^2 \quad + a_3x^3 \quad + \\
+ a_1h - a_1a_1xh + a_1a_2x^2h - \\
+ a_2h^2 \quad + a_1a_2xh^2 + \\
+ a_3h^3 \quad - \\
+
\end{array}\right.$$

and equating the coefficients of the terms involving the same powers of x and h, we have

$$2a_2 = -a_1a_1;\ \text{therefore } a_2 = -\frac{a_1a_1}{2} = -\frac{a_1{}^2}{1.2}$$

$$3a_3 = a_1a_2\ .\ .\ .\ .\ .\ a_3 = \frac{a_1a_2}{3} = -\frac{a_1{}^3}{1.2.3}$$

$$4a_4 = -a_1a_3\ .\ .\ .\ .\ .\ a_4 = -\frac{a_1a_3}{4} = +\frac{a_1{}^4}{1.2.3.4}$$

$$5a_5 = a_1a_4\ .\ .\ .\ .\ .\ a^5 = \frac{a_1a_4}{5} = +\frac{a_1{}^5}{1.2.3.4.5}$$

hence
$$\sin. x = a_1 x - \frac{a_1^3}{1.2.3} x^3 + \frac{a_1^5}{1.2.3.4.5} x^5 - \frac{a_1^7}{1.2.3.4.5.6.7} x^7 +$$
$$\cos. x = 1 - \frac{a_1^2}{1.2} x^2 + \frac{a_1^4}{1.2.3.4} x^4 - \frac{a_1^6}{1.2.3.4.5.6} x^6 + \ldots.$$

and we have only to determine the value of a_1. To effect this, we have

$$\sin. x = a_1 x - \frac{a_1^3}{1.2.3} x^3 + \frac{a_1^5}{1.2.3.4.5} x^5 - \ldots$$
$$= a_1 x \left(1 - \frac{a_1^2}{1.2.3} x^2 + \frac{a_1^4}{1.2.3.4.5} x^4 - \ldots\right.$$

Now the value of x may be assumed so small that the series in the parenthesis, and sin. x, shall differ from 1 and x respectively, by less than any assignable quantities; hence ultimately

$x = a_1 x$, and therefore $a_1 = 1$; whence

$$\sin. x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \ldots$$
$$\cos. x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \ldots.$$

To develop tan. x *and cot.* x *in a series ascending by the powers of* x.

The development may be obtained from those of sin. x and cos. x, already found.

$$\tan. x = \frac{\sin. x}{\cos. x} = \frac{x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5.} - \&c.}{1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.}$$

and the series will therefore be of the form

$$x + a_3 x^3 + a_5 x^5 + a_7 x^7 + \ldots.$$

$$\text{Hence, let } x + a_3 x^3 + a_5 x^5 + \ldots = \frac{x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \ldots}{1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \ldots}$$

$$\therefore\ x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \ldots$$
$$\ldots = \left(1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \ldots\right)\left(x + a_3 x^3 + a_5 x^5 + \ldots\right)$$

$$= x + \left.\begin{array}{r} a_3 \\ -\frac{1}{1.2} \end{array}\right| x^3 + \left.\begin{array}{r} a_5 \\ -\frac{a_3}{1.2} \\ +\frac{1}{1.2.3.4} \end{array}\right| x^5 + \begin{array}{c} \\ - \\ + \\ - \end{array}$$

Hence, equating the coefficients of the like terms, we have

$$a_3 - \frac{1}{1.2} = -\frac{1}{1.2.3} \qquad \therefore a_3 = \frac{2}{1.2.3}$$

$$a_5 - \frac{a_3}{1.2} + \frac{1}{1.2.3.4} = \frac{1}{1.2.3.4.5} \quad \therefore a_5 = \frac{2^4}{1.2.3.4.5} \,\&c.$$

$$\therefore \tan. x = x + \frac{2x^3}{1.2.3} + \frac{2^4 x^5}{1.2.3.4.5} + \ldots\ldots$$

$$\text{Sim. } \cot. x = \frac{1}{x} - \frac{2x}{1.2.3} - \frac{2^4 x^3}{1.2.3.4.5} - \ldots\ldots$$

CHAPTER III.

FORMULÆ FOR THE SOLUTION OF TRIANGLES.

We shall here repeat the enunciations of the two propositions established in Chapter I.

PROPOSITION I.

In any right-angled plane triangle,

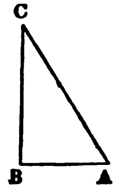
1°. *The ratio which the side opposite to one of the acute angles has to the hypothenuse, is the sine of that angle.*

2°. *The ratio which the side adjacent to one of the acute angles has to the hypothenuse, is the cosine of that angle.*

3°. *The ratio which the side opposite to one of the acute angles has to the side adjacent to that angle, is the tangent of that angle.*

Thus, in any right-angled triangle ABC,

$$
\begin{array}{lll}
= \text{sin. A}, & \frac{BA}{AC} = \text{cos. A}, & \frac{CB}{BA} = \text{tan. A} \\
= \text{cos. C}, & \quad = \text{sin. C}, & \quad = \text{cot. C}
\end{array}
$$



$$
\left.
\begin{array}{l}
\left.\begin{array}{l} CB = AC \text{ sin. A} \\ \quad = AC \text{ cos. C} \end{array}\right\} \\
\left.\begin{array}{l} BA = AC \text{ cos. A} \\ \quad = AC \text{ sin. C} \end{array}\right\} \\
\left.\begin{array}{l} CB = BA \text{ tan. A} \\ \quad = BA \text{ cot. C} \end{array}\right\}
\end{array}
\right\} \text{ - - - - - - - - - - - - - } (\alpha)
$$

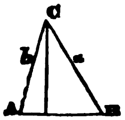
PROPOSITION II.

any plane triangle, the sides are to each other as the sines of the angles opposite to them.

e shall, frequently in treating of triangles, make use of following notation; denoting the angles of the triangle he large letters at the angular points, and the sides he triangle opposite to these angles, by the corresponding small letters.

ius, in the triangle ABC, we shall te the angles, BAC, CBA, BCA, he letters, A, B, C, respectively, the sides BC, AC, AB, by the rs a, b, c, respectively.

ccording to this, we shall have, by proposition,

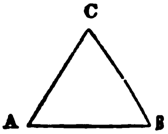


$$
\left.
\begin{array}{l}
\frac{a}{b} = \frac{\text{sin. A}}{\text{sin. B}} \\[4pt]
\frac{a}{c} = \frac{\text{sin. A}}{\text{sin. C}} \\[4pt]
\frac{b}{c} = \frac{\text{sin. B}}{\text{sin. C}}
\end{array}
\right\} \text{ - - - - - - - - - - - - } (\beta)
$$

PROPOSITION III.

In any plane triangle, the sum of any two sides, is to their difference, as the tangent of half the sum of the angles opposite to them, is to the tangent of half their difference.

Let ABC be any plane triangle, then, by Proposition II



$$\frac{a}{b}=\frac{\sin.\ A}{\sin.\ B}$$

$$\frac{a+b}{a-b}=\frac{\sin.\ A+\sin.\ B}{\sin.\ A-\sin.\ B}$$

But, by Trigonometry, Chap. II. (r)

$$\sin.\ A+\sin.\ B=2\sin.\frac{A+B}{2}\cos.\frac{A-B}{2}$$

$$\sin.\ A-\sin.\ B=2\cos.\frac{A+B}{2}\sin.\frac{A-B}{2}$$

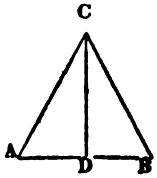
$$\therefore\ \frac{a+b}{a+b}=\frac{2\sin.\frac{A+B}{2}\cos.\frac{A-B}{2}}{\cos.\frac{A+B}{2}\sin.\frac{A-B}{2}}$$

$$=\tan.\frac{A+B}{2}\cot.\frac{A-B}{2}$$

$$\left.\begin{aligned} &=\frac{\tan.\frac{A+B}{2}}{\tan.\frac{A-B}{2}} \\ &\text{And in like manner,} \\ \frac{a+c}{a-c}&=\frac{\tan.\frac{A+C}{2}}{\tan.\frac{A-C}{2}} \\ \frac{b+c}{b-c}&=\frac{\tan.\frac{B+C}{2}}{\tan.\frac{B-C}{2}} \end{aligned}\right\}\ \cdots\cdots\ (\gamma)$$

PROPOSITION IV.

o express the cosine of an angle of a plane triangle in terms of the sides of the triangle.



et ABC be a triangle ; A, B, C, the three les ; a, b, c, the corresponding sides.

. Let the proposed (A) be acute.

rom C draw CD perpendicular to AB, the e of the triangle.

en,

BC²=AC²+AB²—2AB. AD (Prop. XXVI. IV. *El. Geom.*)

$$a^2 = b^2 + c^2 - 2c\,.\,\mathrm{AD}$$

t, since CDA is a right-angled triangle,

$$\mathrm{AD} = \mathrm{AC}\ \cos.\ \mathrm{CAD} = b\ \cos.\ \mathrm{A}$$

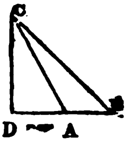
$$\therefore\ a^2 = b^2 + c^2 - 2bc\ \cos.\ \mathrm{A}$$

$$\therefore\ \cos.\ \mathrm{A} = \frac{b^2 + c^2 - a^2}{2bc}$$

ich is the expression required.

2 Let the proposed angle (A) be abtuse.

From C, draw CD perpendicular to AB produced.



hen,

$$\mathrm{BC}^2 = \mathrm{AC}^2 + \mathrm{AB}^2 + 2\mathrm{AB}\,.\,\mathrm{AD}$$

r,

$$a^2 = b^2 + c^2 + 2c\ .\ \mathrm{AD}$$

ut, since CDA is a right angled triangle,

$$\mathrm{AD} = \mathrm{AC}\ \cos.\ \mathrm{CAD}$$

$$= \mathrm{AC} \times -\cos.\ \mathrm{CAB}$$

$\because$ CAB is the supplement of CAD.

$$= -b \cos \mathrm{A}$$

$$\therefore\ a^2 = b^2 + c^2 - 2bc\ \cos.\ \mathrm{A}$$

$$\therefore\ \cos.\ \mathrm{A} = \frac{b^2 + c^2 - a^2}{2bc}$$

It will be seen that this result is identical with that which e deduced in the last case, so that, whether A be acute or btuse, we shall have,

$$\cos. A = \frac{b^2 + c^2 - a^2}{2bc}$$

Proceeding in the same manner for the other angles, we shall find,

$$\left.\begin{aligned} \cos. B &= \frac{a^2 + c^2 - b^2}{2ac} \\ \cos. C &= \frac{a^2 + b^2 - c^2}{2ab} \end{aligned}\right\} \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \quad (\delta)$$

PROPOSITION V.

To express the sine of an angle of a plane triangle in terms of the sides of the triangle.

Let A be the proposed angle; then by last prop.,

$$\cos. A = \frac{b^2 + c^2 - a^2}{2\,b\,c}$$

Adding unity to each member of the equation,

$$\begin{aligned} 1 + \cos. A &= 1 + \frac{b^2 + c^2 - a^2}{2\,b\,c} \\ &= \frac{b^2 + 2bc + c^2 - a^2}{2\,b\,c} \\ &= \frac{(b + c)^2 - a^2}{2\,b\,c} \\ &= \frac{(b+c+a)\,(b+c-a)}{2\,b\,c} \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \quad (1) \end{aligned}$$

Again, $\cos. A = \dfrac{b^2 + c^2 - a^2}{2\,b\,c}$

Substracting each member of the equation from unity,

$$\begin{aligned} 1 - \cos. A &= 1 - \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{2bc - b^2 - c^2 + a^2}{2bc} \\ &= \frac{a^2 - (b^2 - 2bc + c^2)}{2bc} \\ &= \frac{a^2 - (b - c)^2}{2bc} \\ &= \frac{(a+b-c)\,(a+c-b)}{2bc} \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \quad (2) \end{aligned}$$

Multiplying together equations (1) and (2,)

$$(1+\cos A).(1-\cos A)=\frac{(a+b+c)\ (b+c-a)\ (a+c-b)\ (a+b-c)}{4b^2\ c^2}$$

But $(1+\cos. A)\ (1-\cos. A) = 1-\cos.^2 A$

$$= \sin.^2 A \quad \text{(Table I.)}$$

$$\therefore\ \sin.^2 A = \frac{(a+b+c)\ (b+c-a)\ (a+c-b)\ (a+b-c)}{4b^2c^2}$$

Extracting the root on both sides,

$$\sin.\ A = \frac{1}{2bc}\ \sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}..(3$$

The above expresison, for the sine of an angle of a triangle in terms of the sides, is sometimes exhibited under a form somewhat different.

Let s denote the semiperimeter, that is to say, half the sum of the sides of the triangle ; then

$$s = \frac{a+b+c}{2},\ \text{and,} \qquad 2s = a+b+c$$

$$s-a = \frac{b+c-a}{2}\ \ ...\ 2\ (s-a) = b+c-a$$

$$s-b = \frac{a+c-b}{2}\ \ ...\ 2\ (s-b) = a+c-b$$

$$s-c = \frac{a+b-c}{2}\ \ ...\ 2\ (s-c) = a+b-c$$

Substituting $2\ s, 2\ (s-a),......$ for $a+b+c, b+c-a,.....$ in the expression for $\sin.^2 A$, it becomes

$$\sin.^2 A = \frac{16\ s\ (s-a)\ (s-b)\ (s-c)}{4\ b^2c^2}$$

And extracting the root on both sides,

$$\sin.\ A = \frac{2}{bc}.\sqrt{s\ (s-a)\ (s-b)\ (s-c)}$$

Proceeding in the same manner for the other angles, we shall find $\Bigg\}$ - - - (s)

$$\sin.\ B = \frac{2}{ac}.\sqrt{s\ (s-a)\ (s-b)\ (s-c}$$

$$\sin.\ C = \frac{2}{ab}.\sqrt{s\ (s-a)\ (s-b)\ (s-c)}$$

By equation (1) we have

$$1+\cos. A = \frac{(a+b+c)\ (b+c-a)}{2\,b\,c}$$

$$= \frac{4\,s\ (s-a)}{2\,b\,c}$$

But, by Chap. II,

$$1+\cos. A=2\cos.^2\frac{A}{2}$$

$$\therefore\quad 2\cos.^2\frac{A}{2}=\frac{4\,s\,(s-a)}{2\,bc}$$

Extracting the root on both sides,

$$\left.\begin{array}{l} \cos.\dfrac{A}{2}=\sqrt{\dfrac{s\ (s-a)}{bc}} \\ \text{And in like manner,} \\ \cos.\dfrac{B}{2}=\sqrt{\dfrac{s\ (s-b)}{ac}} \\ \cos.\dfrac{C}{2}=\sqrt{\dfrac{s\ (s-s)}{a\,b}} \end{array}\right\}\cdots\cdots\cdots\cdots(\sigma)$$

By equation (2) we have

$$1-\cos. A=\frac{(a+c-b)\ (a+b-c)}{2bc}$$

$$=\frac{4\ (s-b)\ (s-c)}{2bc}$$

But, by Chap. II,

$$1-\cos. A=2\sin.^2\frac{A}{2}$$

$$\therefore\quad 2\sin.^2\frac{A}{2}=\frac{4\ (s-b)\ (s-c)}{2bc}$$

Extracting the root on both sides,

$$\left.\begin{array}{l} \sin.\dfrac{A}{2}=\sqrt{\dfrac{(s-b)\ (s-c)}{bc}} \\ \text{And in like manner,} \\ \sin.\dfrac{B}{2}=\sqrt{\dfrac{(s-a)\ (s-c)}{ac}} \\ \sin.\dfrac{C}{2}=\sqrt{\dfrac{(s-a)\ (s-b)}{ab}} \end{array}\right\}\cdots\cdots\cdots(\zeta)$$

Dividing the formulæ marked (ζ) by those marked (σ) we have

$$\left.\begin{aligned}\tan.\frac{A}{2}&=\sqrt{\frac{(s-b)\,(s-c)}{s\,(s-a)}}\\ \tan.\frac{B}{2}&=\sqrt{\frac{(s-a)\,(s-c)}{s\,(s-b)}}\\ \tan.\frac{C}{2}&=\sqrt{\frac{(s-a)\,(s-b)}{s\,(s-c)}}\end{aligned}\right\}\cdots\cdots\quad(\eta)$$

CHAPTER IV.

ON THE CONSTRUCTION OF TRIGONOMETRICAL TABLES.

Before proceeding to apply the formulæ deduced in the last chapter to the solution of triangles, we shall make a few remarks upon the construction of those tables, by means of which we are enabled to reduce our trigonometrical calculations to numerical results.

It is manifest, from definitions 1°, 2°, 3°, &c. that the various trigonometrical quantities, the sine, the cosine, the tangent, &c. are abstract numbers representing the comparative length of certain lines. We have already obtained the numerical value of these quantities in a few particular cases, and we shall now show how the numbers, corresponding to angles of every degree of magnitude, may be obtained by the application of the most simple principles.

The numbers corresponding to the sine, cosine, &c. of all angles from 1″ up to 90°, when arranged in a table, form what is called the *Trigonometrical canon.*

The first operation to be performed is

To compute the numerical value of the sine and cosine of 1′.

We have seen, Chap. II. formula (j) that

$$\begin{aligned}\sin.\frac{\theta}{2}&=\sqrt{\frac{1-\cos.\theta}{2}}\\ &=\sqrt{\tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-\sin.^2\theta}}\end{aligned}$$

By which formula the sine of any angle is given in terms of the sine of twice that angle.

Now substitute $\frac{\theta}{2}$ for θ and it becomes

$$\text{Sin. } \frac{\theta}{4} \text{ or sin. } \frac{\theta}{2^2} = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \text{sin.}^2 \frac{\theta}{2}}}$$

$$\text{In like manner, sin.} \frac{\theta}{2^3} = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \text{sin.}^2 \frac{\theta}{2^2}}}$$

$$\text{sin. } \frac{\theta}{2^4} = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \text{sin.}^2 \frac{\theta}{2^3}}}$$

$$\&c. = \&c.$$

$$\text{And generally, sin. } \frac{\theta}{2^n} = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \text{sin.}^2 \frac{\theta}{2^{n-1}}}}$$

$$\text{Now let } \theta = 30^\circ \quad \therefore \frac{\theta}{2} = 15^\circ$$

and applying the above formula, we have

$$\text{sin. } 15^\circ = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \text{sin.}^2\, 30^\circ}}$$

But by Chap. II. sin. $30^\circ = \frac{1}{2}$ $\therefore$ sin.^{2} $30^\circ = \frac{1}{4}$

$$\begin{aligned} \therefore \quad \text{sin. } 15^\circ &= \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \tfrac{1}{4}}} \\ &= \tfrac{1}{2}\sqrt{2 - \sqrt{3}} \\ &= .2588190 \text{ - - - -} \end{aligned}$$

$$\begin{aligned} \text{Similarly,} \quad \text{sin. } 7^\circ 30' &= \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \text{sin.}^2\, 15^\circ}} \\ &= \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - (.2588190)^2}} \\ &= .1305268 \text{ - - - - -} \\ \&c. &= \&c. \end{aligned}$$

It is manifest, that, by continuing the process, we shall obtain in succession the sines of 3°45′, of 1°52′30″, &c.

In this way we find

$$\text{Sin. } \frac{30^\circ}{2^{11}} \text{ or sin. } 1'\ 45''\ 28'''\ 7^{\text{iv}}\ 30^{\text{v}} = .0005113269, \&c.$$

$$\text{Sin. } \frac{30^\circ}{2^{12}} \text{ or sin. } 52''\ 44'''\ 3^{\text{iv}}\ 45^{\text{v}} = .0002556634, \&c.$$

From which it appears, that, when the operation above mentioned has been repeated so many times, the sine of the arc is halved at the same time that the arc itself is bisected: that is, *The sines of very small arcs are nearly proportional to the arcs themselves.*

Hence we shall have

$$\text{Sin. } 52'' \, 44''' \, 3^{\text{iv}} \, 45^{\text{v}} : \text{sin. } 1' :: 52'' \, 44''' \, 3^{\text{iv}} \, 45^{\text{v}} : 1'$$

$$:: \frac{60}{2^{12}} : \frac{60}{60 \times 60}$$

$$:: 3600 : 4096$$

$$\therefore \text{ sin. } 1' = \frac{\text{sin. } 52'' \, 44''' \, 3^{\text{iv}} \, 45^{\text{v}} \times 4096}{3600}$$

$$= \frac{.0002556634 \times 4096}{3600}$$

$$= .000290888204 \, ... = \text{cos. } 89^\circ \, 59'$$

$$\because \text{ sin. } \theta = \text{cos. } (90^\circ - \theta)$$

$$\text{Again, } \because \text{ cos. } \theta = \sqrt{1 - \text{sin. } \theta)}$$

$$\text{cos. } 1' = \sqrt{1 - (.000290888204 \, ...)^2}$$

$$= .999999915384$$

The sine and cosine of 1′ being thus determined, we shall roceed to show in what manner we shall now be enabled to ompute the sines and cosines of all superior angles.

By formula (*m*) Chap. II.

$$\text{Sin. } (n+1)\,\theta = 2 \text{ cos. } \theta \text{ sin. } n\,\theta - \text{sin. } (n-1)\,\theta$$

If we suppose $\theta = 1'$ and n to be taken = to the numbers 1, , 3, ... in succession, we find

$\text{in. } 2' = 2 \text{ cos. } 1' \text{ sin. } 1' - \text{sin. } 0 = .0005817764 \, ... = \text{cos. } 89^\circ \, 58'$

$\text{in. } 3' = 2 \text{ cos. } 1' \text{ sin. } 2' - \text{sin. } 1' = .0008726645 \, ... = \text{cos. } 89^\circ \, 57'$

$\text{in. } 4' = 2 \text{ cos. } 1' \text{ sin. } 3' - \text{sin. } 2' = .0011635526 \, ... = \text{cos. } 89^\circ \, 56'$

&c. = &c.

gain, by employing formula (*o*), Chap. II.

$$\text{Cos. } (n+1)\,\theta = 2 \text{ cos. } \theta \text{ cos. } n\,\theta - \text{cos. } (n-1)\,\theta$$

; as before, we suppose $\theta = 1'$ and $n = 1, 2, 3, ...$ in succession,

$\text{Cos. } 2' = 2 \text{ cos.}^2 \, 1' - \text{cos. } 0 = .999999830 \, ... = \text{sin. } 89^\circ \, 58'$

$\text{Cos. } 3' = 2 \text{ cos. } 1' \text{ cos. } 2' - \text{cos. } 1' = .999999619 \, ... = \text{sin. } 89^\circ \, 57'$

$\text{Cos. } 4' = 2 \text{ cos. } 1' \text{ cos. } 3' - \text{cos. } 2' = .999999323 \, ... = \text{sin. } 89^\circ \, 56'$

&c. = &c.

It is manifest, that, by continuing the above processes, we hall obtain the numerical values of the sines and cosines of all ngles from 1′ up to 90°. These being determined, the tanents, cotangents, &c. may be calculated by means of the retions established in table I.

The above operations are exceedingly laborious, but require knowledge of the fundamental rules of arithmetic alone. It manifest that, in employing this method, an error committed the sine or cosine of an inferior arc, will entail errors on the sines or cosines of all succeeding arcs. Hence is created the necessity of some check on the computist, and of some independent mode of examining the accuracy of the computation. For this purpose, formulæ, derived immediately from established properties, are employed; if the numerical results from these formulæ agree with the results obtained by a regular process of computation, then it is almost a certain conclusion that the latter process has been rightly conducted.

Formulæ employed for this purpose are called *formulæ of verification*, and of these any number may be obtained; it will be sufficient for our present purpose to give one.

$$\text{Sin.}^2\,\theta+\cos.^2\,\theta=1 \quad - - - - - - - - - - - \quad \text{tab. I.}$$

And $2\sin.\,\theta\cos.\,\theta=\sin.\,2\,\theta$

Hence $$\sin.\,\theta=\tfrac{1}{2}\sqrt{1+\sin.\,2\,\theta}\pm\tfrac{1}{2}\sqrt{1-\sin.\,2\,\theta}$$

$$\cos.\,\theta=\tfrac{1}{2}\sqrt{1+\sin.\,2\,\theta}\mp\tfrac{1}{2}\sqrt{1-\sin.\,2\,\theta}$$

Now if we suppose $\theta=12^\circ\ 30'$

$$\sin.\,12^\circ\ 30'=\tfrac{1}{2}\sqrt{1+\sin.\,25^\circ}\pm\tfrac{1}{2}\sqrt{1-\sin.\,25^\circ}$$

$$\cos.\,12^\circ\ 30'=\tfrac{1}{2}\sqrt{1+\sin.\,25^\circ}\mp\tfrac{1}{2}\sqrt{1-\sin.\,25^\circ}$$

Hence, if the values of the sine and cosine of 12° 30′, and of the sine of 25° obtained by the method already explained, when substituted in these equations, render the two members identical, we conclude that our operations are correct.

The values of the sine and cosine of 30°, 45°, 60°, &c. which were obtained in Chap. II., may be employed as formulæ of verification.

We can obtain finite expressions, although under an incommensurable form, for the sines of arcs of 3°, and all the multiples of 3°, *i. e.* for

3°, 6°, 9°, 12°, 15°, 18°, 21°, 24°, 27°, 30°, 33°, 36°, 39°, 42°, 45°, 48°, 51°, 54°, 57°, 60°, 63°, 66°, 69°, 72°, 75°, 78°, 81°, 84°, 87°, 90°.

We first obtain the values of the sines 30°, 45°, 60°, 18°, and from these we obtain all the others, by means of the formulæ, for

$$\text{Sin.}\,(\theta+\beta),\ \sin.\,(\theta-\beta),\ \&\text{c.}$$

The numerical value of the trigonometrical functions have been calculated by some to ten places of figures, by others as far as twelve. We must have tables calculated to ten places to have the seconds and tenths of a second with precision, when we make use of the sines of angles which differ but little from 90°, or of the cosines of angles of a few seconds only. Tables in general, however, are calculated as far as seven places only, and these give results sufficiently accurate for all ordinary purposes.

Such is the formation of the trigonometrical canon. The bor of the application of this canon, may be much facilita- d by the application of a system of artificial numbers called garithms, a description of which, forms the subject of the xt chapter.

CHAPTER V.

LOGARITHMS.

DEFINITIONS AND ILLUSTRATIONS.

1. Logarithms are certain functions of natural numbers, by e use of which the tedious operations of multiplication and vision are performed by the addition and subtraction of those nctions; which consist of artificial numbers having such lations to certain natural numbers, that the sum of any two those artificial numbers will be a similar function of the roduct of the natural numbers to which they have such rela- on; and the difference of any two will be a similar func- n of the quotient arising from the division of such natural mbers.

Or more definitely, logarithms are the numerical exponents ratios, being a series of numbers in arithmetical progression rresponding to another series in geometrical progression.

Thus, { 0, 1, 2, 3, 4, 5, 6, Indices or logarithms,
{ 1, 2, 4, 8, 16, 32, 64, Geometrical progression.

Or, { 0, 1, 2, 3, 4, 5, 6, Logarithms,
{ 1, 3, 9, 27, 81, 243, 729, Geometric progression.

Or, { 0, 1, 2, 3, 4, 5 Logarithms,
{ 1, 10, 100, 1000, 10000, 100000 Geom. progress.

Where it is evident that the same indices answer for any ometric series, and therefore there may be an endless variety systems of logarithms to the same natural numbers by anging the second term 2, 3 or 10, &c., of geometrical se- s of whole numbers; and by interpolation the whole system numbers may be made to enter the geometric series and re- ive their proportional logarithms, whether intregers or cimals.

It also appears from the construction of these series, that if y two indices be added together, their sum will be the in- x of that number which is equal to the product of the two rms in the geometric series to which those indices belong. hus, the indices 2 and 3 being added together, make 5, and e product of 4 and 8, being the terms corresponding to those dices is 32, which is the number corresponding to the index 5.

In like manner, if any index be subtracted from another, the difference will be the index of that number which is equal to the quotient of the two terms to which those indices belong. Thus the index 6 — the index 4 is=2, and the terms corresponding to those indices are 64 and 16, whose quotient is=4, which is the number answering to the index 2.

For the same reason, if the logarithm of any number be multiplied by the index of its power, the product will be equal to the logarithm of that power. Thus, the index or logarithm of 4, in the above series is 2; and if this number be multiplied by 3, the product will be=6, which is the logarithm of 64, or the third power of 4.

And, if the logarithms of any number be divided by the index of its root, the quotient will be equal to the logarithm of that root. Thus, the index or logarithm of 64 is 6, and if this number be divided by 2, the quotient will be 3, which is the logarithm of 8, or the equal root of 64.

The logarithms most convenient for practice, are such as are adapted to a series increasing in a tenfold ratio, as in the last of the above forms, and are those which are usually found in most of the common tables on the subject.

2. *In a system of logarithms all numbers are considered as the powers of some one number, arbitrarily chosen, which is called the base of the system, and the exponent of that power of the base which is equal to any given number, is called the logarithm of that number.*

Thus, if a be the base of a system of logarithms, N any number, and x such that

$$N = a^x$$

then x is called the logarithm of N in the system whose base is a.

The base of the common system of logarithms, (called from their inventor "Briggs's Logarithms"), is the number 10. Hence since

$(10)^0=$	1	, 0 is the logarithm of	1	in this system.
$(10)^1=$	10	, 1 ————————	10	————
$(10)^2=$	100	, 2 ————————	100	————
$(10)^3=$	1000	, 3 ————————	1000	————
$(10)^4=$	10000	, 4 ————————	10000	————
&c. =	&c.	&c.		

From this it appears, that in the common system the logarithms of every number between 1 and 10, is some number beween 0 and 1, *i. e.* is 1 plus a fraction. The logarithm of every number between 10 and 100, is some number between 1 and 2, *i. e.* is 1 plus a fraction. The logarithm of every number between 100 and 1000, is some number between 2 and 3, *i. e.* is 2 plus a fraction, and so on.

In the common tables the fractional part alone of the logarithm is registered, and from what has been said above, the rule usually given for finding the *characteristic*, or, *index*, *i. e.* the integral part of the logarithm will be readily understood, viz. *The index of the logarithm of any number greater than unity is equal to one less than the number of integral figures in the given number.* Thus, in searching for the logarithm of such a number as 2970, we find in the tables opposite to 2970 the number 4727564; but since 2970 is a number between 1000 and 10000, its logarithm must be some number between 3 and 4, *i. e.* must be 3 plus a fraction; the fractional part is the number 4727564, which we have found in the tables, affixing to this the index 3, and interposing a decimal point, we have 3,4727564, the logarithm of 2970.

We must not, however, suppose that the number 3.4727564 is the exact logarithm of 2970, or that

$$2970=(10)^{3.4727564}$$

accurately. The above is only an approximate value of the logarithm of 2970; we can obtain the exact logarithm of very few numbers, but taking a sufficient number of decimals, we can approach as nearly as we please to the true logarithm, as will be seen when we come to treat of the construction of tables.

It has been shown that in Briggs' system the logarithm of 1 is 0, consequently, if we wish to extend the application of logarithms of fractions, we must establish a convention by which the logarithms of numbers less than 1 may be represented by numbers less than zero, *i. e.* by *negative numbers.*

Extending, therefore, the above principles to negative exponents, since

$\frac{1}{10}$ or $(10)^{-1}=0.1$, -1 is the logarithm of .1 in this system

$\frac{1}{100}$ or $(10)^{-2}=0.01$, -2 ——— .01 ———

$\frac{1}{1000}$ or $(10)^{-3}=0.001$. -3 ——— .001 ———

$\frac{1}{10000}$ or $(10)^{-4}=0.0001$, -4 ——— .0001. ———

&c. &c.

It appears, then, from the *convention*, that the logarithm of every number between 1 and .1, is some number between 0 and — 1; the logarithm of every number between .1 and .01, is some number between — 1 and — 2; the logarithm of every number between .01 and .001, is some number between — 2 and — 3; and so on.

From this will be understood the rule given in books, of tables, for finding the *characteristic* or *index* of the logarithm of a decimal fraction, viz. *The index of any decimal fraction is a negative number, equal to unity, added to the number of zeros immediately following the decimal point.* Thus, in searching for a logarithm of the number such as .00462, we find in the tables opposite to 462 the number 6646420; but since .00462 is a number between .001 and .0001. its logarithm must be some number between — 3 and — 4, *i. e.* must be — 3 plus a fraction, the fractional part is the number 6646420, which we have found in the tables, affixing to this the index — 3, and interposing a decimal point, we have — 3.6646420, the logarithm of .00462.

General Properties of Logarithms.

Let N and N′ be any two numbers, x and x' their respective logarithms, a the base of the system. Then, by def. (2),

$$N = a^x \text{- - - - - - -} \quad (1)$$
$$N' = a^{x'} \text{- - - - - - -} \quad (2)$$

I. Multiply equations (1) and (2) together,

$$N\,N' = a^x a^{x'}$$
$$= a^{x+x'}$$

∴ by def. 2, $x+x'$ is the logarithm of N N′, that is to say,

The logarithm of the product of the two or more factors is equal to the sum of the logarithms of these factors.

II. Divide equation (1) by (2),

$$\frac{N}{N'} = \frac{a^x}{a^{x'}}$$
$$= a^{x-x'}$$

∴ def. (2), $x - x'$ is the logarithm of $\frac{N}{N'}$, that is to say, *The logarithm of a fraction, or of the quotient of two numbers, is equal to the logarithm of the numerator minus the logarithm of the denominator.*

III. Raise both members of equation (1) to the power of n.

$$N^n = a^{nx}$$

∴ by def. (2), $n\,x$ is the logarithm of N^n, that is to say,

The logarithm of any power of a given number is equal to the logarithm of the number multiplied by the exponent of the power.

IV. Extract the n^{th} root of both members of equation.(1).

$$N^{\frac{1}{n}} = a^{\frac{x}{n}}$$

∴ by def. (2), $\frac{x}{n}$ is the logarithm of $N^{\frac{1}{n}}$ that is to say,

The logarithm of any root of a given number is equal to the logarithm of the number divided by the index of the root.

Combining the two last cases, we shall find,

$$N^{\frac{m}{n}} = a^{\frac{mx}{n}}$$

whence, $\frac{mx}{n}$ is the logarithm of $N^{\frac{m}{n}}$.

It is of the highest importance to the student to make himself familiar with the application of the above principles to algebraic calculations. The following examples will afford a useful exercise:

Ex. 1. $\log.\,(a.b.c.d.\dots) = \log.\,a + \log.\,b + \log.\,c + \log.\,d + \dots$

Ex. 2. $\log.\left(\frac{a\,b\,c}{d\,e}\right) = \log.\,a + \log.\,b + \log.\,c - \log.\,d - \log\,e.$

Ex. 3. $\log.\,(a^m b^n c^p \dots) = m \log.\,a + n \log.\,b + p \log.\,c \dots$

Ex. 4. $\log.\left(\frac{a^m b^n}{c^p}\right) = m \log.\,a + n \log.\,b - p \log.\,c \dots$

Ex. 5. $\log.(a^2 - x^2) = \log.(a+x).(a-x) = \log.(a+x) + \log.(a-x)$

Ex. 6. $\log.\,\sqrt{a^2 - x^2} = \frac{1}{2} \log.\,(a+x) + \frac{1}{2}\,(a-x)$

Ex. 7. $$\log.\,(a^3\sqrt[4]{a^3})=\log.\,a^3+\frac{1}{4}\log.\,a^3=3\log.\,a+\frac{3}{4}\log.\,a$$
$$=\frac{15}{4}\log.\,a$$

Ex. 8. $$\log.\,\sqrt[n]{(a^3-x^3)^m}=\frac{m}{n}\log.\,(a-x)+\frac{m}{n}\log.(a^2+ax+x^2)$$
$$=\frac{m}{n}\{\log.\,(a-x)+\log.\,(a+x+z)+\log.\,(a+x-z)\}$$
where $z^2=ax$

Ex. 9. $$\log.\,\sqrt{a^2+x^2}=\frac{1}{2}\{\log.\,(a+x+z)+\log.\,(a+x-z)\},$$
where $z^2=2ax$

Ex. 10. $$\log.\,\frac{\sqrt{a^2-x^2}}{(a+x)^2}=\frac{1}{2}\{\log.\,(a-x)-3\log.\,(a+x)\}$$

Let us resume the equation,

$$N=a^x$$

1°. If $a>1$, making $x=0$, we have $N=1$; the hypothesis $x=1$ gives $N=a$. As x passes from 0 up to 1, and from 1 up infinity, N will increase from 1 up to a, and from a up to infinity; so that x being supposed to pass through all intermediate values, according to the law of continuity, N increases also, but with much greater rapidity. If we attribute negative values to x. we have $N=a^{-x}$, or $N=\frac{1}{a^x}$ Here, as x increases, N diminishes, so that x being supposed to increase negatively, N will decrease from 1 towards 0, the hypothesis $x=\infty$ gives $N=0$.

2°. If $a<1$, put $a=\frac{1}{b}$, where $b>1$, and we shall then have $N=\frac{1}{b^x}$ or $N=b^x$, according as we attribute positive or negative values to x. We here arrive at the same conclusion as in the former case, with this difference, that when x is positive $N<1$, and when x is negative $N>1$.

3°. If $a=1$, then $N=1$. whatever may be the value x.

From this it appears, that,

1. *In every system of logarithms the logarithm of* 1 *is* 0, *and the logarithm of the base* is 1.

II. *If the base be* >1, *the logarithms of numbers* >1 *are positive, and the logarithms of numbers* <1 *are negative. The contrary takes place if the base be* <1.

III. *The base being fixed, any number has only one real logarithm; but the same number has manifestly a different logarithm for each value of the base, so that every number has an infinite number of real logarithms.* Thus, since $9^2=81$, and $3^4=81$, 2 and 4 are the logarithms of the same number 81, according as the base is 9 or 3.

IV. *Negative numbers have no real logarithms, for attributing to* x *all values from* $-\infty$ *up to* $+\infty$, *we find that the corresponding values of* N *are positive numbers only, from* 0 *up to* $+\infty$.

The formation of a table of logarithms consists in determining and registering the values of x which correspond to $N=1, 2, 3, \ldots$ in the equation,

$$N=a^x$$

If we suppose $m=a^\alpha$, making

$x=0, \alpha, 2\alpha, 3\alpha, 4\alpha, 5\alpha$, - - - - - *logarithms.*
$y=1, m, m^2, m^3, m^4, m^5$, - - - - - *numbers.*

the logarithms increase in arithmetical progression, while the numbers increase in geometrical progression; 0 and 1 being the first terms of the corresponding series, and the arbitrary numbers α and m the common difference and the common ratio.

We may, therefore, consider the systems of values of x and y, which satisfy the equation $N=a^x$, as ranged in these two progressions.

In order to solve the equation

$$c=a^x$$

where c and a are given, and where x is unknown, we equate the logarithms of the two members, which gives us

$$\log. c=x \log. a$$

Whence,

$$x=\frac{\log. c}{\log. a}$$

To determine the value of x in the equation

$$Aa^x+Ba^{x-b}+Ca^{x-c}+\ldots\ldots=P$$

we have

$$a^x\left(A+\frac{B}{a^b}+\frac{C}{a^c}\quad+\ldots\ldots=P\right.$$

Or,

$$Qa^x = P$$

$$\therefore \quad x = \frac{\log. P - \log. Q}{\log. a}$$

If we have an equation $a^z = b$, where z depends u
unknown quantity x, and we have

$$z = Ax^n + Bx^{n-1} + \dots.$$

Since $z = \frac{\log. b}{\log. a} = K$ some known number, the problem d
upon the solution of the equation of the n^{th} degree.

$$K = Ax^n + Bx^{n-1} + \dots.$$

For example, let

$$4\left(\frac{2}{3}\right)^{x^2-5x+4} = 9$$

Hence,

$$(x^2-5x+4)\log.\left(\frac{2}{3}\right) = \log.\frac{9}{4}$$

$$\therefore \quad x^2-5x+4 = -2$$

an equation of the second degree, from which we find
$x = 3$.

To find the value of x from the equation

$$b^{n-\frac{a}{x}} = c^{mx} f^{x-p}$$

Taking the logarithms of each member,

$$(n - \frac{a}{x}) \log. b = m\,x \log. c + (x-p)\log. f$$

Or,

$$(m \log. c + \log. f)\,x^2 - (n \log. b + p \log. f)\,x + a \ \text{lo}$$

a quadratic equation, from which the value of x may
termined.

In like manner, from the equation

$$c^{mx} = ab^{nx-1}$$

we find

$$x = \frac{\log. a - \log. b}{m \log. c - n \log. b}$$

Equations of this nature are called *Exponential Equat*

Let N and N + 1 be two consecutive numbers, the
ence of their logarithms, taken in any system, will be

$$\log. (N+1) - \log. N = \log. \left(\frac{N+1}{N}\right) = \log. (1 + \frac{1}{N}$$

a quantity which approaches to the logarithm of 1, or

ortion as $\frac{1}{N}$ decreases, that is, as N increases. Hence it
ars, that

e difference of the logarithms of two consecutive numbers
s in proportion as the numbers themselves are greater.

hen we have calculated a table of logarithms for
ase a, we can easily change the system, and calculate
er table for a new base b.

$=b^x$, x is the log of c in the system whose base is b;
g the logs. in the known system, whose base is a, we have

$$x=\frac{\log. c}{\log. b}=\log. c\left(\frac{1}{\log. b}\right) \;.\;.\;.\;.\;.\; \text{(A)} \quad \text{hence}$$

e log. of c *in the system whose base is* b, *is the quotient*
g from dividing the log. of c *by the log. of the new base* b,
hese last logs. being taken in the system whose base is a.

order ∴ to have x the log. of c in the new system, we
multiply log. c by $\frac{1}{\log. b}$; this last factor $\frac{1}{\log. b}$ is con-
for all numbers, and is called the *Modulus*; that is to say,
divide the logs. of the same number c taken in two sys-
the quotient will be invariable for these systems, what-
may be the value of c, and will be the modulus, the con-
multiplier which reduces the first system of logs. to the
d.

we find it inconvenient to make use of a log. calculated to
ase 10, we can in this manner, by aid of a set of tables
lated to the base 10, discover the logarithm of the given
er in any required system.

r example, let it be required, by aid of Briggs' tables, to
he log. of $\frac{2}{3}$ in a system whose base is $\frac{5}{7}$

t x be the log. sought, then by (A)

$$x=\frac{\log. \frac{2}{3}}{\log. \frac{5}{7}}$$

$$=\frac{\log. 2-\log. 3}{\log. 5-\log. 7}$$

Taking these logs. in Briggs' system, and reducing, we find.

$$=\frac{-0.17609125}{-0.14612804}$$

$$=1.2050476 = \text{log. } \frac{2}{3} \text{ to base } \frac{5}{7}.$$

Similarly, the log. of $\frac{2}{3}$, in the system whose base is $\frac{3}{2}$, is

$$x=\frac{\text{log. }2-\text{log. }3}{\text{log. }3-\text{log. }2}$$
$$=-1$$

which is manifestly the true result; for in this case the general equation $N=a^x$ becomes $\frac{2}{3}=\left(\frac{3}{2}\right)^x=\left(\frac{2}{3}\right)^{-x}$, and x is evidently $=-1$.

In a system whose base is a, we have

$$n=a^{\text{log. }n}$$

for, by the definition of a log. in the equation $n=a^x$, x is the log. n.

In like manner,

$$n^h = a^{\text{log. }(n^h)} = n^{h\text{ log. }n}.$$

EXAMPLES FOR EXERCISE.

Ex. 1. Given $2^{2x}+2^x=12$ to find the value of x.

Ans. $x=1.584962$, or $x=\text{log. }(-4)\div\text{log. }2$.

Ex. 2. Given $x+y=a$, and $m^{(x-y)}=n$ to find x and y.

Ans. $x=\frac{1}{2}\{a+\text{log. }n\div\text{log. }m\}$ and $y=\frac{1}{2}\{a-\text{log. }n\div\text{log. }m\}$.

Ex. 3. Given $m^x n^x=a$, and $hx=ky$ to find x and y.

$$\text{Ans.}\begin{cases} x=\text{log. }a\div(\text{log. }m+\text{log. }n) \\ \text{and } y=\frac{h}{k}\text{log. }a\div(\text{log. }m+\text{log. }n). \end{cases}$$

To find the logarithm of any given number.

Let N be any given number whose logarithm is x, in a system whose base is a; then

$$a^x=N \text{ and } a^{xz}=N^z;$$

hence, by the exponential theorem, we have from the last equation

$$1+Axz+A^2\frac{x^2z^2}{1.2}+\ldots=1+A_1z+A_1{}^2\frac{z^2}{1.2}+\ldots$$

equating the coefficients of z, we get $Ax=A_1$; hence

$$x=\frac{A_1}{A}=\frac{(N-1)-\frac{1}{2}(N-1)^2+\frac{1}{3}(N-1)^3-\ldots}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\ldots}$$

use $A=(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\quad.\quad.\quad.\quad.$ in the nsion of a^{xz}.

$A_1=(N-1)-\frac{1}{2}(N-1)^2+\frac{1}{3}(N-)^3-\quad.\quad.\quad.$ in the nsion of N^z.

find the logarithm of a number in a converging series.

e have seen that if $a^x=N^1$, then

$$=\frac{(N^1-1)-\frac{1}{2}(N^1-1)^2+\frac{1}{3}(N^1-1)^3-\frac{1}{4}(N^1-1)^4+\ldots}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+\ldots}$$

ow the reciprocal of the denominator is the *modulus* of ystem ; and, representing the modulus by M, we have

$$g.\ N^1=M\{(N^1-1)-\tfrac{1}{2}(N^1-1)^2+\tfrac{1}{3}(N^1-1)^3-\tfrac{1}{4}(N^1-1)^4+\}$$

ut $N^1=1+n$; then $N^1-1=n$, and we have

$$\log.\ (1+n)=M(+n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4+\tfrac{1}{5}n^5-\ldots\}$$

imilarly $\log.\ (1-n)=M(-n-\frac{1}{2}n^2-\frac{1}{3}n^3-\frac{1}{4}n^4-\frac{1}{5}n^5-\ldots)$

$$\log.\ (1+n)-\log.\ (1-n)=2M(n+\tfrac{1}{3}n^3+\tfrac{1}{5}n^5+\tfrac{1}{7}n^7+\ldots)$$

$$\text{or } \log.\frac{1+n}{1-n}=2M(n+\tfrac{1}{3}n^3+\tfrac{1}{5}n^5+\tfrac{1}{7}n^7+\ldots)$$

$$\text{Put } n=\frac{1}{2P+1};\ \text{then } 1+n=\frac{2P+2}{2P+1},\ 1-n$$
$$=\frac{2P}{2P+1},\ \text{and } \frac{1+n}{1-n}=\frac{P+1}{P};$$

sequently

$$.(P+1)-\log.P=2M\left\{\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\right\}$$

$$g(P+1)=\log P+2M\left\{\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\right\}$$

ence, if log. P be known, the log of the next greater num- can be found by this rapidly converging series.

To find the Napierian logarithms of numbers.

the preceding series, which we have deduced for log. 1), we find a number M, called the *modulus* of the sys- ; and we must assign some value to this number before an compute the value of the series. Now, as the value is arbitrary, we may follow the steps of the celebrated Napier, the inventor of logarithms, and assign to M the lest possible value. This value will therefore be unity; we have

$$\text{log. }(P+1)=\text{log. }P+2\left\{\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\right.$$

Expounding P successively by 1,2,3,4, &c., we find

$$2=\text{log. }1+2\left(\frac{1}{3}+\frac{1}{3\cdot3^3}+\frac{1}{5\cdot3^5}+\frac{1}{7\cdot3^7}+\cdots\right)=\ .69$$

$$\text{log. }3=\text{log. }2+2\left(\frac{1}{5}+\frac{1}{3\cdot5^3}+\frac{1}{5\cdot5^5}+\frac{1}{7\cdot5^7}+\cdots\right)=1.09$$

$$\text{log. }4=2\text{ log. }2\quad-\quad-\quad-\quad-\quad-\quad-\quad=1.38$$

$$\text{log. }5=\text{log. }4+2\left(\frac{1}{9}+\frac{1}{3\cdot9^3}+\frac{1}{5.9^5}+\frac{1}{7\cdot9^7}+\cdots\right)=1.60$$

$$\text{log. }6=\text{log. }2+\text{log. }3\quad-\quad-\quad-\quad-\quad=1.79$$

$$\text{log. }7=\text{log. }6+2\left(\frac{1}{3}+\frac{1}{3.13^3}+\frac{1}{5.13^5}+\cdots\cdots\right)=1.94$$

$$\text{log. }8=\text{log. }2+\text{log. }4,\text{ or }3\text{ log. }2\quad-\quad-\quad=2.07$$

$$\text{log. }9=2\text{ log. }3\quad-\quad-\quad-\quad-\quad-\quad=2.19$$

$$\text{log. }10=\text{log. }2+\text{log. }5\quad-\quad-\quad-\quad-\quad=2.30$$

In this manner the Napierian logarithms of all number be computed.

To find the common logarithms of numbers.

Let a^x = N and b^y = N ; then we have

x = log. N to the base a, or x=log. $_a$N

y = log. N to the base b, or y=log. $_b$N

hence, log. $_a$N=log. $_a b_y$ =log. $_a b$ (Gen. properties logarit

$$\therefore\ x=y\text{ log. }_a b$$

$$\text{and }y=\frac{1}{\text{log. }_a b}\cdot x$$

and by means of this equation we can pass from one syst logs. to another, by multiplying x, the log. of any numb the system whose base is a, by the reciprocal of log. b i same system ; and thus we shall obtain the log. of the number in the system whose base is b.

Let the two systems be the Napierian and the comm which the base of the former is ϵ=2.718281828 . . . an base of the latter is b=10, the base of the common syste arithmetic ; then we have b=10, and a=ϵ=2.71828182 and consequently if N denote any number, we shall have

$$\log._{10}N=\frac{1}{\log._{\epsilon}10}\cdot\log\ _{\epsilon}N\text{ ; that is,}$$

$$\text{com. log N}=\frac{\text{nap. log.N}}{\text{nap.log.10}}=\frac{\text{nap. log.N}}{2.3025851}=;43429448\times\text{nap. log.N;}$$

and the modulus of the common system is, therefore,

$$M=\frac{1}{2.3025851}=.43429448\ \therefore\ 2M=.86858896$$

Hence, to construct a table of common logarithms, we have

$$\log(P+1)=\log P+.86858896\left\{\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}\cdots\right.$$

Expounding P successively by 1,2,3, &c., we get

$$\log.\ 2=.86858896\left(\frac{1}{3}+\frac{1}{3^4}+\frac{1}{5.3^6}+\ldots\right)$$

$$=.86858896\times.6931472\ \text{- - -}\ =.3010300$$

$$\log.\ 3=\log.2+.86858896\left(\frac{1}{5}+\frac{1}{3.5^3}+\frac{1}{5^6}+\ldots\right)=.4771213$$

$\log.\ 4=2\log.\ 2$ - - - - - - - $=.6020600$

$\log.\ 5=\log.\ \frac{10}{2}=\log.\ 10-\log 2=1-\log.\ 2$ - $=.6989700$

$\log.\ 6=\log.\ 2+\log.\ 3$ - - - - - - $=.7781513$

$$\log.\ 7=\log.6+\cdot86858896\left(\frac{1}{13}+\frac{1}{3.13^3}+\frac{1}{5.13^6}+..\right)=.8450980$$

$\log.\ 8=\log.2^3=3\log.\ 2$ - - - - $=.9030900$

$\log.\ 9=\log.3^2=2\log.\ 3$ - - - - $=.9542426$

$\log.10=$ - - - - - - - - $=1.0000000$

&c. &c.

$$\text{Since }\log.\frac{1+n}{1-n}=2M\left(n+\tfrac{1}{3}n^3+\tfrac{1}{5}n^5+\tfrac{1}{7}n^7+\ldots\right)$$

$$\text{let }\frac{1+n}{1-n}=P\text{ ; then }1+n=P(1-n)\text{ or }n=\frac{P-1}{P+1}$$

$$\therefore\ \log.\ P=2M\left\{\frac{P-1}{P+1}+\frac{1}{3}\cdot\left(\frac{P-1}{P+1}\right)^3+\frac{1}{5}\cdot\left(\frac{P-1}{P+1}\right)^5+\ldots\right\}$$

and thus we have a series for computing the logs. of all numbers, without knowing the log. of the previous number.

EXAMPLES IN LOGARITHMS.

(1.) Given the log. of $2=0.3010300$, to find the logs. of 25 and .0125.

Here $25=\frac{100}{4}=\frac{10^2}{2^2}$;

therefore $\log. 25=2 \log. 10-2 \log. 2=1.3979400$

Again $.0125=\frac{125}{10000}=\frac{1}{80}=\frac{1}{10\times 2^3}$

$\therefore \log. .0125=\log. 1-\log. 10-3 \log. 2=-1-3 \log. 2$
$=\bar{2}.0969100$

(2.) Calculate the common logarithm of 17.

Ans. 1.2304489.

(3.) Given the logs. of 2 and 3 to find the logarithm of 12.5.

Ans. $1+2 \log. 3-2 \log. 2$.

(4.) Having given the logs. of 3 and .21, to find the logarithm of 83349.

Ans. $6+2 \log. 3+3 \log. .21$.

ON EXPONENTIAL EQUATIONS.

An *exponential equation* is an equation in which the unknown quantity appears in the form of an exponent or index; thus, the following are exponential equations :

$$a^x = b,\ x^x = a,\ a^{b^x} = c,\ x^{x^x} = a, \text{ \&c.}$$

When the equation is of the form $a^x = b$, or $a^{b^x} = c$, the value of x is readily obtained by logarithms, as we have already seen above. But if the equation be of the form $x^x = a$, the value of x may be obtained by a rule of *approximation*, as in the following example :

Ex. Given $x^x = 100$, to find an approximate value of x.

The value of x is evidently between 3 and 4, since $3^3 = 27$ and $4^4 = 256$; hence, taking the logs. of both sides of the equation, we have

$$x \log. x = \log. 100 = 2$$

First, let $x_1 = 3.5$; then	Second, let $x_2 = 3.6$; then
3.5 log. 3.5= 1.9042380	3.6 log. 3.6= 2.0026890
true no. = 2.0000000	true no. = 2.0000000
error =—.0957620	error = +.0026890

Then, as the difference of the results is to the difference of the assumed numbers, so is the least error to a correction of the assumed number corresponding to the least error ; that is,

$$.098451 : .1 :: .002689 : .00273 ;$$

hence $x=3.6-.00273=3.59727$, nearly

Again, by forming the value of x^x for $x=3\cdot5972$, we find the error to be—.0000841, and for $x=3.5973$, the error is + .0000149 ;

hence, as $.000099 : .0001 :: .0001 : .0000151$;

therefore $x=3.5973-.0000151=3.5972849$, the value nearly.

EXAMPLES FOR PRACTICE.

(1.) Find x from the equation $x^x = 5$ Ans. 2.129372.

(2.) Solve the equation $x^x = 123456789$ Ans. 8.6400268.

(3·) Find x from the equation $x^x = 2000$. Ans. 4.8278226.

Since the properties of logarithms afford great facilities in performing complicated arithmetical operations upon large numbers, it becomes desirable to have the logarithms of sines, cosines, tangents, &c. computed and arranged in tables ; but most of these numbers being less than unity, their logarithms would, of course, be negative. To avoid this inconvenience, all the trigonometrical functions calculated in the manner explained in Chap. IV, are multiplied by a large number, and, the operation being performed upon all, their relative value is not altered. This number may, of course, be any whatever, provided it be so large, that, when the numerical values of trigonometrical quantities are multiplied by it, their logarithms may be positive numbers.

The number employed for this purpose in the common tables is 10000000000 or 10^{10}, which is usually represented by the symbol R.

The sine of 1′, as computed above, is

$$\text{Sin. } 1' = .0002908882 \ldots.$$

a number much smaller than unity, and whose logarith would consequently be negative.

When multiplied by 10^{10} it becomes

$$= 2908882 \ldots.$$

a number whose logarithm is 6.4637261, and consequently find in our tables, $\log. \sin. 1' = 6.4637261.$

A table constituted upon this principle is called a *Table* *Logarithmic Sines, Cosines, Tangents,* &c. and by this nearl all the practical operations of trigonometry are usually pe formed.

It is manifest, from these remarks, that before we can appl formulæ deduced in the preceding chapters to practical pu poses, we must transform them in such a manner as to rend the several trigonometrical quantities identical with those re gistered in our tables. The sines, cosines, &c. we have hith erto employed, are called *Trigonometrical qnantities calculate to a radius unity;* those registered in the tables, *Trigonome trical quantities calculated to radius* R.

The problem to be solved therefore is

To transform an expression calculated to a radius unity, t *another calculated to a radius* R

Let us represent sin. θ to radius unity by m.
.. .. .. R by n.

Then the relation between them is

$$n = R\,m$$

$$m = \frac{n}{R}$$

and so for all the other trigonometrical quantities.

Hence, *in order to transform an expression calculated to radius unity, to another calculated to radius* R, *we must divi each of the trigonometrical quantities by* R.

If any of the trigonometrical quantities enter in the squar cube, &c. these must of course be divided by R^2 R^3, &c. —

As observed above, R may be any given number whateve the number usually employed in the ordinary tables being 10^{10} and therefore

$$\log. R = 10$$

Take as an example such an expression as

$$a \sin. \theta = b \tan.^2 \varphi$$

in order to reduce this to an expression which we can com pute by our tables we must, according to the above rule, d vide each of the trigonometrical quantities by the prop power of R: the expression then becomes

$$a \frac{\sin. \theta}{R} = b \frac{\tan.^2 \varphi}{R^2}$$

clearing of fractions,

$$a \text{ R} \sin.\theta = b \tan.^2 \varphi$$

r,

$$g.\ a + \log. \text{R} + \log. \sin. \theta = \log. b + 2 \log. \tan. \varphi$$

pression which may be calculated by the tables.

he expression calculated to radius unity be of the form

$$m = \frac{\sin.\ \theta}{\sin.\ \varphi}$$

ires no modification, for if we divide both terms of the n $\frac{\sin.\ \theta}{\sin.\ \varphi}$ by R, we shall not alter its value.

need not prosecute this subject farther, as numerous ex- s of these transformations will occur at every step in the ding chapters.

CHAPTER VI.

ON THE SOLUTION OF RIGHT-ANGLED TRIANGLES.

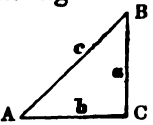
ry plane triangle being considered to consist of 6 parts, ree sides, and the three angles, if any of these three parts en, we can, in general, determine the remaining parts gonometry.

ight-angled triangles, the right angle is always known, erefore any two other parts being given, we can, in ge- determine the rest. We shall thus have five different

When one of the acute angles and the hypothenuse is

When one of the acute angles and a side is given.
When the hypothenuse and one side is given.
When the two sides are given.
When the two acute angles are given.

t ABC be a right-angled triangle, C the right angle.



t the sides opposite to the angles A and B noted respectively by a, b, and let the hy- nuse be called c.

se 1, Given A, c, required B, a. b.

ce C is a right angle

$$\text{A}+\text{B}=90^\circ$$

$$\therefore \quad \text{B}=90^\circ-\text{A} \text{ whence B is known} \cdots (1)$$

Chap. III. prop. 1,

$$a=c \sin.\ \text{A}$$

Adapting this expression to computation by the tabl

$$\log. a = c\,\frac{\sin. A}{R}$$

$\therefore \log. a = \log. c + \log. \sin. A - \log. R$, whence a is kno

In like manner,

$$b = c\,\frac{\cos. A}{R}$$

$\therefore \log. b = \log. c + \log. \cos. A - \log. R$, whence b is kn

If B, c are given, and A, a, b required, we shall ha
cisely in the same manner,

$A = 90^\circ - B$ - - - - - - - - - - - -

$\log. a = \log. c + \log. \cos. B - \log. R$ - - - - - -

$\log. b = \log. c + \log. \sin. B - \log. R$ - - - - - -

Case 2. Given A, a, required B, b, c.

$B = 90^\circ - A$, whence B is known - - - -

$b = a \cot. A$

Adapting the expression to computation by the table

$$b = a\,\frac{\cot. A}{R}$$

$\log. b = \log. a + \log. \cot. A - \log. R$, whence b is kno

Again,

$a = c \sin. A$

$$\therefore\ c = \frac{a}{\sin. A}$$

Adapting it to computation,

$$c = R\,.\,\frac{a}{\sin. A}$$

$\therefore \log. c = \log. R + \log. a - \log. \sin. A$, whence c is kno

If A, b, be given, B, a, c, we shall have in like mann

$B = 90^\circ - A$ - - - - - - - - - - - - -

$\log. a = \log. b + \log. \tan. A - \log. R$ - - - - -

$\log. c = \log. R + \log. b - \log. \cos. A$ - - - - -

If B, b be given, and A, a, c required,

$A = 90^\circ - B$ - - - - - - - - - - -

$\log. a = \log. b + \log. \cot. B - \log. R$ - - - - -

$\log. c = \log. R + \log. b - \log. \sin. B$ - - - - - -

If B, a be given, and A, b, c, required.

$A = 90^\circ - B$ - - - - - - - - - - - -

$\log. b = \log. a + \log. \tan. B - \log. R$ - - - -

$\log. c = \log. R + \log. a - \log. \cos. B$ - - - -

Case 3. Let a, c be given, required b, A, B.

$$b^2=c^2-a^2$$
$$=(c+a)\ (c-a)$$

$\therefore 2\log. b=\log. (c+a)+\log. (c-a)$ whence b is known - - (19)

$$\sin. A=\frac{a}{c}$$

Adapting the expression to computation.

$$\frac{\sin. A}{R}=\frac{a}{c}$$

$\therefore \log. \sin. A=\log. R+\log. a-\log. c$, whence A is known, (20)

So also,

$$\frac{\cos. B}{R}=\frac{a}{c}$$

$\therefore \log. \cos. B=\log. R+\log. a-\log. c$, whence B is known, (21)

If b, c be given, and a, A, B required, we shall have

$2\log. a=\log. (c+b)+\log. (c-b)$ - - - - - - (22)

$\log. \cos. A=\log. R+\log\ b-\log. c$ - - - - - - - (23)

$\log. \sin. B=\log. R+\log. b-\log. c$ - - - - - - - (24)

Case 4. Let a, b be given, required A, B, c.

$$\tan. A=\frac{a}{b}$$

Adapting the expression to computation.

$$\frac{\tan. A}{R}=\frac{a}{b}$$

$\therefore \log. \tan. A=\log. R+\log. a-\log. b$, whence A is known, (25)

So also,

$$\frac{\tan. B}{R}=\frac{b}{a}$$

$\therefore \log. \tan. B=\log. R+\log. b-\log. a$, whence B is known, (26)

$c=\sqrt{a^2+b^2}$, whence c is known, - - - - (27)

Case 5. Given A, B, required a, b, c.

It is manifest that this case does not admit of a solution, for *any number* of unequal similar triangles may be constructed, having their angles equal to the angles A, B, C.

We shall conclude this chapter by giving some numerical examples.

Example 1. Given A=26° 41′ 6″, c=6539.76 yards, required a.

Then by (2).

$$\log. a = \log. c + \log. \sin. A - \log. R.$$

$$\begin{aligned} \text{By the tables} \quad \log. c &= 3.8155618 \\ \log. \sin. A &= 9.6523286 \\ \hline &\ 13.4678904 \\ \log. R &= 10. \\ \hline \therefore \quad \log. a &= 3.4678904 \end{aligned}$$

The number in the tables corresponding to the logarithm 3.4678904 is found to be 2936.91.

$$\therefore \quad a = 2936.91 \text{ yards.}$$

In like manner, the side b may be determined, if required.

Example 2. Given $c = 6539.76$ yards, $a = 2936.91$ yards, required b, A, B.

By (19).

$$2 \log. b = \log. (c+a) + \log. (c-a) \qquad \begin{aligned} c+a &= 9476.67 \\ c-a &= 3602.85 \end{aligned}$$

By the tables,

$$\begin{aligned} \log. (c+a) &= 3.9766557 \\ \log. (c-a) &= 3.5566462 \\ \hline \therefore \quad 2 \log. b &= 7.5333019 \\ \log. b &= 3.7666509 \end{aligned}$$

The number in the tables corresponding to the logarithm 3.7666509 is 5843.2

$$\therefore \quad b = 5843.2 \text{ yards.}$$

To determine A we have (20).

$$\log. \sin. A = \log. R + \log. a - \log. c$$

$$\begin{aligned} \text{By the tables,} \quad \log. a &= 3.4678904 \\ \log. R &= 10. \\ \hline &\ 13.4678904 \\ \log. c &= 3.8155618 \\ \hline \therefore \quad \log. \sin. A &= 9.6523286 \end{aligned}$$

On referring to our tables, we shall find that the angle whose logarithmic sine is 9.6523286 is 26° 41′ 6″, which is consequently the value of A.

A being known, B is determined at once by subtracting the value of A from 90°, or B may be determined independently of A by (21).

$$\log. \cos. B = \log. R + \log. a - \log. c.$$

Example 3. Let a, b, in the last number be given, and c required.

Then by (27).

$$c=\sqrt{a^2+b^2} \text{ or } c^2=a^2+b^2$$

The calculation in this case is not so simple, for the quantity under the radical cannot be easily adapted to logarithmic calculation.

We have, log. $a^2=6.9357808$ $\therefore$ $a^2=$ 8625400

log. $b^2=7.5333019$ $\therefore$ $b^2=$ 34143000

$\therefore$ $c^2=$ 42768400

$\therefore$ log. $c^2=7.6311230$

log. $c=3.8155615$

$c=6539.76$

Example 4. Given $c=6512.4$ yards, $b=6510.6$, to find A.

By (23).

log. cos. A=log. R+log. b—log. c

Now, log. R= 10.

log. b = 3.8136210

13.8136210

log. c= 3.8137411

$\therefore$ log. cos. A= 9.9998799

$\therefore$ A= 1° 20′ 50″

Upon inspecting the tables that are calculated to seven places of decimals only, it will be seen that, when the angles becomes very small, the cosines differ very little from each other. The same remark applies, of course, to the sines of angles nearly 90°. In cases, therefore, where great accuracy is required, we may commit an important error by calculating a small angle from its cosine, or a large one from its sine. We must consequently endeavor to avoid this, by transforming our expression by help of the relations established in chapter first and second.

In the example before us, A is a small angle which has been calculated from its cosine; we must therefore, if possible, calculate this angle by means of its sine, or some other trigonometrical function.

Now, by formula (j), chap. II. we have generally

$$\sin. \frac{A}{2}=\sqrt{\frac{1-\cos. A}{2}}$$

In the present case, cos. $A=\frac{a}{c}$, substituting this in the above equation,

$$\sin.\frac{A}{2}=\sqrt{\frac{c-a}{2a}}$$

$$\therefore \quad \log.\sin.\frac{A}{2}=\tfrac{1}{2}\log.(c-a)-\tfrac{1}{2}\log.2\,a+\log.R.$$

From which we find,

$$\frac{A}{2}=40'\ 24''$$

$$\text{And} \quad \therefore \quad A=1^\circ\ 20'\ 48''$$

Instead of 1° 20′ 50″, as obtained by the former process.

No angle which is nearly 90° ought to be calculated from its tangent, for the tangents of all large angles increase with so much rapidity, that the results derived from the column of proportional parts found in the tables cannot be depended on as accurate.

CHAPTER VII.

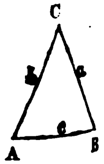
ON THE SOLUTION OF OBLIQUE ANGLED TRIANGLES.

Six different cases present themselves.

1. When two angles and the side between them are given.
2. When two angles and the side opposite to one of them are given.
3. When two sides and the included angle are given.
4. When two sides and the angle opposite to one of them are given.
5. When the three sides are given.
6. When the three angles are given.

Let A, B, C be a plane triangle.

Let the angles be denoted by the large letters A, B, C, and the sides opposite to these angles by the corresponding small letters a, b, c.



Case 1. Given A, B, c, required C, a, b.

Since $A+B+C=180^\circ$

$C=180^\circ-(A+B,)$ whence C is known.

C being thus determined, we have, by chap. III. prop. 2,

$$\frac{a}{c}=\frac{\sin. A}{\sin. C}$$

$$a=c.\frac{\sin. A}{\sin. C}$$

xpression which is in a form adapted to computation by ables.

og. a=log. c+log. sin. A−log. sin. C, whence a is known.

n,

$$\frac{b}{c}=\frac{\sin. B}{\sin. C}$$

$$b=c.\frac{\sin. B}{\sin. C}$$

og. b=log. c+log. sin. B—log. sin. C, whence b is known.

any other two angles and the side between them be given, ay determine the remaining angle and sides in a manner sely similar.

se 2. Given A, B, a, required C, b, c.

Since A+B+C=180°

$$\therefore \quad C=180°—(A+B,) \text{ whence C is known.}$$

Again, $$\frac{b}{a}=\frac{\sin. B}{\sin. A}$$

$$\therefore \quad b=a.\frac{\sin. B}{\sin. A}$$

log. b=log. a+log. sin. B—log. sin. A, whence b is known.

, C being known,

$$\frac{c}{a}=\frac{\sin. C}{\sin. A}$$

$$\therefore \quad c=a\frac{\sin. C}{\sin. A}$$

log. c=log. a+log. sin. C—log. sin. A, whence c is known.

any two other angles and the side opposite to one of them given, the remaining angle and sides may be determined manner precisely similar.

ase 3. Given a, b, C, required A, B, c.

y prop. 3, chap. III.

$$\frac{\tan. \frac{A+B}{2}}{\tan. \frac{A—B}{2}}=\frac{a+b}{a—b} \quad - - - - - - - - - - \quad (1)$$

Now, $A+B+C=180^\circ$

$\therefore \quad \frac{A+B}{2}=90^\circ-\frac{C}{2}$

$\therefore \quad \tan.\frac{A+B}{2}=\tan.(90^\circ-\frac{C}{2})$

$=\cot.\frac{C}{2}$

Substituting this value of $\tan.\frac{A+B}{2}$ in (1.)

$$\frac{\cot.\frac{C}{2}}{\tan.\frac{A-B}{2}}=\frac{a+b}{a-b}$$

$$\therefore \quad \tan.\frac{A-B}{2}=\frac{a-b}{a+b}\cot.\frac{C}{2}$$

$$\log.\tan.\frac{A-B}{2}=\log.(a-b)+\log.\cot.\frac{C}{2}-\log.(a+b)$$

And we can thus calculate the value of the angle $\frac{A-B}{2}$ from our tables; let the angle thus found be called φ $\therefore$ $A-B=2\varphi$.

Now $A+B=180^\circ-C$

And $A-B=2\varphi$

$\therefore$ adding and subtracting $A=90^\circ+\varphi-\frac{C}{2}$

$B=90^\circ-(\varphi+\frac{C}{2})$

The angles A and B will thus become known, and, these being determined, we can find the side c from the relation,

$$\frac{c}{a}=\frac{\sin.C}{\sin.A}$$

$$c=a.\frac{\sin.C}{\sin.A}$$

$$\therefore \quad \log.c=\log.a+\log.\sin.C-\log.\sin.A$$

If a, c, B, or b, c, A be given, the remaining angles and side may be determined in a similar manner by aid of the formula (j) in chap. III.

Case 4. Given a, b, A to determine B, C, c.

$$\frac{\sin.\ B}{\sin.\ A}=\frac{b}{a}$$

$$\therefore \quad \sin.\ B=\sin.\ A\,.\,\frac{b}{a}$$

$$\log.\ \sin.\ B=\log.\ \sin.\ A+\log.\ b-\log.\ a,$$ whence c is known.

B being known, C = 180° —(A+B,) whence C is known.

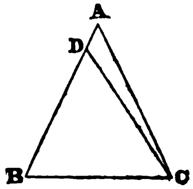
C being known, $$\frac{c}{a}=\frac{\sin.\ C}{\sin.\ A}$$

$$c=a\,.\,\frac{\sin.\ C}{\sin.\ A}$$

$$\therefore \log.\ a=\log.\ \sin.\ C-\log.\ \sin.\ A,$$ whence b is known.

If any two other sides and the angle opposite to one of them given, the remaining angles and side may be determined a manner precisely similar.

It must be remarked, that, in the above case, we determine e angle B from the logarithm of its sine; but since the sine any angle, and the sine of its supplement are equal to one nother, and since it is not always possible for us to ascertain *priori* whether the angle B is acute or obtuse, the solution ill be sometimes ambiguous.



In fact, two different and unequal triangles ay be constructed, having two sides and e angle opposite to one of those sides in e triangle, equal to the corresponding sides d angle of the other; one of these trianes will be obtuse-angled, and the other ute-angled, and the angles opposite the reining given sides in each will be supplental.

Thus let ABC, be a plane triangle.

With centre C and radius equal to CB describe a circle cutg AB in D.

Join CD.

Then it is manifest that the two unequal triangles CBA, A, have the two sides CB, CA of the one, equal to the two es CD, CA of the other, and the angle A, opposite the equal es CB, CD, in each, common.

It is manifest from this, that it is impossible to determine gealty, from the data of this case, which of the two triangles he solution of the problem. There are certain considerais, however, by which the ambiguity may sometimes be reved.

1. If the angle be obtuse, then both of the remaining angles must be acute, and the species of B will be determined.

2. If the given angle be acute, but the side opposite the given angle greater than the given side opposite the required angle, then the required angle is acute. For since in every triangle the greater side has the greater angle opposite to it, and since the side opposite to the given angle, which is acute, is greater than the side opposite to the required angle, it follows, *a fortiori*, that the required angle is acute.

But if the given angle be acute, and the side opposite to the given angle less than the side opposite to the required angle, then we have no means of ascertaining the species of the required angle, and the solution in this case is ambiguous.

Case 5. Given the three sides, a, b, c, required the three angles A, B, C.

By formula (ε) chap. III.

$$\sin. A = \frac{2}{bc} \cdot \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

$$\sin. B = \frac{2}{ac} \cdot \sqrt{s\,(s-a)\,(s-b)\;s-c)}$$

$$\sin. C = \frac{2}{ab} \cdot \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

Adapting these expressions to computation by the tables, and taking the logs.

log. sin. A

$$=\log. R+\log. 2+\tfrac{1}{2}\left\{\log. s+\log. (s-a)+\log. (s-b)+\log. (s-c)\right\}-\log b-\log. c$$

log. sin. B

$$=\log. R+\log. 2+\tfrac{1}{2}\left\{\log. s+\log. (s-a)+\log. (s-b)+\log. (s-c)\right\}-\log. a-\log. c$$

log. sin. C

$$=\log. R+\log. 2+\tfrac{1}{2}\left\{\log. s+\log. (s-a)+\log. (s-b)+\log. (s-c)\right\}-\log. a-\log. b$$

Whence the three angles are known.

The three angles may also be obtained from any of the groups of formulæ (σ), (ζ), (η), in chap. III.

It is manifest, from the remarks made at the conclusion of the last chapter, that, when one or more of the required angles is very small, the group (σ) may be used with the greatest advantage, and when one or more of the angles is nearly 90°, we ought to employ the group (ζ.) The group (η) may be made use of in any case.

Case 6. Given the three angles A, B, C, required the three des a, b, c.

It is manifest that this case does not admit of solution, for y *number* of unequal similar triangles may be constructed, ving their angles equal to the angles A, B, C.

We shall conclude this chapter by giving some numeril examples.

Example 1. Given $A = 68^\circ\ 2'\ 24''$, $B = 57^\circ\ 53'\ 16''.8$, $= 3754$ feet, required C, b, c.

Then by case 2.

$$\begin{aligned} C &= 180^\circ - (A+B) \\ &= 180^\circ - 125^\circ\ 55'\ 40''.8 \\ &= 54^\circ\ 4'\ 19''.2 \\ b &= a \cdot \frac{\sin. B}{\sin. A} \end{aligned}$$

$$\log. b = \log. a + \log. \sin. B - \log. \sin. A$$

Now $\log. a = 3.5744943$

$\log. \sin. B = \underline{9.9278888}$

13.5023831

$\log. \sin. A = \underline{9.9672882}$

$\therefore \quad \log. b = 3.5350949 = \log. 3428.43$

$\therefore \quad b = 3428.43$

Similarly,

$$\log. c = \log. a + \log. \sin. C - \log. \sin. A$$

$\log. a = 3.5744943$

$\log. \sin. C = \underline{9.9083536}$

13.4828479

$\log. \sin. A = \underline{9.9672882}$

$\therefore \quad \log. c = 3.5155597 = \log. 3277.628$

$\therefore \quad c = 3277.628$ feet.

Example 2. Given $a = 145$, $b = 178.3$, $A = 41^\circ\ 10'$, required B, C

This example belongs to case 4, and since the given angle A is acute, and the side b opposite to the required angle B greater than the side a, the solution will be ambiguous.

We have $\log. \sin. B = \log. \sin. A + \log. b - \log. a$

$\log. \sin. A = 9.8183919$

$\log. b = \underline{2.2511513}$

12.0695432

$\log. a = \underline{2.1613680}$

$\therefore \quad \log. \sin. B = 9.9081752$

The angle in the tables corresponding to this logarithm, is 54° 2′ 22″, but we cannot determine *a priori* whether the angle sought be this angle, or its supplement 125° 57′ 38″.

$$\therefore \quad B = 54^\circ\ 2'\ 22''$$

$$\text{Or} \quad B = 125^\circ\ 57'\ 38''$$

If we take the 1st value,

$$\left.\begin{array}{l} C = 84^\circ\ 47'\ 38'' \text{ and the triangle required is ABC} \\ \text{If we take the second value,} \\ C = 12^\circ\ 52'\ 22'' \text{ and the triangle required is ADC} \end{array}\right\} \text{see last fig.}$$

Example 3. Given $a = 178.3$, $b = 145$, $A = 41^\circ\ 10'$, required B.

This example also belongs to case 4, but since the given angle A is acute, and the side b opposite the required angle B less than the side a, it follows that the angle B must be an acute angle, and the solution will not be ambiguous.

$$\text{We have} \quad \log.\sin.B = \log.\sin.A + \log.b - \log.a$$

$$\begin{array}{lrl} \text{But} & \log.\sin.A = & 9.8183919 \\ & \log.b = & \underline{2.1613680} \\ & & 11.9797599 \\ & \log.a = & \underline{2.2511513} \\ \therefore & \log.\sin.B = & 9.7286086 \end{array}$$

The angle in the tables corresponding to this logarithm is 32° 21′ 54″, and since, in the present instance, the supplement of 32° 21′ 54″ cannot belong to the case proposed, the solution is not ambiguous.

Example 4. Given $a = 374$, $b = 3277.628$, and the included angle 57° 53′ 16″.8: required A, C, b.

By case 3 we have

$$\log.\tan.\frac{A-B}{2} = \log.(a-b) + \log.\cot.\frac{C}{2} - \log.(a+b)$$

$$\begin{array}{rl} a - b = 476.372, \ \therefore \ \log.(a-b) = & 2.6779444 \\ \log.\cot.\frac{C}{2} = & \underline{10.2572497} \\ & 12.9351941 \\ a + b = 7031.628, \ \log.(a+b) = & \underline{3.8470543} \\ \therefore \ \log.\tan.\frac{A-B}{2} = & 9.0881398 \end{array}$$

$$\begin{array}{lrl} \text{Whence} & \frac{A-B}{2} = & 6^\circ\ 59'\ 2''.4 \\ \text{And since} & A+B = & 122^\circ\ 6'\ 43''.2 \\ \text{And} & A-B = & 13^\circ\ 58'\ 4''.8 \\ \therefore & 2A = & 136^\circ\ 4'\ 48'' \\ & 2B = & 108^\circ\ 8'\ 38''.4 \end{array}$$

$$\therefore \quad A = 68^\circ\ 2'\ 24'',\ B = 54^\circ\ 4'\ 19''.2$$

The angles A and B being determined, the side c may be eadily found from the equation.

$$\frac{c}{a}=\frac{\text{sin. C}}{\text{sin. A}}$$

$$\text{log. } c = \text{log. } a + \text{log. sin. C} - \text{log. sin. A}$$

Example 5. Given $a=33$, $b=42.6$, $c=53.6$, required A, B, C. Taking the formula marked (ε) in chap. III. we have

log. sin. A

$$\text{log. R}+\text{log. }2+\tfrac{1}{2}\{\text{log. }s+\text{log. }(s-a)+\text{log.}(s-b)+\text{log.}(s-c)\}-\{\text{log.}b+\text{log.}c\}$$

log. sin. B

$$\text{log. R}+\text{log. }2+\tfrac{1}{2}\{\text{log. }s+\text{log. }(s-a)+\text{log.}(s-b)+\text{log.}(s-c)\}-\{\text{log.}a+\text{log.}c\}$$

log. sin. C

$$\text{log. R}+\text{log. }2+\tfrac{1}{2}\{\text{log. }s+\text{log. }(s-a)+\text{log.}(s-b)+\text{log.}(s-c)\}-\{\text{log.}a+\text{log.}b\}$$

ow $\qquad$ $\text{log.R}=10.$

$\text{log.}2=0.3010300$

$a=33 \quad \therefore \text{log.}a \quad =1.5185139$
$b=42.6 \therefore \text{log.}b \quad =1.6294096$
$c=53.6 \therefore \text{log.}c \quad =1.7291648$

$\left.\right\}$ $\text{log.}b+\text{log.}c=3.3585744$
$\therefore \text{log.}a+\text{log.}c=3.2476787$
$\text{log.}a+\text{log.}b=3.1479235$

$s=64.6 \therefore \text{log.}s \quad =1.8102325$
$-a=31.6 \therefore \text{log.}s-a=1.4996871$
$-b=22 \quad \therefore \text{log.}s-b=1.3424227$
$-c=11 \quad \therefore \text{log.}s-c=1.0413927$

$.\ \text{log.}s+\text{log.}(s-a)+\text{log.}(s-b)+\text{log.}(s-c)=5.6937350$

And

$\tfrac{1}{2}\ \{\text{log.}s+\text{log.}(s-a)+\text{log.}(s-b)+\text{log.}(s-c)\} = 2.8468675$

$.\ \text{log.R}+\text{log.}2+\tfrac{1}{2}\{\text{log.}s+\text{log.}(s-a)+\text{log.}(s-b)$
$+\text{log.}(s-c)\}$ - - - - - $-=13.1478975$

Subtracting from this number the values log. b+log. c; og. a+log. c; log. a+log. b; in succession we find,

log. sin. A = 9.7893231 $\therefore$ A = 37° 59′ 53′
log. sin. B = 9.9002188 $\therefore$ B = 52° 37′ 46″$\frac{9}{10}$
log. sin. C = 9.9999740 $\therefore$ C = 89° 22′ 20″$\frac{7}{10}$

Having determined A and B by the above method, we find he above accurate value of C, by subtracting the sum of A nd B from 180°. If, however, it had been required to determine C alone (being an angle nearly equal to 90°) we could ot have found its value with sufficient accuracy from the common tables, for it will be seen, upon referring to them, that the number 9.9999740 may be the logarithm of the sine of any angle from 89° 22″ 20″ up to 89° 22′ 25″ consequently the above method cannot be applied with propriety to determine the exact value of C, unless we previously determine A and B.

The angle C may however be determined directly, and with great accuracy, from any of the three formulæ (σ), (ζ), (η), in Chap. III.

Let us take these in succession, (σ).

$$\log.\cos.\frac{C}{2}=\log.R+\tfrac{1}{2}\{\log.s+\log.(s-c)\}-\tfrac{1}{2}(\log.a+\log.b)$$

$$\left.\begin{array}{r}\log.s=1.8102325\\ \log.(s-c)=1.0413927\\ \hline 2.8516252\end{array}\right\}\therefore\tfrac{1}{2}\{\log.s+\log.(s-c)\}=\begin{array}{r}1.4258126\\ \log.R=10\\ \hline 11.4258126\end{array}$$

$$\log.a+\log.b=3.1479235\ \therefore\ \tfrac{1}{2}(\log.a+\log.b)=\underline{1.5739617}$$

$$\therefore\quad\log.\cos.\frac{C}{2}=9.8518509$$

$$\therefore\quad\frac{C}{2}=44^\circ\ 41'\ 10''\tfrac{6}{33}$$

$$C=89^\circ\ 22'\ 20''\tfrac{12}{33}$$

By (ζ).

$$\log.\sin.\frac{C}{2}=\log.R+\tfrac{1}{2}\{\log.(c-a)+\log.(s-b)\}-\tfrac{1}{2}\{\log.a+\log.b\}$$

$$\left.\begin{array}{r}\log.(s-a)=1.4996871\\ \log.(s-a)=1.3424227\\ \hline 2.8421098\end{array}\right\}\therefore\tfrac{1}{2}\{\log.(s-a)+\log.(s-b)\}=\begin{array}{r}1.4210549\\ \log.R=10.\\ \hline 11.4210640\end{array}$$

$$\log.a+\log.b=3.1479235\quad\therefore\quad\tfrac{1}{2}\{\log.a+\log.b\}=1.5739617$$

$$\therefore\quad\log.\sin.\frac{C}{2}=9{\cdot}8470932$$

$$\therefore\quad\frac{C}{2}=44^\circ\ 41'\ 10''\tfrac{8}{33}$$

$$C=89^\circ\ 22'\ 20''\tfrac{16}{33}$$

By (η).

$$\log.\tan.\frac{C}{2}=\log.R+\tfrac{1}{2}\{\log.(s-a)+\log.(s-b)\}-\tfrac{1}{2}\{\log.s+\log.(s-c)\}$$

$$\left.\begin{array}{r}\log.(s-a)=1.4996871\\ \log.(c-a)=1.3424227\\ \hline 2.8421098\end{array}\right\}\therefore\tfrac{1}{2}\left\{\begin{array}{r}\log.(s-a)+\log.(s-a)\\ \log.R\end{array}\right\}\begin{array}{r}=1.4210549\\ =10.\\ \hline 11.4210549\end{array}$$

$$\left.\begin{array}{r}\log.s=1.8102325\\ \log.s-c=1.0413927\\ \hline 2.8516252\end{array}\right\}\therefore\quad\tfrac{1}{2}\{\log.+\log.(s-c)\quad=\underline{1.4258126}$$

$$\therefore\quad\log.\tan.\frac{C}{2}=9.9952423$$

$$\therefore\quad\frac{C}{2}=44^\circ\ 41'\ 10''\tfrac{8}{33}$$

$$C=89^\circ\ 22'\ 10\tfrac{16}{33}$$

CHAPTER VIII.

ON THE USE OF SUBSIDIARY ANGLES.

sidiary Angles are angles which, although not immediately cted with a given problem, are introduced by the com- in order to simplify his calculations. Their use, and the d in which they are employed, will be understood from follows.

en the two sides of a triangle, and the included angle, ven, according to the method pursued in the last chap- e must determine the two remaining angles before we mpute the third side. It frequently happens, however, ctice, that the side only is required, and it therefore be- desirable to have some direct method of computing the dependently of the two angles.

pose that a, b, C are given, and c is required. By chap. rop. 4,

$$c^2=a^2+b^2-2ab\cos.\ C.$$

le c is determined *theoretically* at once by this expression, e formula is not adapted to logarithmic computation, and , if employed practically, lead to a very tedious and icated calculation. We can, however, put this expres- nder a form adapted to logarithmic calculation, by hav- ourse to an algebraical artifice, and introducing a sub- y angle.

$$c^2=a^2+b^2-2\ ab\cos.\ C$$

ling and substracting 2 ab on the right hand side.

$$\begin{aligned} c^2&=a^2+b^2-2\ ab+2\ ab-2\ ab\cos.\ C\\ &=(a-b)^2+2\ ab\ (1-\cos.\ C)\\ &=(a-b)^2+2\ ab+2\ \sin.^2\frac{C}{2}\\ &=(a-b)^2\left\{1+\frac{4\ ab\sin.^2\frac{C}{2}}{(a-b)^2}\right\}\end{aligned}$$

Assume $\dfrac{4\ ab\sin.^2\frac{C}{2}}{(a-b)^2}=\tan.^2\varphi$.

$$\begin{aligned} c^2&=(a-b)^2\ (1+\tan.^2\varphi)\\ &=(a-b)^2\sec.^2\varphi\\ c&=(a-b)\sec.\varphi\\ \log.\ c&=\log.\ (a-b)+\log.\sec.\varphi-\log.\ R\end{aligned}$$

angle φ is known from the equation.

$$\tan.\ \varphi = \frac{2\sqrt{ab}\ .\ \sin.\ \frac{C}{2}}{(a-b)}$$

Whence,

$$\log.\tan.\varphi = \log.\ 2 + \tfrac{1}{2}\,(\log.a + \log.b) + \log.\ \sin.\frac{C}{2} - \log.(a-b)$$

φ being thus determined, log. sec. φ can be found from the tables, and the value of c becomes known.

The angle φ, which is introduced into the above calculation, in order to render the expression convenient for logarithmic computation, is called a *subsidiary angle.*

The above transformation may be effected in a manner somewhat different, as before.

$$\begin{aligned} c^2 &= a^2 + b^2 - 2\,ab \cos.\ C \\ &= a^2 + b^2 + 2\,ab - 2\,ab - 2\,ab \cos.\ C \\ &= (a+b)^2 - 2\,ab\,(1 + \cos.\ C) \\ &= (a+b)^2 - 2\ ab \times 2 \cos.^2 \frac{C}{2} \\ &= (a-b)^2 \left(1 - \frac{4\ ab \cos.^2 \frac{C}{2}}{(a+b)^2}\right) \end{aligned}$$

$$\text{Assume} \quad \frac{4\,ab \cos.^2 \frac{C}{2}}{(a+b)^2} = \sin.^2\ \varphi$$

$$\begin{aligned} \therefore \quad c^2 &= (a+b)^2\ (1 - \sin.^2\ \varphi) \\ &= (a+b)^2 \cos.^2\ \varphi \\ c &= (a+b)\ \cos.\ \varphi \end{aligned}$$

$$\therefore \quad \log.\ c = (s+b) + \log.\ \cos.\ \varphi - \log.\ R$$

As before the angle φ must be determined from the equation.

$$\sin.\varphi = \frac{2\sqrt{ab}\ .\ \cos.\frac{C}{2}}{(a+b)}$$

In order to prove that we can always assume

$$\frac{2\sqrt{ab}\ .\ \cos.\frac{C}{2}}{(a+b)} = \sin.\ \varphi$$

we must show that $\dfrac{2\sqrt{ab}\ \cos.\frac{C}{2}}{(a+c)}$ is always less than unity, or, in other words, that $2\sqrt{ab}$ is always less than $(a+b)$, this is easily done.

If $a+b > 2\sqrt{ab}$

Then $a^2+2ab+b^2 > 4ab$

$a^2+b^2 > 2ab$

$a^2+b^2-2ab > 0$

Or $(a-b)^2 > 0$

ice $(a-b)^2$ is necessarily a positive quantity, it must e greater than 0 (except in the particular case $a=b$, is=0), and therefore $\frac{2\sqrt{ab}\cos.\frac{C}{2}}{(a+b)}$ is always less than l consequently an angle may always be found whose ual to it.

ing the same case of oblique-angled triangles, we de- the difference of the angles A, B from the equation.

$$\tan.\frac{A-B}{2}=\frac{a-b}{a+b}\cot.\frac{C}{2}$$

$$\text{og.}\tan.\frac{A-B}{2}=\log.(a-b)+\log.\cot.\frac{C}{2}-\log.(a+b)$$

solution of certain astronomical problems, the loga- the sides a, b are given, but not the sides them- d these logarithms being given, we can very easily $\frac{A-B}{2}$ without knowing the sides.

$$\tan.\frac{A-B}{2}=\frac{a-b}{a+b}\cot.\frac{C}{2}$$

$$=\frac{1-\frac{b}{a}}{1+\frac{b}{a}}\cot.\frac{C}{2}$$

e $\frac{b}{a}=\tan.\varphi$

$$\tan.\frac{A-B}{2}=\frac{1-\tan.\varphi}{1+\tan.\varphi}.\cot.\frac{C}{2}$$

$$=\tan.(45^\circ-\varphi)\cot.\frac{C}{2}$$

$$\text{;. }\tan.\frac{A-B}{1}=\log.\tan.(45^\circ-\varphi)+\log.\cot.\frac{C}{2}-\log.R$$

The angle φ is known from the equation.

$$\tan.\ \varphi=\frac{b}{a}$$

Whence $\quad \log.\ \tan.\ \varphi=\log.\ \mathrm{R}+\log.\ b-\log.\ a$

The angle $\frac{\mathrm{A}-\mathrm{B}}{2}$ thus becomes known from the logs. of a and b, without calculating a and b. In the same way we have

$$\cot.\ \frac{\mathrm{A}-\mathrm{B}}{2}=\tan.\ (45^\circ+\varphi)\ \tan.\ \frac{\mathrm{C}}{2}$$

And $\therefore$ $\log.\ \cot.\frac{\mathrm{A}-\mathrm{B}}{2}=\log.\tan.(45^\circ+\varphi)+\log.\ \tan.\frac{\mathrm{C}}{2}-\log.\mathrm{R}.$

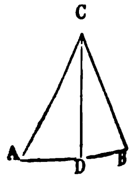
CHAPTER IX.

ON THE SOLUTION OF GEOMETRICAL PROBLEMS BY TRIGONOMETRY.

A great variety of geometrical problems may be solved with much elegance by the introduction of geometrical formulæ. We shall give a few examples.

PROBLEM I.

To express the area of a plane triangle in terms of the sides of the triangle



Let CD be a perpendicular from C upon AB

$$\text{Area of a triangle ABC}=\frac{\mathrm{AB}\times\mathrm{CD}}{2}$$

$$=\frac{c}{2}.\ \mathrm{AC}\ \sin.\ \mathrm{A}$$

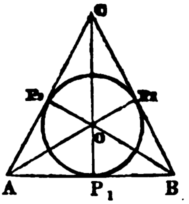
$$=\frac{bc}{2}.\ \sin.\ \mathrm{A}$$

$$=\frac{bc}{2}\cdot\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}..\text{Ch. III.}$$

$$=\sqrt{s\ (s-a)\ (s-b)\ (s-c)}$$

PROBLEM II.

To express the radius of a circle inscribed in a given triangle, in terms of the sides of the triangle.



Let the radius required be called r.

$$\text{Area of AOC}=\frac{rb}{2}$$

$$\text{..}\quad \text{AOB}=\frac{rc}{2}$$

$$\text{..}\quad \text{COB}=\frac{ra}{2}$$

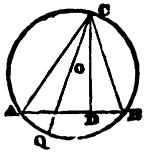
$$\therefore \text{ Whole area of triangle ABC}=\frac{r}{2}\,(a+b+c)=r\,.\,s$$

i.e. $\sqrt{s\,(s-a)\,(s-b)\,(s-c)}=r\,.\,s$ by last problem

$$\therefore \quad r=\frac{\sqrt{s\,(s-a)\,(s-b)\,(s-c)}}{s}$$

PROBLEM III.

To express the radius of a circle circumscribed about a given triangle, in terms of the sides of the triangle.



Let fall CD perpendicular on AB
Let the radius be called R.
By Prop. XXXIX, B. IV. *El. Geom.*

$$\text{CQ}\,.\,\text{CD}=\text{AC}\,.\,\text{CB}$$

$$\therefore \quad \text{CQ}\,.\,\text{CD}\,.\,\text{AB}=\text{AC}\,.\,\text{CB}\,.\,\text{AB}$$

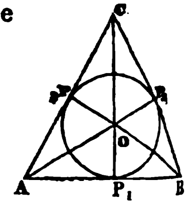
$$2\,\text{R}\times 2\ \text{area}=abc$$

$$\text{R}=\frac{abc}{4\ \text{area}}$$

$$=\frac{abc}{4\sqrt{s\,(s-a)(s-b)(s-c)}}$$

PROBLEM IV.

Given the three angles of a plane triangle, and the radius of the inscribed triangle, to find the sides of the triangle.



Let A, B, C, be the three given angles, r the radius,

$$\begin{aligned} \text{AB or } c &= \text{AP}_1+\text{P}_1\text{B} \\ &= r\left(\text{cot. }\frac{\text{A}}{2}+\text{cot. }\frac{\text{B}}{2}\right) \\ &= r\;.\;\frac{\text{sin. }\left(\frac{\text{A}}{2}+\frac{\text{B}}{2}\right)}{\text{sin. }\frac{\text{A}}{2}\text{ sin. }\frac{\text{B}}{2}} \end{aligned}$$

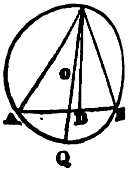
So,

$$\text{AC or } b=r\;.\;\frac{\text{sin. }\left(\frac{\text{A}}{2}+\frac{\text{C}}{2}\right)}{\text{sin. }\frac{\text{A}}{2}\text{ sin. }\frac{\text{C}}{2}}$$

$$\text{BC or } a=r\;.\;\frac{\text{sin. }\left(\frac{\text{B}}{2}+\frac{\text{C}}{2}\right)}{\text{sin. }\frac{\text{B}}{2}\text{ sin. }\frac{\text{C}}{2}}$$

PROBLEM V.

Given the three angles of a plane triangle, and the radius of the circumscribing circle, to find the sides of the triangle.



As in Problem III.

$$\text{CQ . CD}=\text{AC . CB}$$

$$\text{CQ . CB sin. B}=\text{AC . CB}$$

$$\therefore \quad \text{AC}=2\text{ R sin. B}$$

So,

$$\text{BC}=2\text{ R sin. A}$$

$$\text{AB}=2\text{ R sin. C}$$

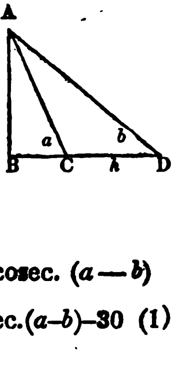
CHAPTER X.

PROBLEMS IN TRIGONOMETRICAL SURVEYING.

THE DETERMINATION OF TOPOGRAPHICAL DATA BY GEOMETRICAL CONSTRUCTION AND TRIGONOMETRICAL ANALYSIS.

PROBLEM I.

To determine the height of an inaccessible object.



Let AB be the object, and in a straight line towards it measure any distance DC, and observe the angles of elevation ADB, ACB at the stations D, C. Put CD$=h$, ACB$=a$, ADB $=b$: then DAC$=a-b$; hence we have

$$\frac{AB}{AC}=\sin. a;\ \frac{AC}{CD}=\frac{\sin. b}{\sin. (a-b)}$$

and, multiplying these two equations, we have

$$\frac{AB}{CD}=\frac{\sin. a \sin. b}{\sin. (a-b)},\text{ or, } AB=h \sin. a \sin. b \,\text{cosec.}\ (a-b)$$

$$\therefore \log.AB=\log.h+\log.\sin.a+\log.\sin.b+\log.\text{cosec.}(a-b)-30 \quad (1)$$

Cor. Since DB=AB cot. b, and CB=AB cot. a; therefore, by subtraction, $CD=AB(\cot.b-\cot.a,)$ or $AB=\frac{h}{\cot.b-\cot.a}$ (2)

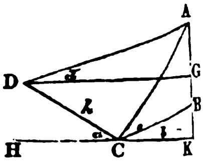
Ex. Let DC$=h=200$, BDA$=b=31°$, BCA$=a=46°$; to find AB and CB.

log. h - - -	=	log. 200 =	2.3010300
log. sin. a - -	=	log. sin. 46°=	9.8569341
log. sin. b - -	=	log. sin. 31°=	9.7118393
log. cosec. $(a-b)$	=	log. cosec. 15°=	10.5870038
log. AB - -	=	log. 286.29 =	2.4568072 ∴ AB=286.29

Also, BC=AB cot. a, ∴ log. BC=log. AB+log. cot. a — 10.

PROBLEM II.

To determine the height af an inaccessible object, which has no level ground before it.



Let AB be the object, and C, D, two stations in a vertical plane passing through AB; measure the distance CD, and at C take the angles of elevation or depression of the station D, and the top and bottom of the object. Also at D take the elevation of the top of AB.

Put $CD=h$, $DCH=a$, $BCK=b$, $ACK=c$, $ADG=d$; then $ACB=c-b$, $ADC=a+d$, and $CAD=c-d$.

Now, $\dfrac{AB}{AC}=\dfrac{\sin. ACB}{\sin. ABC}=\dfrac{\sin. (c-b)}{\sin. ABK}=\dfrac{\sin. (c-b)}{\cos. b}$;

$\dfrac{AC}{CD}=\dfrac{\sin. ADC}{\sin. CAD}=\dfrac{\sin. (a+d)}{\sin. (c-d)}$; hence, multiply these equations,

we have $\dfrac{AB}{CD}=\dfrac{\sin. (c-b) \sin. (a+d)}{\cos. b \sin. (c-d)}$; and, therefore,

$$AB=h \sin. (c-b) \sin. (a+d) \sec. b \operatorname{cosec.} (c-d)$$

$\therefore$ log. $AB=\log. h+\log. \sin. (c-b)+\log. \sin. (a+d)+\log. \sec. b+\log. \operatorname{cosec.} (c-d)-40$ - - - - - - - - - - (1)

Cor. When $a=90°$, and $b=0°$, then we have

$\log. AB=\log. h+\log. \sin. c+\log. \cos. d+\log. \operatorname{cosec.} (c-d)-30$ (2)

Ex 1. Let $h=18$ feet; $c=40°$; $d=37°30'$, $a=90°$ and $b=0°$; to find AB.

log. h . . . =log. 18 . . . = 1.2552725
log. sin. c . . =log. sin. 40° . = 9.8080675
log. cos. d. . =log. cos. 37° 30′= 9.8994667
log. cosec. $(c-d)$=log. cosec. 2° 30′=11.3603204

log. AB . . =log. 210.4394 = 2.3231271

$\therefore AB=210.4394$.

Ex. 2. The angle of elevation of the top of a tower, standing on a hill, was 33° 45′, and measuring on level ground 300 yards directly towards the tower, the angles of elevation of the top and bottom of the tower were 51° and 40° respectively. What is the height of the tower? Ans. 140 yds.

Remark.—When the station D is higher than A, the top of the tower, then the angle d must be considered negative, and therefore we should have

$$AB= h \sin. (c-b) \sin. (a-d) \sec. b \operatorname{cosec.} (c+d)$$

PROPOSITION I. LEMMA.

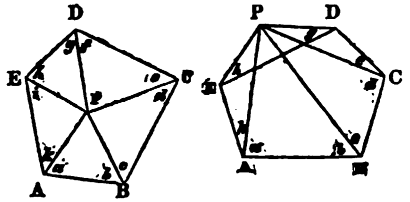
'*straight lines be drawn from any point, either within, or*
f, a polygon, to all the angular points, the continued pro-
s of the sines of the alternate angles, made by the sides of
olygon, and the lines so drawn, will be equal.

Let angle CDP$=f$, and PEA$=i$;

$$\text{then } \frac{\text{PA}}{\text{PB}} = \frac{\sin. b}{\sin. a} \qquad \frac{\text{PB}}{\text{PC}} = \frac{\sin. d}{\sin. c}$$

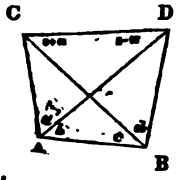
$$\frac{\text{PC}}{\text{PD}} = \frac{\sin. f}{\sin. e} \qquad \frac{\text{PD}}{\text{PE}} = \frac{\sin. h}{\sin. g}$$

$$\frac{\text{PE}}{\text{PA}} = \frac{\sin. k}{\sin. i.}$$



ıt PA . PB . PC . PD . PE$=$PB . PC . PD . PE . PA;
is, the product of the numerators$=$product of the denomi-
rs, in the first members of these equations ; hence, this
g true in the second members also, sin. b sin. d sin. f sin.
. $k=$sin. a sin. c sin. e sin. g sin. i.

PROBLEM III.



iven AB, *and the angles* a, b, c, d, *to find*
ıd thence CD.
ıt BCD$+$ADC$=b+c=2s$
BCD$-$ADC$=$ . . . $2x$

ı BCD $= s+x$, ADC $= s-x$; also, sin.
} $=$ sin. $(b+c+d)$, and sin. ACB $=$ sin.
$b+c$); hence, by Lemma, (I) we have
a sin c sin. $(b+c+d)$ sin. $(s+x)=$sin. b sin. d sin. $(a+b+c)$
$(s-x)$, or sin. a sin. c sin. $(b+c+d)$ $\{$sin. s cos. $x+$cos. s
$x\}$ $=$ sin. b sin. d sin. $(a+b+c)$ $\{$sin. s cos. $x-$cos. s
$x\}$.
hen, dividing by cos. s cos. x, we have

$\sin. a \ \sin. c \ \sin. (b+c+d) \ (\tan. s+\tan. x)=\sin. b \sin. d \sin. (a+b+c) \ (\tan. s-\tan. x)$,

$$\therefore \tan. x=\frac{\sin. b \sin. d \sin. (a+b+c)-\sin. a \sin. c \sin. (b+c+d)}{\sin. b \sin. d \sin. (a+b+c)+\sin. a \sin. c \sin. (b+c+d)} \tan. s.$$

Dividing numerator and denominator by $\sin. a \ \sin. c \sin. (b+c+d)$, and putting $\dfrac{\sin. b \sin. d \sin. (a+b+c)}{\sin. a \sin. c \sin. (b+c+d)} = \tan. \beta$ and $1 = \tan. 45^\circ$; then

$$\tan. x = \frac{\tan. \beta - \tan. 45^\circ}{\tan. \beta + \tan. 45^\circ} \tan. s = \frac{\sin. (\beta - 45^\circ)}{\sin. (\beta + 45^\circ)} \tan. s;$$

hence x, $s+x$, $s-x$ are all known, and thence CD is known. For $\dfrac{\text{CD}}{\text{BD}}= \dfrac{\sin. d}{\sin. (s+x)}$ and $\dfrac{\text{BD}}{\text{AB}} = \dfrac{\sin. b}{\sin. (b+c+d)}$; therefore,

$$\text{CD}=\text{AB} \sin. b \sin. d \operatorname{cosec.} (s+x) \operatorname{cosec.} (b+c+d).$$

Cor. When CD is given, and the same angles, to find AB, we have

$$\text{AB}=\text{CD} \sin. (b+c+d) \sin. (s+x) \operatorname{cosec.} b \operatorname{cosec.} d.$$

EXAMPLE.

Given AB=600 yards, $a=37^\circ$, $b=58^\circ \ 20'$, $c=53^\circ \ 30'$, $d=45^\circ \ 15'$, to find CD.

Here, $\tan. \beta=\operatorname{cosec.} a \sin. b \operatorname{cosec.} c \sin. d \sin. (a+b+c) \operatorname{cosec.} (b+c+d)$,

log. cosec. a
= log. cosec. 37° . . . . . = 10.2205370
log. sin. b
= log. sin. $58^\circ \ 20'$ . . . = 9.9299891 9.9299891
log. cosec. c
= log. cosec. 53 30 . . . = 10.0948213
log. sin. d
= log. sin. 45 15 . . . = 9.8513717 9.8513717
log. sin. $(a+b+c)$
= log. sin. 148 50 . . . = 9.7139349
log. cosec. $(b+c+d)$
= log. cosec. 157 5 . . . = 10.4096131 10.4096131

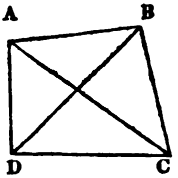
log. tan. β
= log. tan. $58^\circ \ 56' \ 39''$ = $\underline{10.2202671}$

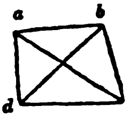
log. sin. $(\beta - 45^\circ)$
= log. sin. 13 56 39 = 9.3819742
log. cosec. $(\beta+45^\circ)$
= log. cosec. 103 56 39 = 10.0129906

log. tan. s
= log. tan. 55 55 . . . = 10.1696508
log. tan. x
= log. tan. 20 9 3 = 9.5646156
log. cosec. $(s+x)$
= log. cosec. 76 4 3 = ... 10.0129687
log. AB.
= log. 100 . . . . . . = ... 2.7781513
∴ CD . . . = 959.608 . . . = ... 2.9820939

PROBLEM IV.

The distance of two objects at B *being known, I find, by observation, the angles* ACD ADB BDC, *and* BCA, *taken at the stations* D, C, *required the distances* DC, AD, BD, BC, *and* CA; *both by construction and calculation.*





Assume *dc* at pleasure, and make the angles *adb*, *bdc* and *bca*, respectively equal to the angles ADB BDC and BCA; join *a*, *b*, and *abcd* will be similar to ABCD, and if *ab* represent the side AB, then will *dc* repersent DC. &c.

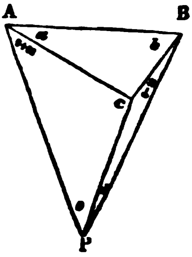
By analysis,

In the triangle *adc*, all the angles and the assumed side, *dc* are given to find *ad* and *ac*. Then in the triangle *bcd*, all the angles and the assumed side *dc*, are given to find *bc*, *bd*. Lastly, in the triangle *adb*, we have the sides *ad*, *db*, and the angle *adb*, to find the side *ab*. Then we have,

ab : AB : : *dc* : DC : : *ad* : AD : : *bc* : BC : : *ac* : AC : : *bd* : BD, whence we have the distances, DC, AD, BC, AC and BD.

PROBLEM V.

Given AB, a, b, *and the angles* c, d, *taken at some point* P *in the same plane* ABC, *to find* x; *and thence* PA, PB, PC.



Put PAC+PBC$=180°-(a+b+c+d)=2s$
PAC—PBC= . . . . $=2x$;
Then, PAC$=s+x$, PBC$=s-x$, and, by lemma 1, sin. a sin. c sin $(s-x)=$sin. b sin. d sin. $(s+x)$

$$\therefore \frac{\text{sin. } b \text{ sin. } d}{\text{sin. } a \text{ sin. } c}=\frac{\text{sin. }(s-x)}{\text{sin. }(s+x)}=\frac{\text{tan. } s-\text{tan. } x}{\text{tan. } s+\text{tan. } x}$$

Put $\tan \beta = \dfrac{\text{sin. } b \text{ sin. } d}{\text{sin. } a \text{ sin } c}=$ cosec a sin b cosec. c sin. d; then we have

$$\frac{\text{tan. } s-\text{tan. } x}{\text{tan. } s+\text{tan. } x}=\text{tan. } \beta \therefore \frac{\text{tan. } x}{\text{tan. } s}=\frac{1-\text{tan. } \beta}{1+\text{tan. } \beta}=\frac{\text{tan. } 45°-\text{tan. } \beta}{\text{tan. } 45°+\text{tan. } \beta};$$

$$\therefore \text{ tan. } x=\frac{\text{tan. } 45°-\text{tan. } \beta}{\text{tan. } 45°+\text{tan. } \beta} \text{ tan. } s=\frac{\text{sin. }(45°-\beta)}{\text{sin. }(45°+\beta)}\text{tan. } s.$$

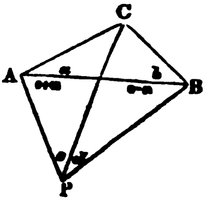
Hence x is known, and thence $s+x$ and $s-x$ are known.

$$\text{Then } \frac{\text{PC}}{\text{AC}}=\frac{\text{sin. }(s+x)}{\text{sin. } c}, \frac{\text{AC}}{\text{AB}}=\frac{\text{sin. } b}{\text{sin. }(a+b)}; \text{ hence}$$

$$\text{PC} = \text{AB cosec. }(a+b) \text{ sin. } b \text{ cosec. } c \text{ sin. } (s+x).$$

PROBLEM VI.

When the points P *and* C *are on opposite sides of* AB.



Put PAB+PBA$=180°-(c+d)=2s$
PAB—PBA=. . . . . $=2x$;
then, PAB$=s+x$, PBA $= s-x$; and, by lemma 1, sin. a sin. c sin. $(s-x)=$ sin. b sin. d sin. $(s+x)$;
hence, as in the last problem, we have

$$\text{tan. } x=\frac{\text{sin. }(45°-\beta)}{\text{sin. }(45°+\beta)} \text{ tan. } s;$$

where tan. $\beta =$ cosec. a sin. b cosec. c sin. d; and $2s = 180°-(c+d)$.

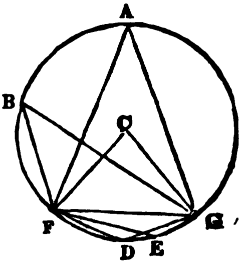
PART II.

Many curious and highly useful problems in trigonometrical urveying, may be elegantly solved through the properties of he circle, by geometrical construction, and by analyzing this onstruction by trigonometrical analysis; a few of which we ill give in the continuation of this chapter.

PROPOSITION II. LEMMA.

f two points be assumed in the circumference of a circle, they will subtend the same angle from any point whatever of the circumference on the same side of the chord joining these two points, which will be half the angle at the centre when the centre is on the same side of that chord as the point of observation, and will be equal to half its complement to 360° when on the opposite side.

This proposition is evident from Prop. XIX. *Cors.* 1 and 3, . III. *El. Geom.*



Hence, if F, G are two points assumed n the circumference ABFG, then will , G appear under the same angle from ny points A and B situated in the cir- umference, and on the same side of the hord FG; which will be half the angle at the centre; and the points F, G will lso appear under equal angles at every oint D, E on the side DE of the chord G, which will be equal to the angle mea- red by half the arc FBAG equal the complement of the angle CG as enunciated.

Cor. 1. Hence, any two objects in the circumference of a rcle will always appear under the same angle, in any point the arc of either segment, and in no other point situated out that circumference, on the same side of the objects, will the gle be the same.

Cor. 2. If the angle under which any two objects appear less than 90 degrees, the place of observation will be some here in the arc of the greater segment; and if the angle be reater than 90 degrees, the place of observation must be some here in the arc of a segment less than a semicircle, and the gles under which the objects appear, will be the same in y point whatever of those arcs.

Scholium. Hence, having the angles subtended by any two objects from any two given positions, not all in the same circumference, the positions of the objects may be determined by the intersections of the circumferences of two circles, each of which is so described as to pass through the two objects and one of the given positions.

PROBLEM VII.

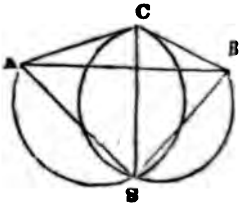
Three points in the same plane being given in position, to determine the position of any other point or place of observation in reference to the given points.

This problem admits of six cases.

The three given points may be the vertices of a triangle, and the required point, or station, may be without the triangle, and opposite one of its sides; it may fall in the same right line with two of the given points, it may fall directly between two of them, it may fall within the triangle or it may fall without the triangle but opposite one of the angles; and lastly, the given points may be all in the same straight line.

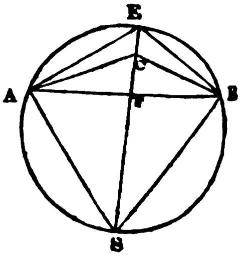
Case 1.

When the given points are the vertices of a given triangle, and the station regarded, falls without the triangle and opposite one of its sides.



Let A, B, C, be the given points whose positions in reference to each other are known, and let S be the point required. Having taken the angles ASC, ASB, describe on AC the segment of a circle that shall contain an angle equal the observed angle ASC; and on CB describe a segment that shall contain an angle equal to the angle CSB, and the point of intersection of the arcs of those segments will determine the position S.

Or,



Make the angle EBA = the observed angle ASC, and the angle BAE = the angle BSE; through A, B, and the intersection at E, describe the circle AEBS; through E and C draw EC, which produce to meet the circumference at S. Join AS, BS, and the distances AS, CS, BS will be the required distances of the station S, from the points A, B, and C.

By trigonometrical analysis.

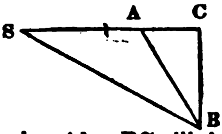
In the triangle ABC, the three sides are given to find the angle BAC. And in the triangle AEB, we have *angle* EAB = *angle* BSE, *angle* ABE = *angle* ASE, and therefore the angle AEB, with the side AB, to find AE and BE. Also in the triangle AEC, we have the sides AC, AE, and the included angle to find the angle AEC. Whence the sum of the angles AES, and the observed angles ASE, subtracted from 180° gives the angle SAE. Then in the triangle AES, we have all the angles and the side AE, to find the side AS. And the *angle* ACS = 180°— *angle* SAC — *angle* ASC; then in the triangle ACS we have all the angles and the side AC to find CS. *Angle* AEB — *angle* AEC = *angle* BES, whence we have the side BE of the triangle BES, and the angles E and S, to find BS; then AS, CS, and BS are the station distances required.

Scholium. 1st. If the angle BSC, be less than CAB, the point E will be below the point C.

2. When the points E and C fall so near each other that the production of EC toward S is attended with uncertainty, the former method of construction is preferred.

Case 2.

Let it be required to determine the position of an observer at S in reference to the three objects ABC, when SAC are in the same right line.



Having taken the angle at S, and calculated the angle CAB from the sides of the triangle ABC which are known by hypothesis, we have the angle ABS = *ang.* CAB—*ang.* ASB. Then at B with the side BA, construct the angle ABS. produce the side, BS till it meets the production of CA in S, and SA, SC, SB will be the several distances of S from the points A, C, B, whence having the angles S, C and B and the side BC, the sides SB, SC and SA may be obtained.

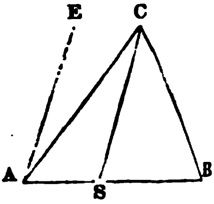
By trigonometrical analysis.

Angle SAB = 180°—BAC, angle SBA = CAB—ASB. Hence in the triangle SAB, we have all the angles and the side AB to find the distances AS, BS, &c.

Case 3.

To determine the position of S in reference to three given objects ABC, where the required point is directly between A and B.

Having constructed the triangle ABC which is given by hypothesis, and having observed the angle BSC, construct from any point A on the line AB an angle BAE=the observed angle and draw CS parallel to EA, and S is the required station.



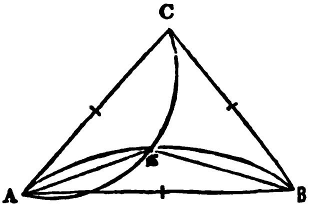
By Analysis.

In the triangle ABC all the sides being known, let the angle A, B be obtained. Then in the triangle BCS having the angle S and B, and the side CB we may proceed to find the distances BS, CS. Then we shall find AS=AB—SB.

Case 4.

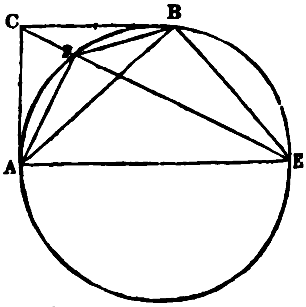
To determine the position of a station S in reference to three given objects when the required station falls within the triangle formed by connecting those given objects.

Let the given objects be three towns A, B and C which are all visible from a station S, which is included in the triangle formed by lines drawn from A to B, from B to C, and from C to A.



First take the angles ASB, BSC, CSA, then on either side AC describe an arc ASC, which shall contain the observed angle ASC, and one either of the other sides AB, describe an arc which will contain the angle ASB, and the point of intersection of those arcs is the station S, all of which is evident from Lemma II.

Otherwise, on AB make an angle ABE=the supplement of the angle ASC; and make an angle BAE equal to the supplement of the angle BSC, then will ABE=ASE and BAE = BSE, since ASE and BSE are respectively the supplements of the angles ASC and BSC, through the points A, B, and E describe a circle, join EC cutting the circumference in the point S which is the station required.



nalysis.

In the triangle AEB the angles B and A being the sup-nts of the observed angles CSA, CSB, are therefore ı, and consequently the angle E and the side AB, to find les AE, BE.

In the triangle ECA we have the sides EA, AC and ncluded angle EAC=BAE+CAB, to find the angle AEC.

And the triangle CEB the sides CB, BE, and the in-d angle CBE are given to find the angle CEB.

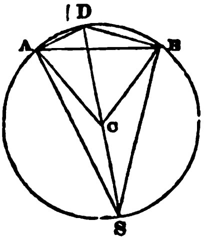
. Therefore in the triangle SAB the angle A, being the angle E, since they are angles in the same segment, known and also the angle ASB, hence we have all the s and the side AB to find the sides AS, BS which are f the station distances required.

. If from the angle CAB we take the angle SAB, we have the angle CAS. Therefore in the triangle ACS ave all the angles with the sides AC and AS to find the station distance CS.

5.

be required to find the distance of any station S from each hree objects A, B, C, when one of the angles C of the tri-gle formed by connecting the three objects falls toward the ion S.

ke the angle DAB=the observed CSB and the angle DBA, equal to bserved angle CSA. On AB de- a circle that shall contain in its er segment the observed angle ASB, gh D and C draw the line DC, till it ects the circle at S, which intersec-letermines the position S.



nalysis.

. In the triangle DAB, all the angles and side AB are n to find AD, DB.

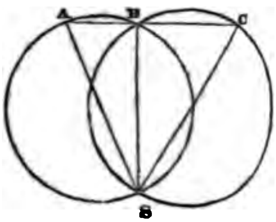
In the triangle ADC are given the sides AD, AC and ncluded angle, to find the angle ACD.

. The angle CAB=ACD—ASD, since ACD, the out-angle is equal to the sum of two inward opposite angles , CAS. Therefore in the triangle ACS we have all the s and the side AC to find the distances SA, SC.

stly, in the triangle BSC the sides CB CS, and the angle are given to find the distance CB.

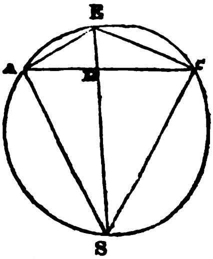
Case 6.

To determine the position of any station S in reference to three objects ABC all in the same straight line.



On AB describe an arc of a segment to contain an angle equal to the angle ASB, and on BC describe an arc containing an angle CSB=the observed angle, subtended by BC, and the point of intersection of those arcs will determine the position of S; whence if we draw the lines SA, SB, SC, those lines will the several distances of S from the objects A, B, C.

For (Lemma II) the points A, B appear under the same angle in every point in the arc BSA, and in no point out of that arc, and B, C, appears under the same angle in every part of the arc CSB and no point out of the arc. Hence the point of intersection of those arcs is the only point where both of those conditions are united, or where both of the objects appear under the observed angles.



Otherwise, at A and on the line AB make BAE=the observed angle CSB and at C an angle=the observed angle ASB; on AC describe a circle that shall contain an angle=the sum of the observed angles at S or which is the same, describe a circle which shall pass through the three angles A, E and C, from E through the point B, drawn EBS to cut the circle in S, and SA, SB, SC determine the relative position of the station S in reference to the three A, B and C.

By Trigonometrical analysis.

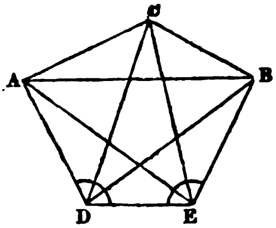
1st. In the triangle CAE all the angles are given and the side AC to find AE.

2d. In the triangle AEB the sides AE, AB, and their included angle are given to find the angles AEB and ABE.

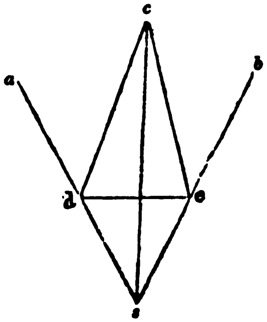
3rd. In the triangle BSC we have the angle CSB and the angle SBC=ABE, and the angle SCB=angle AEB since they are both angles in the same segment ACS, hence all the angles and the sides BC are given to find the side, SC, SB. and SA.

PROBLEM VIII.

he distances of three objects A, B *and* C *being given and consequently the angles which they form with each other. There are also two stations* D, E, *such that at* D *the objects* A, C *and* E *may be seen but not* B *; at* E *the object,* B, C *and* D *may be seen but not* A. *Hence the angles*CDE ADC BED BEC *and consequently the angles* CDA *and* CEB *are given or known from observation to find the distances* DA, DC, DE, EC, *and* EB.



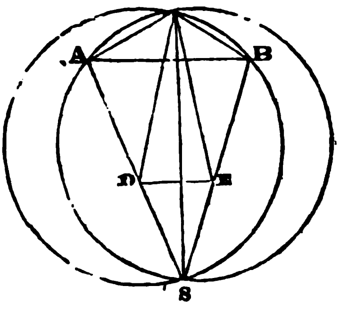
Draw *cd* at pleasure, and at *d* make n angle *cde*=the given angle CDE ake also the angle *dec*=DEC, the gle *ceb*=CEB, and *cda* = CDA roduced *ad*, *bc*, till they meet in *s*, d draw *sc*.



y Analysis.

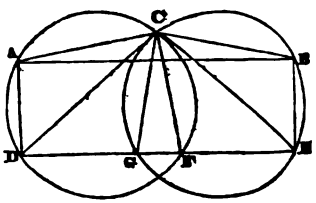
Assume any value for *de* ; then in the triangle *cde* all the ngles are given and the side *de*, to find the sides *cd*, *ce*, the ngle $eds=180^\circ-ade$ and the angle $des=180^\circ-bed$; hence e have in the triangle *dse* all the angles and side *de*, to find , and *es*. In the triangle *cds* the angle $cds=180^\circ-adc$, hence e have two sides *cd ds*, and their included angle to find the ngle $dsc=csa$ and side *cs*, then from the angle *dsc* take the ngle *dsc* and we have the angle $cse=csb$.

Then with the angles *csa*, *b* and the triangle ABC, as ata, make the following construction by case first Prop. II, and find the station distances, SA, SB, SC. Then e shall have



cs : CS : : *de* DE : : *dc* : DC : : *ec* : EC : : *ds* : DS : : *se* : SE from SA take SD and there remains DA, also from SB take SE and there remains BE, hence DA, DC, DE, EC and EB are found.

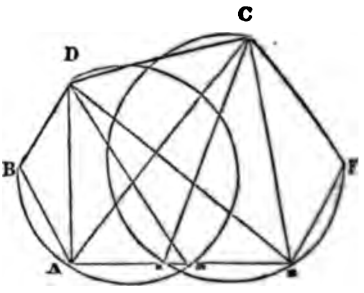
Scholium.



When AD and BE are parallel, the foregoing method of solution fails. In which case, on AC describe a segment to contain an angle equal to the observed angle CDE, and on CB a segment to contain an angle equal to CEB, draw the chord CF to cut off the segment CADF containing the angle CDE, and another chord CG cutting off a segment CBEG, containing the angle CED, the points F and G will be in the same right line with D and E, join GF which produce both ways till it cuts the circumference in D and E, and the points D and E will be the stations required.

PROBLEM IX.

The relation of the four points B,C,D,F, *to each other are known, or the four sides of a quadrilateral figure and its angles are known, there are also two stations* A, E, *such that at* A *only* B, C, E, *are visible and at* E *only the points* D, F, A, *so that the angles* BAC, BAE, AED *and* DEF *and consequently* AEF *may be known, required the distances* AB, AC, ED, EF, EC, *and* AD.



1st. On BC describe a segment to contain the angle BAC, and draw the chord C*m* that shall cut off an angle BC*m* = the supplement of the angle BAE.

2nd. On DF describe a segment that shall contain an angle DEF, and draw the chord D*n*, that shall cut off an angle FD*n* = the supplement of the angle AEF, and the intersections *m* and *n* will be in the same right line with the stations A and E.

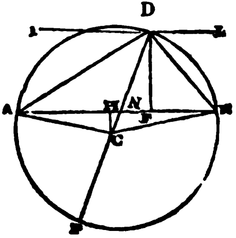
3d. Through the points of intersection *m* and *n*, draw the line *mn* which produce to E at its intersection with the arc of the segment and the points of intersection, A and E will be the points of station.

The analysis of this may be supplied by the student.

PROBLEM X.

Given the base AB, the perpendicular FD, and vertical angle ADB of a triangle to find the sides AD, DB.

On the base AB describe an isosceles triangle ACB whose ertical angle C shall be double the given angle ADB, if that ngle is less than 90°, but double its supplement if the angle ADB is greater than 90°, and from this vertice as a centre, with the radius CA or CB describe a ircle, and the vertice D of the triangle will be found somewhere in the circumerence (Lemma II.) At a distance FD qual to the altitude of the given triangle, raw a right line IL parallel to AB, and he point where this line cuts the circumerence will determine the position of he vertical angle; hence the sides DA nd DB may be drawn.



y Analysis.

Draw the diameter DCP, and from C draw CH perpenicular to AB. Hence having the angle ACB we have also he angles CAB and ABC each equal to $\frac{180°—ACB}{2}$; therefore in the triangle ABC having all the angles and the side AB, he two equal sides AC, BC, become known also. And in the ight-angled triangle AHC, CH$=\sqrt{AC^2—AH^2}$.

And in the similar triangles DFN, CHN we have DF: DN : : CH : CN

nd by composition : : DF+CH : DN+CN

r DF+CH : DC : : DF : DN

lso HN$=\sqrt{CN^2—CH^2}$;

nd BN$=\frac{1}{2}$ AB—HN

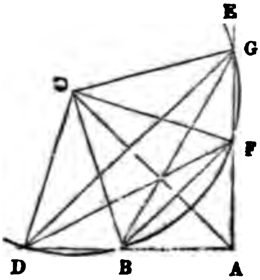
N$=\sqrt{DN^2—DF^2}$

nd BF=BN—NF

Whence we have the right-angled triangled BFD with the ides BF and DF including the right angle, to determine the hird side BD which also becomes known, and consequently the ide AD as required.

PROBLEM XI.

At the distance AB from the bottom of a tower is an object whose length is BD, how far must I ascend the tower that the object may appear under any angle T?



Draw the line AB=the given distance, and produce it to D, then will BD represent the object; draw the indefinite line AE perpendicular to AB, which will represent the side of the tower. Then on DB make an angle BCD=2T, and from the vertex C as a centre, describe an arc of a circle passing through the points D and B, cutting the tower in F, and AF will be the distance required.

For since DCB is an angle in the centre of a circle, and AFB is an angle in the circumference subtending the same arc, hence (Prop. II.) the *angle* DFB=½ *angle* DCB=T.

Cor. Since by continuing the arc of the circle the perpendicular is cut also in G; this point also answers the condition of the question, for *angle* DGB is evidently=the *angle* DFB, hence, the question admits of two answers.*

By Analysis.

First, in the isosceles triangle CBD we have the side BD and the angle C by construction, and since the triangle is isosecles, the angles B and D are each = (180° — 2T)÷2 =90°—T. Hence, having all the angles and one side, the sides CB or CD are also known.

Second, in the triangle ABC we have the sides AB and BC, and the angle B=CBD+½ angle BCD=90°+T, to find the side AC and the angle CAB, which thereby become known.

Third, in the triangle ACF we have the sides AC and FC, and the angle CAF=90°—CAB, to find the side AF, the height required. But since the same data given to determine this triangle, apply also to the triangle ACG, the point may be also in G; hence, the problem is ambiguous both by construction and analysis, as explained in case 4th, chap. VII.

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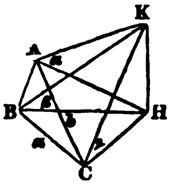
* This elegant construction was received from Mr. Joseph Gallup, of Norwich, Connecticut, whose mathematical talent is acknowledged to be of a high order.

Scholium. If the circumference which passes through the points D, B should not cut the edge of the tower or perpendicular AE, but only touch it, it would admit of only one solution, and that point which would answer the conditions would be the point of contact; but if the circle should not reach the perpendicular, the question would be impossible.

EXAMPLES FOR PRACTICE.

Ex. 1. Given the angles of elevation of any distant object, taken at three places on a level plane, no two of which are in the same vertical plane with the object; to find the height of the object, and its distance from either station.



Let A, B, C, be the three stations, K the object, and KH perpendicular to the plane of the triangle ABC.

Put BC$=a$, AC$=b$, AB$=c$, HAK$=\alpha$, HBK$=\beta$, HCK$=\gamma$, and HK$=x$; then the angles AHK, BHK, CHK being right angles, we have AH$=x$ cot. α, BH$=x$ cot. β, CH$=x$ cot. γ; whereby from the given data the required may be found.

Ex. 2. Given $\alpha=30° 40'$, $\beta=40° 33'$, $\gamma=50° 23'$; find x, when the three stations are in the same straight line, AB being$=50°$ and BC$=60$ yards. Ans. 77.7175 yards.

Ex. 3. Demonstrate that sin. 18°=cos. 72° is $=\frac{1}{4}$ R $(-1+\sqrt{5})$, and sin. 54°=cos. 36° is$=\frac{1}{4}$ R $(1+\sqrt{5})$.

Ex. 4. Demonstrate that the sum of the sines of two arcs which together make 60°, is equal to the sine of an arc which is greater than 60°, by either of the two arcs: Ex. gr. sin. 3'+sin. 59° 57' =sin. 60° 30'; and thus that the tables may be continued by addition only.

Ex. 5. Show the truth of the following proportion: As the sine of half the difference of two arcs, which together make 60°, or 90°, respectively, is to the difference of their sines; so is 1 to $\sqrt{2}$, or $\sqrt{3}$, respectively.

Ex. 6. Demonstrate that the sum of the square of the sine and versed sine of an arc, is equal to the square of double the sine of half the arc.

Ex. 7. Demonstrate that the sine of an arc is a mean proportional between half the radius and the versed sine of double the arc.

Ex. 8. Show that the secant of an arc is equal to the sum of the tangent and the tangent of half its complement.

Ex. 9. Prove that, in any plane triangle, the base is to the difference of the other two sides, as the sine of half the sum of the angles at the base, to the sine of half their difference: als that the base is to the sum of the other two sides as the cosi of half the sum of the angles at the base, to the cosine of ha their difference.

Ex. 10. How must three trees, A, B, C, be planted, so th the angle at A may double the angle at B, the angle at B do ble that at C; and so that a line of 400 yards may just round them?

Ex. 11. In a certain triangle, the sines of the three angl are as the numbers 17, 15, and 8, and the perimeter is 16 What are the sides and angles?

Ex. 12. The logarithms of two sides of a triangle a 2.2407293 and 2.5378191, and the included angle, is 37° 2 It is required to determine the other angles, without first fin ing any of the sides?

Ex. 13. The sides of a triangle are to each other as t fractions ⅓, ¼, ⅕: what are the angles?

Ex. 14. Show that the secant of 60°, is double the tange of 45°, and that the secant of 45° is a mean proportional b tween the tangent of 45° and the secant of 60°.

Ex. 15. Demonstrate that four times the rectangle of t sines of two arcs, is equal to the difference of the squares the chords of the sum and difference of those arcs.

Ex. 16. Convert formulæ ζ, Chap. III, into their equiv lent logarithmic expressions; and by means of them and fo mulæ β, Chap. III, find the angles of a triangle whose sid are 5, 6, and 7.

Ex. 17. Being on a horizontal plane, and wanting to asce tain the height of a tower, standing on the top of an inacce sible hill, there were measured, the angle of elevation of t top of the hill 40°, and the top of the tower 51°: then measu ing in a direct line 180 feet farther from the hill, the angle elevation of the top of the tower was 33° 45′: required t height of the tower. Ans. 83.9983 feet.

Ex. 18. From a station P there can be seen three objec A, B, and C, whose distance from each other are known, vi AB=800, AC=600, and BC=400 yards. There are al measured the horizontal angles APC=33° 45′, BPC=22° 3 It is required, from these data, to determine the three di tances PA, PC, and PB.

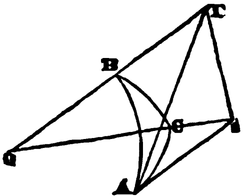
Ans. PA=710.193, PC=1042.522, PB=934.191 yards.

SPHERICAL TRIGONOMETRY.

Having demonstrated in the treatise on Spherical Geometry, several important properties of the circle of the sphere, and of spherical triangles, we shall now proceed to deduce various relations which exist between the several parts of a spherical triangle. These constitute what is called *Spherical Trigonometry;* and enables us, when a certain number of the parts are given, to determine the rest. The first formula which we shall establish, serves as a key to the rest, and is to spherical trigonometry what the expression for the sine of the sum of two angles is to plane trigonometry.

CHAPTER I.

1. *To express the cosine of an angle of a spherical triangle in terms of the sines and cosines of the sides.*



Let ABC be a spherical triangle, O the centre of the sphere.

Let the angles of the triangles be denoted by the large letters A, B, C, and the sides opposite to them by the corresponding small letters, a, b, c.

At the point A, draw AT a tangent to the arc AB; and At a tangent to the arc AC.

Then the spherical angle A is equal to the angle TAt between the tangents, (Spher. Geom. Prop. VII.)

Join OB, and produce it to meet AT in T.

Join OC, and produce it to meet At in t.

Join T, t;

Then,

$$\frac{OT}{OC}=\text{sec. }AB=\text{sec. }c.$$

$$\frac{Ot}{OC}=\text{sec. }AC=\text{sec. }b$$

$$\frac{AT}{OC}=\text{tan. }AB=\text{tan. }c$$

$$\frac{At}{OC}=\text{tan. }AC=\text{tan. }b$$

Then in triangle TO*t*

$$Tt^2=OT^2+Ot^2-2OT\,.\,Ot\text{ cos. }TOt$$

$$\therefore\quad \frac{Tt^2}{OC^2}=\frac{OT^2}{OC^2}+\frac{Ot^2}{OC^2}-2\,.\,\frac{OT}{OC}\,.\,\frac{Ot}{OC}\text{ cos. }TOt$$

$$=\text{sec.}^2\,c+\text{sec.}^2\,b-2\text{ sec. }c\text{ sec. }b\text{ cos. }a,\cdot\ (1)$$

Again, in triangle TA*t*

$$Tt^2=AT^2+At^2-2AT\,.\,At\text{ cos. }TAt$$

$$\therefore\quad \frac{Tt^2}{OC^2}=\frac{AT^2}{OC^2}+\frac{At^2}{OC^2}-2\,.\,\frac{AT}{OC}\,.\,\frac{At}{OC}\text{ cos. }TAt$$

$$=\text{tan.}^2\,c+\text{tan.}^2\,b-2\text{ tan. }c\text{ tan. }b\text{ cos. }A.\cdot\cdot\ (2)$$

Equating (1) and (2)

$$\text{tan.}^2\,c+\text{tan.}^2\,b-2\text{ tan. }c\text{ tan. }b\text{ cos. }A$$

$$=\text{sec.}^2\,c+\text{sec.}^2 b-2\text{ sec. }c\text{ sec. }b\text{ cos. }a$$

$$=1+\text{tan.}^2\,c+1+\text{tan.}^2\,b-2\text{ sec. }c\text{ sec. }b\text{ cos }a$$

$$\therefore\ -2\text{ tan. }c\text{ tan. }b\text{ cos. }A=2-2\text{ sec. }c\text{ sec. }b\text{ cos. }a$$

$$\text{or, }-\frac{\text{sin. }c}{\text{cos. }c}\,.\,\frac{\text{sin. }b}{\text{cos. }b}\text{ cos. }A=1-\frac{1}{\text{cos. }c}\,.\,\frac{1}{\text{cos. }b}\text{ cos. }a$$

$$\left.\begin{array}{l}\therefore\quad \text{cos. }A=\dfrac{\text{cos. }a-\text{cos. }b\text{ cos. }c}{\text{sin. }b\text{ sin. }c}\\ \text{Similarly we shall have,}\\ \text{cos. }B=\dfrac{\text{cos. }b-\text{cos. }a\text{ cos. }c}{\text{sin. }a\text{ sin. }c}\\ \\ \text{cos. }C=\dfrac{\text{cos. }c-\text{cos. }a\text{ cos. }b}{\text{sin. }a\text{ sin. }b}\end{array}\right\}(\alpha)$$

2. *To express the cosine of a side of a spherical triangle, in terms of the sines and cosines of the angles.*

Let A, B, C, *a*, *b*, *c*, be the angles and sides of a spherical triangle; A′, B′, C′, *a*′, *b*′, *c*′, the corresponding qualities in the Polar triangle,

Then by (*α*),

$$\text{cos. } A'=\frac{\text{cos. } a'-\text{cos. } b' \text{ cos. } c'}{\text{sin. } b' \text{ sin. } c'.}$$

But (Spherical Geometry, Prop. X.), A'=(180°—a'), =(180°—A), b'=(180°—B), c'=(180°—C),

$$\text{cos.}(180°-a)=\frac{\text{cos.}(180°-A)-\text{cos.}(180°-B)\text{cos.}(180°-C)}{\text{sin. }(180°-B)\text{ sin. }(180°-C)}$$

$$\left.\begin{aligned}\therefore\quad \text{cos. } a&=\frac{\text{cos. } A+\text{cos. } B \text{ cos. } C}{\text{sin. } B \text{ sin. } C}\\ \text{Similarly,}\quad \text{cos. } b&=\frac{\text{cos. } B+\text{cos. } A \text{ cos. } C}{\text{sin. } A \text{ sin. } C}\\ \text{cos. } c&=\frac{\text{cos. } C+\text{cos. } A \text{ cos. } B}{\text{sin. } A \text{ sin. } B}\end{aligned}\right\}(\beta.)$$

To express the sine of an angle of a spherical triangle, in terms of the sines of the sides of the triangle.

By (α) we have,

$$\text{cos. } A=\frac{\text{cos. } a-\text{cos. } b \text{ cos. } c}{\text{sin. } b \text{ sin. } c}$$

$$\begin{aligned}1+\text{cos. } A&=\frac{\text{cos. } a-\text{cos. } b \text{ cos. } c+\text{sin. } b \text{ sin. } c}{\text{sin. } b \text{ sin. } c}\\ &=\frac{\text{cos. } a-(\text{cos. } b \text{ cos. } c-\text{sin. } b \text{ sin. } c)}{\text{sin. } b \text{ sin. } c}\\ &=\frac{\text{cos. } a-\text{cos. }(b+c)}{\text{sin. } b \text{ sin. } c}\\ &=\frac{2\text{ sin.}\frac{a+b+c}{2}\text{ sin.}\frac{b+c-a}{2}}{\text{sin. } b \text{ sin. } c}\text{ (Plane Trig. Ch. II.)}\end{aligned}$$

$$\text{Let } s=\frac{a+b+c}{2}$$

$$\begin{aligned}\therefore\ s-a&=\frac{b+c-a}{2}\\ s-b&=\frac{a+c-b}{2}\\ s-c&=\frac{a+b-c}{2}\end{aligned}$$

$$1+\text{cos. } A=\frac{2\text{ sin. } s \text{ sin. }(s-a)}{\text{sin. } b \text{ sin. } c}\quad .\ .\ .\ .\ .\ .\ .\quad (1.)$$

Again, resuming the expression for cos. A,

$$1-\cos. A=\frac{\cos. b\cos. c+\sin. b\sin. c-\cos. a}{\sin. b\sin. c}$$

$$=\frac{\cos. (b-c)-\cos. a}{\sin. b\sin. c}$$

$$=\frac{2\sin.\frac{a+b-c}{2}\sin.\frac{a+c-b}{2}}{\sin. b\sin. c}$$

$$=\frac{2\sin.(s-c)\sin.(s-b)}{\sin. b\sin. c}\quad .\ .\ .\ .\ (2.)$$

Multiplying equations (1) and (2)

$$1-\cos.^2 A=\frac{4\sin. s\sin.(s-a)\sin.(s-b)\sin.(s-c)}{\sin.^2 b\sin.^2 c.}$$

$$\left.\begin{array}{l}
\therefore \sin. A=\dfrac{2}{\sin. b\sin. c}\sqrt{\sin. s\sin. (s-a)\sin.(s-b)\sin.(s-c)}\\
\text{Similarly,}\\
\sin. B=\dfrac{2}{\sin. a\sin. c}\sqrt{\sin. s\sin. (s-a)\sin.(s-b)\sin.(s-c)}\\
\sin. C=\dfrac{2}{\sin. a\sin. b}\sqrt{\sin. s\sin (s-a)\sin.(s-b)\sin.(s-c)}
\end{array}\right\}(\gamma.$$

Now, by equation (1) we have,

$$1+\cos. A=\frac{2\sin. s\sin.(s-a)}{\sin. b\sin. c}$$

or

$$2\cos.^2\frac{A}{2}=\frac{2\sin. s\sin. (s-a)}{\sin. b\sin. c}$$

$$\left.\begin{array}{l}
\therefore \cos.\dfrac{A}{2}\sqrt{\dfrac{\sin. s\sin. (s-a)}{\sin. b\sin. c}}\\
\text{Similarly,}\\
\cos.\dfrac{B}{2}\sqrt{\dfrac{\sin. s\sin. (s-b)}{\sin. a\sin. c}}\\
\cos.\dfrac{C}{2}\sqrt{\dfrac{\sin. s\sin. (s-c)}{\sin. a\sin. b}}
\end{array}\right\}(\gamma.2)$$

Next, by equation (2),

$$1-\cos. A=\frac{2\sin. (s-b)\sin. (s-c)}{\sin. b\sin. c}$$

$$2\sin.^2\frac{A}{2}=\frac{2\sin.(s-b)\sin.(s-c)}{\sin.b\sin.c};$$

$$\left.\begin{aligned}\therefore\ \sin.\frac{A}{2}&=\sqrt{\frac{\sin.(s-b)\sin.(s-c)}{\sin.b\sin.c}}\\ \text{arly,}\qquad&\\ \sin.\frac{B}{2}&=\sqrt{\frac{\sin.(s-a)\sin.(s-c)}{\sin.a\sin.c}}\\ \sin.\frac{C}{2}&=\sqrt{\frac{\sin.(s-a)\sin.(s-b)}{\sin.a\sin.b}}\end{aligned}\right\}(\gamma.\ 3)$$

y, dividing the expressions (γ. 3 by those γ. 2), we obtain,

$$\left.\begin{aligned}\tan.\frac{A}{2}&=\sqrt{\frac{\sin.(s-b)\sin.(s-c)}{\sin.s\sin.(s-a)}}\\ \tan.\frac{B}{2}&=\sqrt{\frac{\sin.(s-a)\sin.(s-c)}{\sin.s\sin.(s-b)}}\\ \tan.\frac{C}{2}&=\sqrt{\frac{\sin.(s-a)\sin.(s-b)}{\sin.s\sin.(s-c)}}\end{aligned}\right\}(\gamma.\ 4)$$

To express the sine of a side of a spherical triangle in of the sines and cosines of the angles.

(β) we have,

$$\cos.a=\frac{\cos.A+\cos.B\cos.C}{\sin.B\sin.C}$$

$$\therefore 1+\cos.a=\frac{\cos.A+\cos.B\cos.C+\sin.B\sin.C}{\sin.B\sin.C}$$

$$=\frac{\cos.A+\cos.(B-C)}{\sin.B\sin.C}$$

$$=\frac{2\cos.\frac{A+B-C}{2}\cos.\frac{A+C-B}{2}}{\sin.B\sin.C}\ \text{(PlaneTrig.Ch.II.)}$$

$=\frac{A+B+C}{2}$, $\therefore s'-A=\frac{B+C-A}{2}$, $s'-B=\frac{A+C-B}{2}$,

$-C=\frac{A+B-C}{2}$

Hence,

$$1+\cos. a=\frac{2\cos.(s'-C)\cos.(s'-B)}{\sin. B\sin. C}\quad\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\quad(1.)$$

Resuming expression for cos. *a*,

$$\begin{aligned}1-\cos. a&=-\frac{\cos. B\cos. C-\sin. B\sin. C+\cos. A}{\sin. B\sin C}\\&=-\frac{\cos.(B+C)+\cos. A}{\sin. B\sin. C}\\&=-\frac{2\cos.\dfrac{A+B+C}{2}\cos.\dfrac{B+C-A}{2}}{\sin. B\sin. C}\\&=-\frac{2\cos. s'\cos.(s'-A)}{\sin. B\sin. C}\quad\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\quad(2.)\end{aligned}$$

Multiplying Equations (1.) and (2.).

$$1-\cos.^2 a=-\frac{4\cos. s'\cos.(s'-A)\cos.(s'-B)\cos.(s'-C)}{\sin.^2 B\sin.^2 C}$$

$$\left.\begin{array}{l}\therefore\quad \sin. a=\dfrac{2}{\sin. B\sin. C}\\ \times\sqrt{-\cos. s'\cos.(s'-A)\cos.(s'-B)\cos.(s'-C)}\\ \text{Similarly,}\\ \sin. b=\dfrac{2}{\sin. A\sin. C}\\ \times\sqrt{-\cos. s'\cos.(s'-A)\cos.(s'-B)\cos.(s'-C)}\\ \sin. c=\dfrac{2}{\sin. A\sin. B}\\ \times\sqrt{-\cos. s'\cos.(s'-A)\cos.(s'-B)\cos.(s'-C)}\end{array}\right\}(\delta. 1.)$$

By Equation (1) we have,

$$1+\cos. a=\frac{2\cos.(s'-B)\cos.(s'-C)}{\sin. B.\sin. C}$$

$$\therefore 2\cos.^2\frac{a}{2}=\frac{2\cos.(s'-B)\cos.(s'-C)}{\sin. B\sin. C}$$

$$\left.\begin{array}{l}\therefore\quad\cos.\dfrac{a}{2}=\sqrt{\dfrac{\cos.(s'-B)\cos.(s'-C)}{\sin. B\sin. C}}\\ \text{Similarly,}\\ \cos.\dfrac{b}{2}=\sqrt{\dfrac{\cos.(s'-A)\cos.(s'-C)}{\sin. A\sin. C}}\\ \cos.\dfrac{c}{2}=\sqrt{\dfrac{\cos.(s'-A)\cos.(s'-B)}{\sin. A\sin. B}}\end{array}\right\}(\delta. 2.)$$

By equation (2.)

$$1-\cos.\ a = -\frac{2\cos.\ s'\cos.\ (s'-A)}{\sin.\ B\sin.\ C}$$

$$\therefore 2\sin.^2\frac{a}{2} = -\frac{2\cos.\ s'\cos.\ (s'-A)}{\sin.\ B\sin.\ C}$$

$$\left.\begin{array}{l} \therefore \sin.\ \dfrac{a}{2} = \sqrt{\dfrac{-\cos.\ s'\cos.\ (s'-A)}{\sin.\ B\sin.\ C}} \\ \sin.\ \dfrac{b}{2} = \sqrt{\dfrac{-\cos.\ s'\cos.\ (s'-B)}{\sin.\ A\sin.\ C}} \\ \sin.\ \dfrac{c}{2} = \sqrt{\dfrac{-\cos.\ s'\cos.\ (s'-C)}{\sin.\ A\sin.\ B}} \end{array}\right\} (\delta.\ 3.)$$

inally, dividing the expressions (δ. 3.), by the expressions (δ. 2.)

$$\left.\begin{array}{l} \tan.\ \dfrac{a}{2} = \sqrt{\dfrac{-\cos.\ s'\cos.\ (s'-A)}{\cos.\ (s'-B)\cos.\ (s'-C)}} \\ \tan.\ \dfrac{b}{2} = \sqrt{\dfrac{-\cos.\ s'\cos.\ (s'-B)}{\cos.\ (s'-A)\cos.\ (s'-C)}} \\ \tan.\ \dfrac{c}{2} = \sqrt{\dfrac{-\cos.\ s'\cos.\ (s'-C)}{\cos.\ (s'-A)\cos.\ (s'-B)}} \end{array}\right\} (\delta.\ 4.)$$

It is to be remarked that although the expressions (δ. 1.)
3.), (δ. 4.), appear under an impossible form, they are in re-
y always possible.

'or by Prop. XVIII. of Spherical Geometry, the sum of the
les of a spherical triangle, is always greater than two
ht angles, and less than six right angles.

$$\therefore A+B+C > 180^\circ \text{ and } < 540^\circ$$

$$\therefore \frac{A+B+C}{2} \text{ or } s' > 90^\circ \text{ and } < 270^\circ$$

nce, cosine s' is always negative, and ∴ — cos. s' is always
itive.

Again, if a', b', c', be the three sides of the polar triangle,
ce the sum of any two sides of a spherical triangle is
ater than the third side:

$$b' + c' > a'$$

$$\therefore 180^\circ - B + 180^\circ - C > 180^\circ - A$$

$$\therefore B + C - A < 180^\circ$$

$$\frac{B + C - A}{2} < 90^\circ$$

$\therefore$ cos. $(s' - \text{A})$ is always positive, and in like manner, cos. $(s' - \text{B})$, cos. $(s' - \text{C})$, are always positive; hence the above expressions are in every case possible.

5. *The sines of the angles of a spherical triangle are to each other as sines of the two sides opposite to them.*

Taking the expressions (γ. 1.) and calling the common radical quantity N for the sake of brevity:

$$\sin.\ \text{A} = \frac{2\ \text{N}}{\sin.\ b \sin.\ c}$$

$$\sin.\ \text{B} = \frac{2\ \text{N}}{\sin.\ a \sin.\ c}$$

Dividing the first of these by the second:

$$\left.\begin{array}{l} \dfrac{\sin.\ \text{A}}{\sin.\ \text{B}} = \dfrac{\sin.\ a \sin.\ c}{\sin.\ b \sin.\ c} = \dfrac{\sin.\ a}{\sin.\ b} \\ \text{Similarly,} \\ \dfrac{\sin.\ \text{A}}{\sin.\ \text{C}} = \dfrac{\sin.\ a \sin.\ b}{\sin.\ c \sin.\ b} = \dfrac{\sin.\ a}{\sin.\ c} \\ \\ \dfrac{\sin.\ \text{B}}{\sin.\ \text{C}} = \dfrac{\sin.\ b \sin.\ a}{\sin.\ c \sin.\ a} = \dfrac{\sin.\ b}{\sin.\ c} \end{array}\right\} (\epsilon.)$$

6. *To express the tangent of the sum and difference of two angles of a spherical triangle, in terms of the sides opposite to these angles, and the third angle of the triangle.*

By (α) we have,

$$\cos.\ \text{A} = \frac{\cos.\ a - \cos.\ b \cos.\ c}{\sin.\ b \sin.\ c} \quad . \ . \ . \ . \ . \ . \ . \ . \ (1.)$$

And,

$$\cos.\ \text{C} = \frac{\cos.\ c - \cos.\ a \cos.\ b}{\sin.\ a \sin.\ b}$$

$$\therefore \cos.\ c = \cos.\ a \cos.\ b + \sin.\ a \sin.\ b \cos.\ \text{C} \quad . \ . \ . \ (2.)$$

Substituting this value of cos. c in Equation (1.):

$$\cos.\ \text{A} = \frac{\cos.\ a - \cos.\ a \cos.^2 b - \cos.\ b \sin.\ a \sin.\ b \cos.\ \text{C}}{\sin.\ b \sin.\ c}$$

$$= \frac{\cos.\ a\ (1 - \cos.^2 b) - \cos.\ b \sin.\ a \sin.\ b \cos.\ \text{C}}{\sin.\ b \sin.\ c}$$

$$= \frac{\cos.\ a \sin.\ b - \cos.\ b \sin.\ a \cos.\ \text{C}}{\sin.\ c} \quad . \ . \ . \ (3.)$$

ike manner, substituting the value of cos. c in Equation (2), he expression for cos. B, we shall find,

$$\text{os. B} = \frac{\cos. b \sin. a - \cos. a \sin. b \cos. C}{\sin. c} \quad . \; . \; . \quad (4.)$$

dding equations (3) and (4):

$$\text{s. A} + \cos. B$$

$$\frac{\text{in. } a \cos. b + \sin. b \cos. a - (\sin. a \cos. b + \sin. b \cos. a) \cos. C}{\sin. c}$$

$$= \frac{\sin. (a+b) - \sin. (a+b) \cos. C}{\sin. c}$$

$$= \frac{\sin. (a+b)(1 - \cos. C)}{\sin. c} \quad . \; . \; . \; . \; . \; . \; . \quad (5.)$$

gain, by Equation (e) we have,

$$\frac{\sin. A}{\sin. B} = \frac{\sin. a}{\sin. b}$$

$$\frac{\sin. A + \sin. B}{\sin. B} = \frac{\sin. a + \sin. b}{\sin. b}$$

$$\sin. A \pm \sin. B = (\sin. a \pm \sin. b) \frac{\sin. B}{\sin. b}$$

$$= (\sin. a \pm \sin. b) \frac{\sin. C}{\sin. c} \quad . \; . \; . \quad (6.)$$

Dividing Equation (6) by Equation (5), and taking first the itive sign :

$$\frac{\text{in. A} + \sin. B}{\text{os. A} + \cos. B} = \frac{\sin. a + \sin. b}{\sin. (a+b)} \; \frac{\sin. C}{1 - \cos. C}$$

$$\frac{2 \sin. \frac{A+B}{2} \cos. \frac{A-B}{2}}{\cos. \frac{A+B}{2} \cos. \frac{A-B}{2}} = \frac{2 \sin. \frac{a+b}{2} \cos. \frac{a-b}{2}}{2 \sin. \frac{a+b}{2} \cos. \frac{a+b}{2}} \cot. \frac{C}{2}$$

$$\therefore \quad \tan. \frac{A+B}{2} = \frac{\cos. \frac{a-b}{2}}{\cos. \frac{a+b}{2}} \cot. \frac{C}{2}$$

gain, dividing Equation (6) by Equation (5) and taking negative sign.

$$\frac{\sin. A - \sin. B}{\cos. A + \cos. B} = \frac{\sin. a - \sin. b}{\sin. (a+b)} \; \frac{\sin. C}{1 - \cos. C}$$

$$\frac{2\ \text{sin.}\ \frac{A-B}{2}\ \text{cos.}\ \frac{A+B}{2}}{2\ \text{cos.}\ \frac{A-B}{2}\ \text{cos.}\ \frac{A+B}{2}} = \frac{2\ \text{sin.}\ \frac{a-b}{2}\ \text{cos.}\ \frac{a+b}{2}}{2\ \text{sin.}\ \frac{a+b}{2}\ \text{cos.}\ \frac{a+b}{2}} \qquad \frac{\text{sin.}\,(}{1-\text{cos}}$$

$$\therefore\ \text{tan.}\ \frac{A-B}{2} = \frac{\text{sin.}\ \frac{a-b}{2}}{\text{cos.}\ \frac{a+b}{2}}\ \text{cot.}\ \frac{C}{2}.$$

We have thus obtained the required expression, viz.

$$\left.\begin{aligned}
\text{tan.}\ \frac{A+B}{2} &= \frac{\text{cos.}\ \frac{a-b}{2}}{\text{cos.}\ \frac{a+b}{2}}\ \text{cot.}\ \frac{C}{2}\\
\text{tan.}\ \frac{A-B}{2} &= \frac{\text{sin.}\ \frac{a-b}{2}}{\text{sin.}\ \frac{a+b}{2}}\ \text{cot.}\ \frac{C}{2}\\
&\text{Similarly,}\\
\text{tan.}\ \frac{B+C}{2} &= \frac{\text{cos.}\ \frac{b-c}{2}}{\text{cos.}\ \frac{b+c}{2}}\ \text{cot.}\ \frac{A}{2}\\
\text{tan.}\ \frac{B-C}{2} &= \frac{\text{sin.}\ \frac{b-c}{2}}{\text{sin.}\ \frac{b+c}{2}}\ \text{cot.}\ \frac{A}{2}\\
\text{tan.}\ \frac{A+C}{2} &= \frac{\text{cos.}\ \frac{a-c}{2}}{\text{cos.}\ \frac{a+c}{2}}\ \text{cot.}\ \frac{B}{2}\\
\text{tan.}\ \frac{A-C}{2} &= \frac{\text{sin.}\ \frac{a-c}{2}}{\text{sin.}\ \frac{a+c}{2}}\ \text{cot.}\ \frac{B}{2}
\end{aligned}\right\}\ (\zeta)$$

. *To express the tangent of the sum and difference of two s of a spherical triangle, in terms of the angles opposite to ı and the third side of the triangle.*

et A, B, C, a, b, c, be the sides and angles of a spherical ngle, A′, B′, C′, a', b', c', the corresponding parts of the po- triangle then by expression (ζ),

$$\tan.\frac{A'+B'}{2}=\frac{\cos.\frac{a'-b'}{2}}{\cos.\frac{a'+b'}{2}}\cot.\frac{C'}{2}$$

herefore,

$$.\frac{180°-a+180°-b}{2}=\frac{\cos.\frac{(180°-A)-(180°-B)}{2}}{\cos.\frac{(180°-A)+(180°-B)}{2}}\cot.\frac{(180°-c)}{2}$$

$$\text{an.}\left(180-\frac{a+b}{2}\right)=\frac{\cos.\left(-\frac{A-B}{2}\right)}{\cos.\left(180°-\frac{A+B}{2}\right)}\cot.\ (90°-\frac{c}{2}$$

$$\therefore\quad\tan.\frac{a+b}{2}=\frac{\cos.\frac{A-B}{2}}{\cos.\frac{A+B}{2}}\tan.\frac{c}{2}$$

$$\tan.\frac{A'-B'}{2}=\frac{\sin.\frac{a'-b'}{2}}{\sin.\frac{a'+b'}{2}}\cot.\frac{C'}{2}$$

Therefore,

$$\text{n.}\frac{(180°-a)-(180°-b)}{2}=\frac{\sin.\frac{(180°-A)-(180°-B)}{2}}{\sin.\frac{(180°-A)+(180°-B)}{2}}\cot.\frac{(180°-c)}{2}$$

$$\therefore\quad\tan.\frac{a-b}{2}=\frac{\sin.\frac{A-B}{2}}{\sin.\frac{A+B}{2}}\tan.\frac{c}{2}$$

We shall thus obtain another group of formulæ analog to the last.

$$\left.\begin{array}{l}
\left.\begin{array}{l}
\tan. \dfrac{a+b}{2} = \dfrac{\cos. \dfrac{A-B}{2}}{\cos. \dfrac{A+B}{2}} \tan. \dfrac{c}{2} \\[3ex]
\tan. \dfrac{a-b}{2} = \dfrac{\sin. \dfrac{A-B}{2}}{\sin. \dfrac{A+B}{2}} \tan. \dfrac{c}{2}
\end{array}\right\} \\[8ex]
\left.\begin{array}{l}
\tan. \dfrac{b+c}{2} = \dfrac{\cos. \dfrac{B-C}{2}}{\cos. \dfrac{B+C}{2}} \tan. \dfrac{a}{2} \\[3ex]
\tan. \dfrac{b-c}{2} = \dfrac{\sin. \dfrac{B-C}{2}}{\sin. \dfrac{B+C}{2}} \tan. \dfrac{a}{2}
\end{array}\right\} \\[8ex]
\left.\begin{array}{l}
\tan. \dfrac{a+c}{2} = \dfrac{\cos. \dfrac{A-C}{2}}{\cos. \dfrac{A+C}{2}} \tan. \dfrac{b}{2} \\[3ex]
\tan. \dfrac{a-c}{2} = \dfrac{\sin. \dfrac{A-C}{2}}{\sin. \dfrac{A+C}{2}} \tan. \dfrac{b}{2}
\end{array}\right\}
\end{array}\right\} (\zeta')$$

8. *To express the cotangent of an angle of a spherical t gle, in terms of the side opposite one of the other sides an angle contained between these two sides.*

By (α)

$$\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c} \quad . \; . \; . \; . \; . \; . \; . \; .$$

and,

$$\cos. C = \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b}$$

Hence,

$$\text{cos. } c = \text{cos. } b + \text{sin. } a \text{ sin. } b \text{ cos. C}.$$

Substituting this value of cos. c in equation (1), it becomes

$$\text{cos. A} = \frac{\text{cos. } a - \text{cos. } a \text{ cos.}^2 b - \text{sin. } a \text{ sin. } b \text{ cos. } b \text{ cos. C}}{\text{sin. } b \text{ sin. } c}$$

$$= \frac{\text{cos. } a\,(1 - \text{cos.}^2 b) - \text{sin. } a \text{ sin. } b \text{ cos. } b \text{ cos. C}}{\text{sin. } b \text{ sin. } c}$$

$$\therefore \text{ cos. A} = \frac{\text{cos. } a\,(1 - \text{cos.}^2 b) - \text{sin. } a \text{ sin. } b \text{ cos. } b \text{ cos. C}}{\text{sin. } b \text{ sin. } c}$$

$$\therefore \text{ cos. A sin. } c = \text{cos. } a \text{ sin. } b - \text{sin. } a \text{ cos. } b \text{ cos. C}$$

But,

$$\text{sin. c} = \frac{\text{sin. C}}{\text{sin. A}} \text{ sin. } a, \text{ by } (\epsilon),$$

$$\therefore \quad \text{cos. A } \frac{\text{sin. C}}{\text{sin. A}} \text{ sin. } a = \text{cos. } a \text{ sin. } b - \text{sin. } a \text{ cos. } b \text{ cos C}$$

$$\therefore \qquad \text{cot. A} = \text{cot. } a \text{ sin. } b \text{ cosec. C} - \text{cos. } b \text{ cot. C}.$$

In which the cotangent of A is expressed in the required manner.

If in Equation (1), instead of substituting for cos. c, we had substituted for cos. b, the value derived from the Equation.

$$\text{cos. B} = \frac{\text{cos. } b - \text{cos. } a \text{ cos. } c}{\text{sin. } a \text{ sin. } c},$$

we should have found a value for cot. A in terms of a, c, B, or

$$\text{cot. A} = \text{cot. } a \text{ sin. } c \text{ cosec. B} - \text{cos. } c \text{ cot. B}.$$

Proceeding in like manner for the other angles, we shall obtain similar results, and presenting them at one view, we have

$$\left.\begin{array}{l} \left.\begin{array}{rl} \text{cot. A} = & \text{cot. } a \text{ sin. } b \text{ cosec. C} - \text{cos. } b \text{ cot. C} \\ = & \text{cot. } a \text{ sin. } c \text{ cosec. B} - \text{cos. } c \text{ cot. B} \end{array}\right\} \\ \left.\begin{array}{rl} \text{cot. B} = & \text{cot. } b \text{ sin. } a \text{ cosec. C} - \text{cos. } a \text{ cot. C} \\ = & \text{cot. } b \text{ sin. } c \text{ cosec. A} - \text{cos. } c \text{ cot. A} \end{array}\right\} \\ \left.\begin{array}{rl} \text{cot. C} = & \text{cot. } c \text{ sin. } a \text{ cosec. B} - \text{cos. } a \text{ cot. B} \\ = & \text{cot. } c \text{ sin. } b \text{ cosec. A} - \text{cos. } b \text{ cot. A} \end{array}\right\} \end{array}\right\} (\eta)$$

9. *To express the cotangent of a side of a spherical triangle, in terms of the opposite angle, one of the other angles, and the side interjacent to those two angles.*

Let A, B, C, a, b, c, be the angles and sides of a spherical triangle, and A′, B′, C′, a', b', c', the corresponding parts in the polar triangle.

Then by (η)

$$\cot. A'=\cot. a' \sin. b' \operatorname{cosec.} C'-\cos. b' \cot. C'$$

$$\therefore \cot.(180°-a)=\cot.(180°-A) \sin.(180°-B) \operatorname{cosec.}(180°-c)$$
$$-\cos.(180°-B) \cot.(180°-c)$$
$$-\cot. a=-\cot. A \sin. B \operatorname{cosec.} c-\cos. B \cot. c$$
$$\therefore \cot. a=\cot. A \sin. B \operatorname{cosec.} c+\cos. B \cot. c.$$

Applying the same process to each of the expressions in (η), we shall obtain analogous results, and thus have a new set of formulæ:

$$\left.\begin{array}{l} \left.\begin{array}{l}\cot. a=\cot. A \sin. B \operatorname{cosec.} c+\cos. B \cot. c \\ \quad =\cot. A \sin. C \operatorname{cosec.} b+\cos. C \cot. b\end{array}\right\} \\ \left.\begin{array}{l}\cot. b=\cot. B \sin. A \operatorname{cosec.} c+\cos. A \cot. c \\ \quad =\cot. B \sin. C \operatorname{cosec.} a+\cos. C \cot. a\end{array}\right\} \\ \left.\begin{array}{l}\cot. c=\cot. C \sin. A \operatorname{cosec.} b+\cos. A \cot. b \\ \quad =\cot. C \sin. B \operatorname{cosec.} a+\cos. B \cot. a\end{array}\right\} \end{array}\right\} (\theta)$$

By aid of the nine groups of formulæ marked, (α), (β), (γ), (δ), (ε), (ζ), (ζ′), (η), (θ), we shall be enabled to solve all the cases of spherical triangles, whether right-angled, or oblique-angled; and we shall proceed in the next chapter to apply them.

CHAPTER II.

ON THE SOLUTION OF RIGHT-ANGLED SPHERICAL TRIANGLES.

Spherical triangles, that have one right angle only, are the subject of the investigation of this chapter; those that have two or three right angles are excluded.

A spherical triangle consists of 6 parts, the 3 sides and 3 angles, and any 3 of these being given, the rest may be found. In the present case, one of the angles is by supposition a right angle; if any other two parts be given, the other three may be determined. Now the combination of 5 quantities taken, 3 and $3=\frac{5.\ 4.\ 3}{1.\ 2.\ 3}=10$; therefore ten different cases present themselves in the solution of right-angled triangles.

The manner in which each case may be solved individually, by applying the formulæ already deduced, will be pointed out at the conclusion of this chapter; but we shall in the first place explain two rules, by aid of which the computist is enabled to solve every case of right-angled triangles. These are known by the name of *Napier's Rules for Circular Parts;* and it has been well observed by the late Professor Woodouse, that, in the whole compass of mathematical science, here cannot be found rules which more completely attain that hich is the proper object of all rules, namely, facility and revity of computation.

The rules and their descriptions are as follow:

Description of the Circular parts.

The right angle is thrown altogether out of consideration. he two sides, the complements of the two angles, and the mplement of the hypothenuse, are called *the circular parts.* nd one of these circular parts may be called a *middle part*), and then the two circular parts immediately adjacent to e right and left of M are called *adjacent parts;* the other o remaining circular parts, each separated from M the iddle part by an adjacent part, are called *opposite parts,* or *posite extremes.*

This being premised, we now give

Napier's Rules.

1. *The product of* sin. M *and tabular radius=product of the ngents of the adjacent parts.*

2. *The product of* sin. M *and tabular radius=product of the sines of the opposite parts.*

These rules will be clearly understood if we show the man-r in which they are applied in various cases.

Let A, B, C, be a spherical triangle, right angle at C.

Let a be assumed as the middle part.

Then $(90°-B)$ and b are the adjacent parts.

And $(90°-c)$ and $(90°-A)$ are the opposite parts.

Then by rule (1)

$$R \times \sin. a = \tan. (90° - B) \tan. b$$
$$= \cot. B \tan. b \quad - - - - - - - - - \quad (1)$$

By Rule (2)

$$R. \sin. a = \cos. (90° - A) \cos. (90° - c)$$
$$= \sin. A \sin. c, \quad - - - - - - - - - \quad (2)$$

2. Let b be the middle part,

Then $(90°-A)$ and a are adjacent parts,

Then $(90°-c)$ and $(90°-B)$ are opposite parts.

Then by Rule I,

$$\therefore\ R. \sin. b = \tan. (90° - A) \tan. a$$
$$= \cot. A \tan. a \quad - - - - - - - - - \quad (3)$$

And Rule II,

and $R.\sin. b=\cos.(90^\circ-B)\cos.(90^\circ-c)$

$=\sin. B \sin. c$ - - - - - - - - - - (

3. Let $(90^\circ-c)$ be the middle part.

Then $(90^\circ-A)$, and $(90^\circ-B)$ are adjacent parts,

And b and a are opposite parts.

Then,

$R \sin.(90^\circ-c)=\tan.(90^\circ-A)\tan.(90^\circ-B)$

$R.\cos. c=\cot. A \cot. B$ - - - - - - - -

And,

$R.\sin.(90^\circ-c)=\cos. a \cos. b.$

$R.\cos. c=\cos. a \cos. b$ - - - - - - - - - -

4. Let $(90^\circ-A)$ be the middle part.

Then $(90^\circ-c)$ and b are adjacent parts,

And $(90^\circ-B)$ and a are opposite parts.

Then Rule I.

$R.\sin.(90^\circ-A)=\tan.(90^\circ-c)\tan. b.$

$\therefore$ $R.\cos. A=\cot. c \tan. b$ - - - - - - - - - -

And Rule II.

$R.\sin.(90^\circ-A)=\cos.(90^\circ-B)\cos. a,$

$\therefore$ $R.\cos. A=\sin. B \cos. a$ - - - - - - - - -

5. Let $(90^\circ-B)$ be the middle part.

Then $(90^\circ-c)$ and a are the adjacent parts,

And $(90^\circ-A)$ and b are the opposite parts.

Then Rule I.

$\cos. B=\tan.(90^\circ-c)\tan. a,$

$=\tan. a \cot. c$ - - - - - - - - - -

$\cos. B=\cos.(90^\circ-A)\cos. b,$

$=\sin. A \cos. b$ - - - - - - - - - (

Collecting the above results, and making $R=1$, we shall b

$\sin. a=\cot. B \tan. b$ - - - - - - - - - - - -

$\sin. a=\sin. A \sin. c$ - - - - - - - - - - - -

$\sin. b=\cot. A \tan. a$ - - - - - - - - - - - -

$\sin. b=\sin. B \sin. c$ - - - - - - - - - - - -

$\cos. c=\cot. A \cot. B$ - - - - - - - - - -

$\cos. c=\cos. a \cos. b$ - - - - - - - - - - - -

$\cos. A=\tan. b \cot. c$ - - - - - - - - - - -

$\cos. A=\sin. B \cos. a$ - - - - - - - - - - -

$\cos. B=\tan. a \cot. c$ - - - - - - - - - - -

$\cos. B=\sin. A \cos. b$ - - - - - - - - - - - - (

now remains for us to show that these conclusions are rate, and in accordance with the formulæ already deduced.

ow by (α).

$$\cos.\,C=\frac{\cos.\,c-\cos.\,a\,\cos.\,b}{\sin.\,a\,\sin.\,b}$$

But when C=90°, then cos. C=0.

$$\therefore \quad 0=\frac{\cos.\,c-\cos.\,a\,\cos.\,b}{\sin.\,a\,\sin.\,b}$$

os. $c=\cos.\,a\,\cos.\,b$, which is formula (6) in the above table.

gain by (ε)

$$\frac{\sin.\,a}{\sin.\,c}=\frac{\sin.\,A}{\sin.\,C}$$

But when C=90° sin. C=1

$\sin.\,a=\sin.\,A\,\sin.\,c$, which is formula (2) above.

imilarly,

$$\frac{\sin.\,b}{\sin.\,c}=\frac{\sin.\,B}{\sin.\,C}$$

$\sin.\,b=\sin.\,B\,\sin.\,c$, which is formula (4).

ext since by (α)

$$A=\frac{\cos.\,a-\cos.\,b\,\cos.\,c}{\sin.\,b\,\sin.\,c}$$, substitute for cos. c its value in (6).

$$=\frac{\cos.\,a-\cos.\,a\,\cos.^2 b}{\sin.\,b\,\sin.\,c}$$

$$=\frac{\cos.\,a\,\sin.\,b}{\sin.\,c}$$, substitute for sin. c, its value as found in (2.)

$$=\frac{\cos.\,a\,\sin.\,b}{\frac{\sin.\,a}{\sin.\,A}}$$

n.$b=\cot.\,A\,\tan.\,a$, which is formula (3.)

in,

$$A=\frac{\cos.\,a-\cos.\,b\,\cos.\,c}{\sin.\,b\,\sin.\,c}$$ substitute for cos. a, its value in (6.)

$$=\frac{\frac{\cos.\,c}{\cos.\,b}-\cos.\,b\,\cos.\,c}{\sin.\,b\,\sin.\,c}$$

$$=\frac{\cos.\,c\,\sin.\,b}{\sin.\,c\,\cos.\,b}$$

$=\tan.\,b\,\cot.\,c$, which is formula (7.)

Again by (α.) cos. B

$$=\frac{\cos. b-\cos. a\cos. c}{\sin. a\sin. c}\text{ substitute for cos. } c \text{ its value from (6)}$$

$$=\frac{\cos. b-\cos. b\cos.^2 a}{\sin. a\sin. c}$$

$$=\frac{\cos. b\sin. a}{\sin. c}\text{ substitute for sin. } c\text{, its value from (4.)}$$

$$=\frac{\cos. b\sin. a}{\frac{\sin. b}{\sin. B}}$$

Sin. a=cot. B tan. b, which is formula (1.)

Again, cos. B

$$=\frac{\cos. b-\cos. a\cos. c}{\sin. a\sin. c}\text{, substitute for cos. } b\text{, its value in (6.)}$$

$$=\frac{\frac{\cos. c}{\cos. a}-\cos. a\cos. c}{\sin. a\sin. c}$$

$$=\frac{\cos. c\sin. a}{\sin. c\cos. a}$$

=tan. a cot. c, which is formula (9.)

Next by (β.)

$$\cos. a=\frac{\cos. A+\cos. B\cos. C}{\sin. B\sin. C}$$

But C=90° ∴ cos. C=0, and sin. C=1.

$$\therefore\ \cos. a=\frac{\cos. A}{\sin. B}$$

∴ cos. A=sin. B cos. a, which is formula (8.)

Again,

$$\cos. b=\frac{\cos. B+\cos. A\cos. C}{\sin. A\sin. C}\text{ and when C=90°.}$$

$$=\frac{\cos. B}{\sin. A}$$

∴ cos. B=sin. A cos. b, which is formula (10.)

Lastly,

$$\cos. c=\frac{\cos. C+\cos. A\cos. B}{\sin. A.\sin. B}\text{ and in this case,}$$

$$=\frac{\cos. A \cos. B}{\sin. A \sin. B}$$

$$=\cot. A \cot. B, \text{ which is formula (5.)}$$

We have thus proved the truth of the results derived from the application of Napier's rules, and may therefore apply these rules without scruple to the solution of various cases of right-angled triangles.

Let us then take each combination of the two data, and determine in each case the other three quantities, adapting our formulæ to computation by tables.

1. Given A, B, required *a*, *b*, *c*.

$$R \cos. A=\sin. B \cos, a \therefore \cos. a=R\frac{\cos. A}{\sin. B} \quad - \ - \quad (1)$$

$$R \cos. B=\sin. A \cos. b \therefore \cos. b=R\frac{\cos. B}{\sin. A} \quad - \ - \quad (2)$$

$$R \cos. c=\cot. A \cot. B \quad - \ - \ - \ - \ - \ - \ - \ - \ - \ - \quad (3)$$

2. Given *a*, *b*, required A, B, *c*,

$$R \sin. a=\cot. B \tan. b \therefore \cot. B=R \sin. a \cot. b \quad - \quad (4)$$

$$R \sin. b=\cot. A \tan. a \therefore \cot. A=R \sin. b \cot. a \quad - \quad (5)$$

$$R \cos. c=\cos. a \cos. b \quad - \ - \ - \ - \ - \ - \ - \ - \ - \ - \quad (6)$$

3. Given *a*, *c*, required A, B, *b*

$$R \sin. a=\sin. A \sin. c \therefore \sin. A=R\frac{\sin. a}{\sin. c} \quad - \ - \ - \quad (7)$$

$$R \cos. c=\cos. a \cos. b \therefore \cos. b=R\frac{\cos. c}{\cos. a} \quad - \ - \ - \quad (8)$$

$$R \cos. B=\tan. a \cot. c \quad - \ - \ - \ - \ - \ - \ - \ - \ - \ - \quad (9)$$

4. Given *b*, *c*, required A, B, *a*.

$$R \sin. b=\sin. B \sin. c \therefore \sin. B=R\frac{\sin. b}{\sin. c} \quad - \ - \quad (10)$$

$$R \cos. c=\cos. a \cos. b \therefore \cos. a=R\frac{\cos. c}{\cos. b} \quad - \ - \quad (11)$$

$$R \cos. A=\tan. b \cot. c \quad - \ - \ - \ - \ - \ - \ - \ - \ - \ - \quad (12)$$

5. Given A, *c*, required B, *a*, *b*.

$$R \cos. A=\tan. b \cot. c \therefore \tan. b=R \cos. a \tan. c \quad - \quad (13)$$

$$R \cos. c=\cot. A \cot. B \therefore \cot. B=R \tan. A \cos. c \quad - \quad (14)$$

$$R \sin. a=\sin. A \sin. c \quad - \ - \ - \ - \ - \ - \ - \ - \ - \ - \quad (15)$$

6. Given B, *c*, required A, *a*, *b*.

$$R \cos. B=\cot. c \tan. a \therefore \tan. a=R \cos. B \tan. c \quad - \quad (16)$$

$$R \cos. c = \cot. A \cot. B \therefore \cot. A = R \tan. B \cos. c \tag{17}$$
$$R \sin. b = \sin. B \sin. c \tag{18}$$

7. Given A, b, required B, c, a.

$$R \cos. A = \cot. c \tan. b \therefore \cot. c = R \cos. A \cot. b \tag{19}$$
$$R \sin. b = \cot. A \tan. a \therefore \tan. a = R \tan. A \sin. b \tag{20}$$
$$R \cos. B = \sin. A \cos. b \tag{21}$$

8. Given B, a, required A, c, b.

$$R \cos. B = \cot. c \tan. a \therefore \cot. c = R \cos. B \cot. a \tag{22}$$
$$R \sin. a = \cot. B \tan. b \therefore \tan. b = R \tan. B \sin. a \tag{23}$$
$$R \cos. A = \sin. B \cos. a \tag{24}$$

9. Given A, a, required B, b, c.

$$R \cos. A = \sin. B \cos. a \therefore \sin. B = R \frac{\cos. A}{\cos. a} \tag{25}$$
$$R \sin. a = \sin. A \sin. c \therefore \sin. c = R \frac{\sin. a}{\sin. A} \tag{26}$$
$$R \sin. b = \cot. A \tan. a \tag{27}$$

10. Given B, b, required A, a, c.

$$R \cos. B = \sin. A \cos. b \therefore \sin. A = R \frac{\cos. B}{\cos. b} \tag{28}$$
$$R \sin. b = \sin. B \sin. c \therefore \sin. c = R \frac{\sin. b}{\sin. B} \tag{29}$$
$$R \sin. a = \cot. B \tan. b \tag{30}$$

CHAPTER III.

ON THE SOLUTION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES.

The different cases which present themselves are contained in the following enumerations.

1. When two sides and the included angle are given.
2. When two angles and the side between them are given.
3. When two sides and the angle opposite to one of them are given.
4. When two angles and the side opposite to one of them are given.
5. When three sides are given.
6. When three angles are given.

I. When two sides and the included angle are given.

The remaining angles may be determined from the formula (σ.)

Thus, let a, b, C, be given, A, B, c, required.

$$\tan.\frac{A+B}{2}=\frac{\cos.\frac{a-b}{2}}{\cos.\frac{a+b}{2}}\cot.\frac{C}{2}$$

$$\tan.\frac{A-B}{2}=\frac{\sin.\frac{a-b}{2}}{\sin.\frac{a+b}{2}}\cot.\frac{C}{2}$$

Whence $\frac{A+B}{2}$ and $\frac{A-B}{2}$ are known from the tables.

Let $\frac{A+B}{2}=\theta$

$\frac{A-B}{2}=\varphi$

$\therefore\quad A=\theta+\varphi$

$B=\theta-\varphi$

A and B being known, c may be obtained from (ε.)

For $\frac{\sin.c}{\sin.a}=\frac{\sin.C}{\sin.A}$

$\therefore\quad \sin.c=\sin.a\frac{\sin.C}{\sin.A}$

And, in like manner, if any two other sides and the included angle be given, the remaining parts may be determined.

II. When two angles and the side between them are given.

The remaining sides may be determined from the formula (ζ′.)

Thus, let A, B, c, be given; a, b, C, required.

$$\tan.\frac{a+b}{2}=\frac{\cos.\frac{A-B}{2}}{\cos.\frac{A+B}{2}}\tan.\frac{c}{2}$$

$$\tan.\frac{a-b}{2}=\frac{\sin.\frac{A-B}{2}}{\sin.\frac{A+B}{2}}\tan.\frac{c}{2}$$

Whence $\frac{a+b}{2}$ and $\frac{a-b}{2}$ are known from the tables.

Let $$\frac{a+b}{2}=\theta'$$

$$\frac{a-b}{2}=\varphi'$$

$$\therefore \quad a=\theta'+\varphi'$$

$$b=\theta'-\varphi'$$

a and b being known, C may be obtained by (ε.)

For $$\frac{\sin.\ C}{\sin.\ A}=\frac{\sin.\ c}{\sin.\ a}$$

$$\therefore \quad \sin.\ C=\sin.\ A\ \frac{\sin.\ c}{\sin.\ a}$$

And, in like manner, if any two other angles and the cluded side are given, the remaining parts may be determi

III. When two sides and the angle opposite to one of th are given.

The angle opposite to the other side may be found fr formula (ε.)

Thus, let a, b, A be given, B, C, c, required.

$$\frac{\sin.\ B}{\sin.\ A}=\frac{\sin.\ b}{\sin.\ a}$$

$$\therefore \quad \sin.\ B=\sin.\ A\ \frac{\sin.\ b}{\sin.\ a}$$

The angle B being determined, the remaining angle C w be found from (σ.)

For $$\tan.\ \frac{A+B}{2}=\frac{\cos.\ \frac{a-b}{2}}{\cos.\ \frac{a+b}{2}}\cot.\ \frac{C}{2}$$

$$\cot.\ \frac{C}{2}=\frac{\cos.\ \frac{a+b}{2}}{\cos.\ \frac{A-B}{2}}\tan.\ \frac{A+B}{2}$$

The angle C being determined, the remaining side c will l found from (ε.)

For $$\frac{\sin.\ c}{\sin.\ a}=\frac{\sin.\ C}{\sin.\ A}$$

$$\therefore \quad \sin.\ c=\sin.\ a\ \frac{\sin.\ C}{\sin.\ A}$$

or c may be found from (ζ'.)

And, in like manner, if any other two sides and the ang opposite to one of them be given, the remaining parts may t determined.

IV. When two angles and the side opposite to one of them re given.

The side opposite to the other angle may be found from for- ula (ε.)

Thus, let A, B, a, be given; b, c, C, required.

$$\frac{\text{sin. } b}{\text{sin. } a} = \frac{\text{sin. B}}{\text{sin. A}}$$

$$\therefore \qquad \text{sin. } b = \text{sin. } a \, \frac{\text{sin. B}}{\text{sin. A}}$$

The side b being determined, the remaining side c will be und from (ζ')

$$\text{For} \qquad \text{tan.} \frac{a+b}{2} = \frac{\text{cos. } \frac{\text{A}-\text{B}}{2}}{\text{cos. } \frac{\text{A}+\text{B}}{2}} \, \text{tan.} \frac{c}{2}$$

$$\therefore \qquad \text{tan.} \frac{c}{2} = \frac{\text{cos. } \frac{\text{A}+\text{B}}{2}}{\text{cos. } \frac{\text{A}-\text{B}}{2}} \, \text{tan.} \frac{a+b}{2}$$

The side c being determined, the remaining angle C will be und from (ε.)

$$\text{For} \qquad \frac{\text{sin. C}}{\text{sin. A}} = \frac{\text{sin. } c}{\text{sin. } a}$$

$$\therefore \qquad \text{sin. C} = \text{sin. A} \, \frac{\text{sin. } c}{\text{sin. } a}$$

c may be found from (σ.)

And, in like manner, any other two sides being given and angle opposite to one of them, the remaining parts may be termined.

V. When three sides are given.

The three angles may be immediately determined from any of the formulæ (γ 1,) (γ 2,) (γ 3,) (γ 4.)

The choice of the formula, which it will be advantageous to ploy in practice, will depend upon the consideration already iced in the solution of the analogous case in plane trigonetry.

VI. When three angles are given.

The three sides may be immediately determined from any he groups of formulæ (δ 1,) (δ 2,) (δ 3,) (δ 4.)

CHAPTER IV.

ON THE USE OF SUBSIDIARY ANGLES.

We have already explained in plane Trigonometry, the meaning of Subsidiary Angles, and the purpose for which they are introduced; we shall now proceed to point out under what circumstances they may be employed with advantage, in Spherical Trigonometry.

In the solution of case I., where two sides and the included angle were given, we first determined the two remaining angles, and having found these, we were enabled to find the side also. It frequently happens, however, that the side alone is the object of our investigations, and it is therefore convenient to have a method of determining it, independently of the angle.

Thus, for example, let b, c, A be given, and let it be required to determine a, independently of the angles B, c.

By $(\alpha,)$ we have

$$\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c}$$

Whence $\cos. a = \cos. A \sin. b \sin. c + \cos. b \cos. c$.

From which equation a is determined, but the expression is not in a form adapted to the logarithmic computation; we can, however, effect the necessary transformation by the introduction of a subsidiary angle.

Add and subtract $\sin. b \sin. c$ on the right hand side of the equation.

Then $\cos. a$

$$= \cos. A \sin. b \sin. c + \cos. b \cos. c + \sin. b \sin. c - \sin. b \sin. c$$

$$= \cos. b \cos. c + \sin. b \sin. c + \sin. b \sin. c \cos. A - \sin. b \sin. c$$

$$= \cos. (b - c) - \sin. b \sin. c \text{ vers. } A$$

$$1 - \cos. a = 1 - \cos. (b - c) + \sin. b \sin. c \text{ vers. } A$$

$$\text{vers. } a = \text{vers. } (b - c) + \sin. b \sin. c \text{ vers. } A$$

$$= \text{vers. } (b - c) \left\{ 1 + \frac{\sin. b \sin. c \text{ vers. } A}{\text{vers. } (b - c)} \right\}$$

Let $\tan.^2 \theta = \dfrac{\sin. b \sin. c \text{ vers. } A}{\text{vers. } (b - c)}$

$$\therefore \text{ vers. } a = \text{vers. } (b - c) \{ 1 + \tan.^2 \theta \}$$

$$= \text{vers. } (b - c) \sec.^2 \theta$$

from which a may be determined by the tables, θ being known from the equation

$$\tan.^2 \theta = \frac{\sin. b \sin. c \text{ vers. } A}{\text{vers. } (b - c)}$$

In like manner in case II, where two angles and the included side were given, we first determined the remaining sides, and then we were enabled to find the remaining angle. Now, let us suppose that A, B, *c*, are given, and that we are required to find C independently of *a* and *b*.

From (β) $\cos. c = \dfrac{\cos. C + \cos. A \cos. B}{\sin. A \sin. B}$

$$\therefore \quad \cos. C = \cos. c \sin. A \sin. B - \cos. A \cos. B$$

$$\therefore \quad 1 - \cos. C = 1 - \sin. A \sin. B\,(1 - \text{vers. } c) + \cos. A \cos. B.$$

$$= 1 + \cos.\,(A+B) + \sin. A \sin. B \text{ vers. } c.$$

$$\text{or} \quad 2 \sin.^2 \frac{C}{2} = 2 \cos.^2 \frac{A+B}{2} + \sin. A \sin. B \text{ vers. } c$$

$$= 2 \cos.^2 \frac{A+B}{2} \left\{ 1 + \frac{\sin. A \sin. B \text{ vers. } c}{2 \cos.^2 \dfrac{A+B}{2}} \right\}$$

$$\therefore \sin.^2 \frac{C}{2} = \cos.^2 \frac{A+B}{2} \sec.^2 \theta$$

If we assume

$$\tan.^2 \theta = \frac{\sin. A \sin. B \text{ vers. } c}{2 \cos^2 \dfrac{A+B}{2}}$$

In case III, where two sides and the angle opposite to one of them were given, we first determined the angle opposite to the other side, and then the remaining angles and the remaining side in succession. Now, let us suppose the side *c*, independently of the angle B and of each other, under a form adapted for logarithmic computation.

To find C, we have (η.)

$$\cot. A = \cot. a \sin. b \operatorname{cosec.} C - \cos. b \cot. C$$

$$\text{or} \quad \cot. A \sin. C = \cot. a \sin. b - \cos. b \cos. C$$

$$\text{or} \quad \sin. C = \cot. a \sin. b \tan. A - \cos. b \cos. C \tan. A$$

$$\therefore \sin. C + \cos. C \cos. b \tan. A = \cot. a \sin. b \tan. A.$$

$$\text{Let} \quad \cos. b \tan. A = \tan. \theta = \frac{\sin. \theta}{\cos. \theta}$$

$$\therefore \quad \sin. C + \frac{\sin. \theta}{\cos. \theta} \cos. C = \cot. a \sin. b \tan. A$$

$$\therefore \sin. C \cos. \theta + \cos. C \sin. \theta = \cot. a \sin. b \tan. A \cos. \theta$$

$$\sin. (C+\theta) = \cot. a \sin. b \tan. A \frac{\sin. \theta}{\cos. b \tan. A}$$

$$= \cot. a \tan. b \sin. \theta$$

whence C is known, θ being previously determined from equation

$$\tan.\ \theta=\cos.\ b\ \tan.\ A.$$

To find c, we have from (α.)

$$\cos.\ A=\frac{\cos.\ a-\cos.\ b\ \cos.\ c}{\sin.\ b\ \sin.\ c}$$

$$\therefore\quad \sin.\ c\ \sin.\ b\ \cos.\ A=\cos.\ a-\cos.\ b\ \cos.\ c$$

$$\sin.\ c\ \tan.\ b\ \cos.\ A=\frac{\cos.\ a}{\cos.\ b}-\cos.\ c$$

$$\sin.\ c\ \tan.\ b\ \cos.\ A=\frac{\cos.\ a}{\cos.\ b}-\cos.\ c$$

Let $\tan.\ b\ \cos.\ A=\tan.\ \theta=\dfrac{\sin.\ \theta}{\cos.\ \theta}$

$$\therefore\quad \sin.\ c\frac{\sin.\ \theta}{\cos.\ \theta}+\cos.\ c=\frac{\cos.\ a}{\cos.\ b}$$

$$\cos.\ (c-\theta)=\frac{\cos.\ a\ \cos.\ \theta}{\cos.\ b}$$

whence c may be found, θ being previously determined from the equation

$$\tan.\ \theta=\tan.\ b\ \cos.\ A.$$

In like manner, in case IV, when two angles and the side opposite to one of them were given, we first determined the side opposite to the other angle, then the remaining side and the remaining angle in succession. Now, let A, B, a, be given, and let it be required to determine c and C, independently of b and of each other, and under a form adapted to logarithmic computations. If we take the formula (θ.)

$$\cot.\ a=\cot.\ A\ \sin.\ B\ \text{cosec.}\ c+\cos.\ B\ \cot.\ c$$

or $\cot.\ a\ \sin.\ c=\cot.\ A\ \sin.\ B+\cos.\ B\ \cos.\ c$

or $\sin.\ c=\cot.\ A\ \sin.\ B\ \tan.\ a+\cos.\ B\ \cos.\ c\ \tan.\ a$

$$\therefore\ \sin.\ c-\cos.\ c\ \cos.\ B\ \tan.\ a=\cot.\ A\ \sin.\ B\ \tan.\ a.$$

Let $\cos.\ B\ \tan.\ a=\tan.\ \theta=\dfrac{\sin.\ \theta}{\cos.\ \theta}$

$$\sin.\ c-\frac{\sin.\ \theta}{\cos.\ \theta}\cos.\ c=\cot.\ A\ \sin.\ B\ \tan.\ a$$

$$\begin{aligned}\sin.\ (c-\theta)&=\cot.\ A\ \sin.\ B\ \tan.\ a\ \cos.\ \theta\\&=\cot.\ A\ \sin.\ B\ \tan.\ a\frac{\sin.\ \theta}{\cos.\ B\ \tan.\ a}\\&=\cot.\ A\ \tan.\ B\ \sin.\ \theta\end{aligned}$$

whence c may be determined, θ being previously known from equation $\tan.\ \theta=\cos.\ B\ \tan.\ a.$

To find C, we have from (β)

$$\cos. a = \frac{\cos. A + \cos. B \cos. C}{\sin. B \sin. C.}$$

$$\therefore \quad \sin. B \sin. C \cos. a = \cos. A + \cos. B \cos. C$$

$$\therefore \quad \sin. C \tan. B \cos. a = \frac{\cos. A}{\cos. B} + \cos. C$$

$$\therefore \sin. C \tan. B \cos. a - \cos. C = \frac{\cos. A}{\cos. B}$$

Let $\tan. B \cos. a = \tan. \theta = \frac{\sin. \theta}{\cos. \theta}$

$$\therefore \quad \sin. C \times \frac{\sin. \theta}{\cos. \theta} - \cos. C = \frac{\cos. A}{\cos. B}$$

$$\therefore \quad -\cos. (C+\theta) = \frac{\cos. A \cos. \theta}{\cos. B}$$

whence C may be found, θ being known from equation

$$\tan. \theta = \tan. B \cos. a.$$

In the fifth and sixth cases, any one of the angles or sides required, may be found independently of the rest by the formulæ referred to.

EXAMPLES IN SPHERICAL TRIGONOMETRY.

Ex. 1. In the right-angled spherical triangle ABC, the hypothenuse AB is 65° 5′, and the angle A is 48° 12′; find the sides AC, CB, and the angle B.

Ans. AC=55° 7′ 32″
BC=42 32 19
angle B=64 46 14

Ex. 2. In the oblique-angled spherical triangle ABC, given AB=76° 20′, BC=119° 17′, and angle B=52° 5′; to find AC and the angles A and C.

Ans. AC=66° 5′ 36″
angle A=131 10 42
angle C= 56 58 58

Ex. 3. In an oblique spherical triangle the three sides are

$$a=81° 17', b=114° 3', c=59° 12';$$

required the angles A, B, C.

Ans. A= 62° 39′ 42″
B=124 50 50
C= 50 34 42.

APPLICATION OF ALGEBRA TO GEOMETRY.

ON THE GEOMETRICAL CONSTRUCTION OF ALGEBRAICAL QUANTITIES.

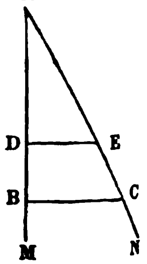
As lines, surfaces, and solids are quantities which admit of increase and decrease, like other quantities, they may, like others, be made the subjects of algebraical operations, either by their numerical representatives, or by symbols expressing such quantities. It is only necessaryfor this purpose, that their representatives should possess values in relation to each other corresponding to the magnitudes of the quantities which they represent; and that those values should be expressed according to the properties or relations of the lines, surfaces, or solids, to each other; subject to geometrical construction and algebraical notation.

We are enabled, also, to express by lines, surfaces, and solids, the solutions furnished by Algebra. This is founded on the known properties of geometrical figures, corresponding to similar properties of the quantities algebraically expressed. In this view of the subject, all quantities under algebraic expressions, may be conceived to be susceptible of some kind of geometrical construction.

2. We will proceed to explain the manner of constructing those expressions, and representing, under a geometrical form, the conditions of an equation. This is called *constructing* the algebraic quantities.

Ex. 1. Let it be proposed to construct such a quantity as the following: $\frac{ab}{c}$ the value of the letters composing the quantity being known.

From any point A draw two indefinite lines AM, AN, making any angle with each other; upon one of these lines AM take AB$=c$, and AD$=a$; then upon the line AN take AC$=b$. Having drawn the line BC, draw also DE parallel to BC, this will determine AE as the value of $\frac{ab}{c}$. For the parallels DE, BC, give this proportion AB : AD : : AC : AE, (Prop. XIV. Cor. 2 B. IV. *El. Geom.*) or $c : a : : b :$ AE.



Therefore, $AE=\frac{ab}{c}$ - - - - - - - - - - - - ∴ 1

Hence, the line AE being the fourth proportional to the three lines represented by c, a, b, it may be used for the construction of the quantity $\frac{ab}{c}$.

2d. Hence, also if it were proposed to construct the quantity $\frac{a^2}{c}$ it may evidently be done in a similar manner, since in this case the lines b and a would be equal, for if $\frac{a^2}{c}=\frac{ab}{c}$ then $a=b$.

3d. If it were proposed to construct $\frac{ab+bd}{cd}$; it may be observed that the quantity may be resolved into the expression $\frac{(a+d)\,b}{c+d}$, hence representing $a+d$ by m, and $c+d$ by n, we shall have $\frac{mb}{n}$ to be constructed, which may be referred to the former case.

4th. Let the quantity to be constructed be $\frac{a^2-b^2}{c}$; it may be observed that a^2-b^2 is equivalent to $(a+b)\times(a-b)$; hence, $\frac{c^2-b^2}{c}$ may be represented under the form $\frac{(a+b)\times(a-b)}{c}$; and we have only to find the fourth proportional to $c, a+b, a-b$.

5th. If the quantity to be constructed be $\frac{abc}{de}$, it may be put under the form $\frac{ab}{d}+\frac{c}{e}$ and having constructed $\frac{ab}{d}$ in the manner just explained, we call the line given by this construction, m; then $\frac{ab}{d}+\frac{c}{e}$ becomes $\frac{mc}{e}$ which may also be constructed as above shown.

6th. It will be presumed, therefore, that in order to construct $\frac{a^2b}{c^2}$ it may be represented under the form $\frac{a^2}{c}+\frac{b}{c}$; whence if we construct $\frac{a^2}{c}$ and represent its value by m, we may proceed to construct $\frac{mb}{c}$

Thus the whole art consists in decomposing the quantity into portions, each of which retains the form $\frac{ab}{c}$ or $\frac{a^2}{c}$; and although this process may appear sometimes difficult, yet we may easily arrive at the object proposed by employing transformations.

7th. If, for example, $\frac{a^3+b^3}{a^2+c^2}$ is to be constructed, we may take $b^3=a^2m$, and $c^2=an$; then, $\frac{a^3+b^3}{a^2+c^2}$ becomes $\frac{a^3+a^2m}{a^2+an}$ which may be reduced to $\frac{a^2+am}{a+n}$ or $\frac{(a+m)\,a}{a+n}$, a quantity easy to be constructed after what has been said, when m and n are known.

Now to determine m and n, the equations $b^3=a^2m$, $c^2=an$, give, $m=\frac{b^3}{a^2}$ and $n=\frac{c^2}{a}$ which may be constructed by the methods already explained.

Thus, while the quantity is rational, that is, without radical expressions, if the dimensions of the numerator do not exceed those of the denominator except by unity, we may always reduce the construction to the finding of a fourth proportional to three given lines.

It sometimes happens, that quantities present themselves under a form, that renders recourse to transformations of no use; this is when the quantity is not *homogeneous*, that is, when each of the terms of the numerator and denominator is not composed of the same number of factors; when the quantity, for example, is such as $\frac{a^2+b}{c^2+d}$.

But it should be observed, that we never arrive at a result of this kind, except when, in the course of an investigation, we suppose, with a view of simplifying the calculation, some one of the quantities equal to unity. If, for example, in $\frac{a^2+b^2c}{a^2+c^2}$, we suppose b equal to 1, we shall have $\frac{a^2+c}{a^2+c^2}$. But, as we never undertake to construct a quantity without knowing the elements which we are to use for this construction, we always know in each case what is the quantity which is supposed equal to unity. We can always therefore restore it, and the above difficulty cannot occur; because, as the number of dimensions must be the same in each term of the numerator, and also of the denominator, although the number of terms may be different in the one from what it is in the other, we restore in each term a power of the line, which is taken for unity, sufficiently raised to complete the number of dimensions; thus, if we construct $\frac{a^3 + b + c^2}{a + b^2}$; d being supposed to be the line which is taken for unity, we may write formula $\frac{a^3 + b\,d^2 + c^2\,d}{a\,d + b^2}$, which may be constructed by making $b^2 = d\,m$, $c^2 = d\,n$, and $a^3 = d^2\,p$, which will change it into

$$\frac{d^2\,p + b\,d^2 + d^2\,n}{a\,d + d\,m}$$

or $\frac{d\,p + b\,d + d\,n}{a + m}$, or $\frac{(p + b + n)\,d}{a + m}$, a quantity easily constructed, when we have constructed the value of m, n, and p; namely, $m = \frac{b^2}{d}$, $n = \frac{c^2}{d}$, $p = \frac{a^3}{d^2}$, which is readily done after what has been said.

Hitherto we have supposed that the number of factors, or the dimensions of each term of the numerator exceeds the number of factors, or the dimensions of the denominator only by unity. It may exceed that number by two or even three, but never by more than three, unless some line has been supposed equal to unity, or some of the factors do not represent numbers.

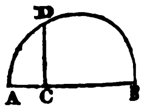
3. When the dimensions of the numerator of the proposed quantity exceed by two the dimensions of the denominator, the quantity expressed is a surface, the construction of which can always be referred to that of a rhomboid. and consequently to that of a square. If, for example, the quantity to be constructed be $\frac{a^3 + a^2\,b}{a + c}$

it may be considered as $a \times \frac{a^2 + a\,b}{a + c}$. Now $\frac{a + a\,b}{a + c}$ is easily constructed, after what has been laid down, by considering it as $a \times \frac{a + b}{a + c}$. Let us suppose therefore that m is the value of the line thus obtained; then $a \times \frac{a^2 + a\,b}{a + c}$ will become $a \times m$. Now if we make a the altitude and m the base of a rhomboid, we shall have $a \times m$ for the surface of this rhomboid, (Prop. VI, B. IV, *El. Geom.*) therefore, reciprocally, this surface will represent $a \times m$, or $\frac{a^3 + a^2 b}{a + c}$.

In like manner, the quantity $\frac{a^3 + b\,c^2 + d^3}{a + c}$ may be reduced to a similar construction by making $bc = am$, and $d^2 = an$; for it will then become $\frac{a^3 + amc + and}{a + c}$ or $a\left(\frac{a^2 + mc + nd}{a + c}\right)$ Now the factor $\frac{a^2 + mc + nd}{a + c}$ refers itself to the preceding constructions, as also the values of m, n. Having found the value of this factor, if we represent it by p, we have only to construct $a \times p$, that is, to make a rhomboid whose altitude is a and base p.

4. Lastly, if the dimensions of the numerator exceed the dimensions of the denominator by three, the quantity expresses a solid, the construction of which may always be reduced to a parallelopiped. If, for example, we were to construct $\frac{a^3 b + a^2 b^2}{a + c}$, we might consider this quantity as the same as $ab \times \frac{a^2 + ab}{a + c}$; and, having constructed $\frac{a^2 + ab}{a + c}$ in the manner already explained, if we represent by m, the line given by this construction, the question will be reduced to this, namely, to construct $ab \times m$. Now ab represents, as we have seen, a rhomboid; if, therefore, we conceive a parallelopiped, having for its base this rhomboid, and for its altitude the line m, the solidity of this parallelopiped will represent $ab \times m$, that is, $\frac{a^3 b + a^2 b^2}{a + c}$.

5. What has been said will suffice for constructing any rational quantity; we proceed now to rational quantities of the second degree.



1st. In order to construct $\sqrt{ab}$, let us draw an indefinite line AB, upon which we may take the part CA, equal to a, and the part BC, equal to b; upon the whole AB as a diameter, describe a semicircle, cutting in D, the perpendicular CD, raised upon AB at the point C; then CD will be the value of $\sqrt{ab}$; that is, the value of $\sqrt{ab}$ is obtained by finding a mean proportional between the two quantities represented by a, b. Indeed, we have

$$\mathrm{AC} : \mathrm{CD} :: \mathrm{CD} : \mathrm{CB},$$

or

$$a : \mathrm{CD} :: \mathrm{CD} : b;$$

whence

$$\mathrm{CD}^2 = ab, \text{ or } \mathrm{CD} = \sqrt{ab}$$

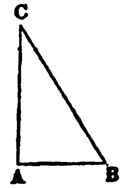
2nd. If we were to construct $\sqrt{3ab + b^2}$, or which is the same thing, $\sqrt{(3a + b)\,b}$, we should find a mean proportional between $3a + b$ and b.

3d. In like manner, if the quantity to be constructed were

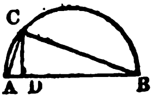
$$\sqrt{a^2 - b^2},$$

we might consider this the same as $\sqrt{(a+b)(a-b)}$; and then find a mean proportional between $a + b$ and $a - b$. If the quantity were $\sqrt{a^2 + bc}$, and we make $bc = am$, then we shall have $\sqrt{a^2 + am}$, or $\sqrt{(a + m)\,a}$, which is constructed by finding a mean proportional between $a + m$ and a after having constructed the value of $m = \frac{bc}{a}$ by the rules already given.

4th. To construct $\sqrt{a^2 + b^2}$, we can in like manner make $b^2 = am$, and construct $\sqrt{a^2 + am}$, in the manner just explained. But the property of a right angled triangle furnishes a more simple construction. If we draw the line AB, equal to a, and at its extremity A erect a perpendicular AC, equal to b, joining BC, we shall have $BC^2 = AB^2 + AC^2 = a^2 + b^2$, and, consequently, $BC = \sqrt{a^2 + b^2}$.



5th. We can also, by means of a right angled triangle, construct $\sqrt{a^2 - b^2}$ in a manner different from that above given; draw a line, AB, equal to a, and having described upon AB, as a diameter, the semicircle ACB, draw from the point A a chord AC, equal to b; then, if we draw BC, this line will be the value of $\sqrt{a^2 - b^2}$; for the triangle ABC being right angled, we shall have $AB^2 = AC^2 + BC^2$; consequently, $BC^2 = AB^2 - AC^2 = a^2 - b^2$; therefore $BC = \sqrt{a^2 - b^2}$.



6th. Hence, also, $\sqrt{a^2 + bc}$ admits of a different construction from the above. Make $bc = m^2$, and construct $\sqrt{a^2 + m^2}$, as just shown, first finding for m a mean proportional between b and c, as indicated by the equation $bc = m^2$, which gives $m = \sqrt{bc}$.

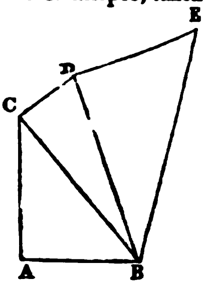
7th. If there are more than two terms under the radical sign, the construction is to be reduced to one of the preceding methods by means of transformations. If, for example, we have $\sqrt{a^2 + bc + ef}$ we may make $bc = am$, $ef = an$, and we have $\sqrt{a^2 + am + an}$, or $\sqrt{(a + m + n)\,a}$, which may be constructed by finding a mean proportional between a and $a + m + n$, after having constructed the values of m and n, namely, $m = \frac{bc}{a}$, $n = \frac{ef}{a}$. We might, moreover, make

$$bc = m^2,\ ef = n^2,$$

and then we should have to construct $\sqrt{a^2+m^2+n^2}$. Now, when there are several positive squares contained under the radical sign, as $\sqrt{a^2+m^2+n^2+p^2+ \&c.}$ we may make

$$\sqrt{a^2+m^2}=h,$$

$\sqrt{h^2+n^2}=i$, $\sqrt{i^2+p^2}=k$, and so on; and, as each of the quantities is determined by the preceding, the last will give the value of $\sqrt{a^2+m^2+n^2+p^2+\&c.}$ In order to construct these quantities in the most simple manner, each hypothenuse is to be regarded successively as a side; having, for example, taken AB $= a$, and raised the perpendicular AC $= c$, we may join BC, which will be h; then at the point C if we raise upon BC the perpendicular CD $= n$; and having drawn BD, which will be i, at the extremity D, we may raise upon BD the perpendicular DE $= p$, and BE will be k, and equal to $\sqrt{a^2+m^2+n^2+p^2}$.



If some of the squares are negative, we may combine the method just given with that for constructing $\sqrt{a^2-b^2}$

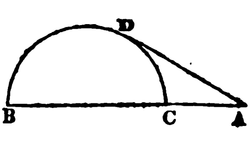
8th. Lastly, if the quantity to be constructed be of this form

$$a\frac{\sqrt{b+c}}{\sqrt{d+e}},$$

multiplying by $\sqrt{d+e}$, will change it into $a\,\frac{\sqrt{(b+c)\,(d+e)}}{d+e}$; then, by finding a mean proportional between $b+c$ and $d+e$, and calling it m, we have $\frac{am}{d+e}$, which is easily constructed.

The construction often becomes much more simple by setting out always from the same principles; but these simplifications are derived from certain considerations which are peculiar to each question, and consequently can be made known only as the occasion presents itself. We will merely remark, in concluding, that although the construction of the radical quantities, which we have been considering, reduces itself to finding fourth proportionals, mean proportionals, and constructing right-angled triangles, still we can arrive at constructions more or less simple or elegant by the method employed for finding these mean proportionals; we shall now, therefore, introduce two other methods of finding a mean proportional between two given lines.

The first consists in describing upon the greater AB (see 3d diagram to Art. 5) of two given lines a semicircle ACB, and, having taken a part AD equal to the less, raising a perpendicular DC and drawing the chord AC, which will be a mean proportional between AB and AD; for, by drawing CB, the triangle ACB is right angled, (Prop. XIX, *Cor.* 2, B. III *El. Geom.*) and consequently AC is a mean proportional between the hypothenuse AB and the segment AD. (Prop. XVII, *Cor.* 5, B. IV, *El. Geom.*)



The second method consists in drawing a line AB, equal to the greater given line, and having taken a part AC equal to the less, describing upon the remainder BC a semicircle CDB, to which if we draw the tangent AD; this tangent is a mean proportional between AB and AC. (Prop. XXVII, B. IV, *El. Geom.*)

It is evident, therefore, that rational quantities may always be constructed by means of straight lines, and radical quantities of the second degree may be constructed by means of the circle and straight line united.

As to radical quantities of higher degrees, their construction depends upon the combination of different curved lines.

We will now proceed to the consideration of questions, the solution of which depends either upon rational quantities or radical quantities of the second degree.

Geometrical questions, and modes of forming equations therefrom, and their solutions.

6. The precepts usually given in algebra for putting questions into equations, are equally applicable to questions in geometry. Here, also, the thing sought is to be represented by some symbol; and the equation is to be constructed in such manner, as to express the relations of the quantity represented by such symbol, in quantities that are known, or in those whose values are attainable; and the reasoning is to be conducted by the aid of this symbol, and of those which represent the other quantities, algebraically, as if the whole were known, and we were proceeding to verify it; this method of proceeding is called *analysis.*

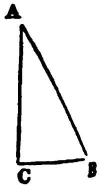
Although in expressing geometrical questions by algebraic equations, we have more resources and more facilities according as we are acquainted with a greater number of the properties of lines, surfaces, &c., still, as algebra itself furnishes the means of discovering these properties, the number of propositions really necessary is very limited. The two propositions that *similar triangles have their homologous sides proportional;* and, that, *in a right angled triangle the square of the* *hypothenuse is equivalent to the sum of the squares of the two other sides*, are the fundamental propositions and the basis of the application of algebra to geometry.

But there are many ways of making use of these propositions, according to the nature of the question, and there is always a discretion to be exercised in the choice of the means, and manner of applying them; and this discertion can only be acquired by practice.

When a geometrical question is to be resolved algebraically, it will be necessary to construct a figure that shall represent the several parts or conditions of the problem under consideration, and, if possible, get such expressions for the unknown quantities in terms of those that are known, as may easily be determined, according to the known properties of the figure. But if it so happens, that the required quantity can have no expression which will render it available under the present construction, we may, frequently, by drawing lines having certain relations to the known parts of the figure and also to the unknown, so connect the known to those that are required, as to get available expressions for their values. Having proposed a figure as above, we may, by means of the proper geometrical theorems, proceed to make out as many independent equations as there are unknown quantities; and the resolutions of these will give the solution.

PROBLEM I.

The base BC, *and the sum of the hypothenuse* AB *and perpendicular* AC, *of a right angled triangle being given, to determine the triangle.*



Let BC = b, and AC = x, and if AB + AC be represented by s, then will the hypothenuse AB be represented by $s - x$.

Therefore, by the properties of the right angled triangle (Prop. XXIV, B. IV, *El. Geom.*) we have

$$AC^2 + BC^2 = AB^2$$

Or, $$x^2 + b^2 = s^2 - 2\,sx + x^2,$$

omitting x^2 which is common to both sides of the equation, and transposing the other numbers we have,

$$2\,sx = s^2 - b^2$$

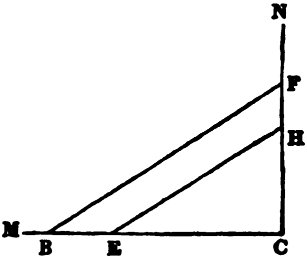
Or, $$x = \frac{s^2 - b^2}{2s}$$

which is the value of the perpendicular AC; where s and b may be any numbers whatever, provided s be greater than b. (Prop. X, B. II, *El. Geom.*)

s quantity is to be constructed, it may be resolved
Ex. 4,) into the form

$$x = \frac{(s+b)\,(s-b)}{2s}$$

structed as follows. From any point C, draw an in-
line, CM, and perpendicular thereto another indefinite
I,
ff on CM, CB = $2s$, and
o CE = one of the factors
umerator, as $s+b$, and take
he other factor, viz: $s-b$,
BF and also EH parallel
and CH will be the value
quired.
e manner, if the base and
rence of the hypothenuse
rpendicular be given, we
ve by putting d for the difference and the other letters
as before; $d+x$ for the hypothenuse.
we have,



$$x^2 + 2dx + d^2 = b^2 + x^2$$
$$x = \frac{b^2 - d^2}{2d}$$

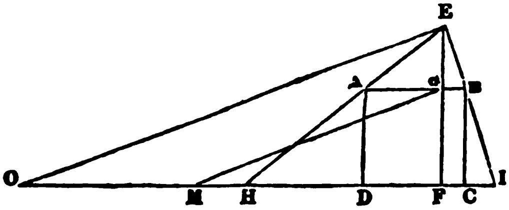
may be constructed as before

PROBLEM II.

To desribe a square in a given triangle.

given triangle is understood a triangle in which the
tion is known, viz: one whose sides, angles, altitude,
known.)

l be perceived that this question resolves itself into the
nation of some point, G, in the altitude EF, through
line AB, drawn parallel to HI, shall be equal to GF;
, therefore, determine in algebraic expression, for AB,
for GF, and put them equal to each other and we
ve a solution.



Let us therefore designate the known altitude EF by a, the known base HI by b, and the unknown line, GF by x; then will EG $= a - x$.

Since AB is parallel to HI, we shall have

$$EF : EG :: FI : GB :: EI : EB :: HI : AB$$

consequently, $EF : EG :: HI : AB$

Or, $a : a - x :: b : AB$,

whence $AB = \dfrac{ab - bx}{a}$

But, $AB = GF = x$,

therefore, $\dfrac{ab - bx}{a} = x$

and $ab - bx = ax$

Or, $ab = ax + bx = (a + b)\,x$,

hence, $x = \dfrac{a + b}{ab}$

In order to construct this quantity, it is necessary to find a fourth proportional to $a+b$, b, and a (Art. 2,) which may be done as follows:

From F to O apply a line FO $= a+b$, that is, $=$ EF + HI, and join EO; then, having taken FM = HI $= b$, draw MG parallel to EO, which, by its meeting with EF, gives the determination of GF, or the value of x; for the similar triangles EFO, GFM, give FO : FM : : FE : FG

or, $a + b : b :: a : FG$

therefore, $FG = \dfrac{ab}{a + b}$

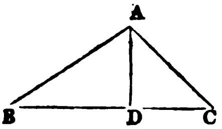
PROBLEM III.

Given the base BC, *and the angles* B *and* C *of the triangle* ABC, *to determine the altitude* AD.

(Angles are made to enter into an algebraic expression by the aid of lines employed in trigonometry, viz, sines, tangents, &c. Thus when it is said that an angle is given, it may be understood that the value of its sine or tangent is given.)

If we designate BC by a, and AB by y, we shall have CD : AD : : *radius* : *tan.* ACD, (Trigonometry) or if we designate the radius by r, and the tangent of the angle C by t, we have,

$$CD ; y :: r : t$$



whence $\quad CD = \frac{ry}{t}$

In like manner designating the tangent of the angle B by t' we shall have,

$$BD : y : : r : t'$$

whence $\quad BD = \frac{ry}{t'}$

But, $CD + BD = BC = a$

therefore, $\frac{ry}{t} + \frac{ry}{t'} = a$

whence by changing the construction we may present

$$y = \frac{at\,t'}{rt' + rt}$$

This expression is susceptible of greater simplicity, by introducing, instead of the tangents of the angles C, B, their cotangents; which, let us designate by q and q'; observing that the tangent is to radius, as radius to the cotangent, (Trigonometry) and we shall have

$$t : r : : r\ q, \text{ and } t' : r : : r : q' ;$$

whence $t = \frac{r^2}{q}$, and $\frac{t' = r^2}{q'}$

substituting these values for t and t' in the former equations we have,

$$y = \frac{ar^4 \div qq'}{r^2 \div q + r^2 \div q'} = \frac{ar^4}{qq'} \times \frac{qq'}{qr^2 + q'r^2} = \frac{ar}{q + q'}$$

Scholium

From the above it may be perceived that when among quantities that are given those employed do not lead to results so simple as may be desired, it is not always necessary to commence the work anew in order to arrive at a more simple result; but it may be sufficient to express, by equations, the ratios of the quantities first employed to those which we would introduce, as we have expressed t and t' by the equations $t = \frac{r^2}{q}$ and $t' = \frac{r^2}{q'}$ by which a solution dependent upon q and q' is obtained.

PROBLEM IV.

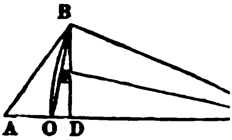
Given the three sides of a triangle ABC, *to find the segments* AD, DC, *formed by the perpendicular* BD, *and also to find the perpendicular* BD.

Let BD $= y$, CD $= x$, BC $= a$, AB $= b$, AC $= c$, the or AC — CD $= c - x$.

Hence, we have

$$x^2 + y^2 = a^2, \text{ and } c^2 - 2\,cx + x^2 + y^2 = b^2$$

Let the second equation be subtracted from the first, and we have



$$2\,cx - c^2 = a^2 - b^2;$$

whence, we have

$$x = \frac{a^2 - b^2 + c^2}{2c} = \frac{a^2 - b^2}{2c} + \tfrac{1}{2}c,$$

which may be resolved into

$$x = \tfrac{1}{2}\,\frac{(a+b)\,(a-b)}{c} + \tfrac{1}{2}c$$

By reference to article 2 it will be perceived that to the value of x, we have to find a fourth proportional to c and $a - b$, to take one half of this and add it to $\frac{1}{2}c$, or on the side AC.

Scholium.

Several important conclusions may be drawn fro solution, some of which we will notice; showing a different modes of putting geometrical questions into equ and how, by varying the propositions of these equations propositions may be discovered.

1st. The equation $2\,cx - c^2 = a^2 - b^2$ is resolveabl $c\,(2x - c) = (a + b)\,(a - b)$.

Now since the product of the first two factors is equal product of the last two, we may consider the first two extremes, and the last two as the means of a proportion; we have

$$c : a + b :: a - b : 2x - c, \text{ or } x - (c - x);$$

or, $\quad$ AC : BC + AB : : BC — AB : CD — AD

2nd. If from the point C as a centre, and with a radiu we describe the arc BO, and draw the chord BO we ha

$$\text{BD}^2 + \text{DO}^2 = \text{BO}^2;$$

now $\quad$ DO = CO — CD = BC — CD $= a - x$,

therefore $\quad \text{BO}^2 = y^2 + a^2 - 2\,ax + x^2$;

but we have found above $\quad y^2 + x^2 = a^2$;

consequently $\quad \text{BO}^2 = 2\,a^2 - 2\,ax = 2\,a\,(a - x)$.

Putting for x its value $\dfrac{a^2 - b^2 + c^2}{2c}$,

since $2\,ac - a^2 - c^2 = -(a^2 - 2\,ac + c^2) = -(c - a$

we shall have

$$\text{BO}^2 = 2\,a\left(a + \frac{b^2 - a^2 - b^2}{2\,c}\right) = 2\,a\left(\frac{2\,ac - a^2 - c^2 +}{2c}\right.$$

$$= \frac{a}{c}\left(b^2 - (c - a)^2\right).$$

Now, by considering $c - a$ as a single quantity, we find
$b^2 - (c - a)^2 = (b + c - a)\ (b - c + a)$,
hence

$$BO^2 = \frac{a}{c}\ (b + c - a)\ (b - c + a),$$

which may be put under this form,

$$BO^2 = \frac{a}{c}\ (a + b + c - 2\ a)\ (a + b + c - 2\ c).$$

If, therefore, we designate the sum of the three sides by $2s$, we shall have

$$BO^2 = \frac{a}{c}\ (2\ s - 2\ a)\ (2\ s - 2\ c) = 4\frac{a}{c}\ (s - a)\ (s - c).$$

Letting fall from the point C upon OB the perpendicular CI, we obtain from the right angled triangle CIO this proportion,

CO : OI : : R : sin. OCI, (Trigonometry,)

that is, $a : \frac{1}{2}$ BO : : R : sin. OCI,

whence $\frac{1}{2}\ BO = \frac{a \text{ sin. OCI},}{R}$, or $BO = \frac{2\ a \text{ sin. OCI}}{R}$;

consequently $BO^2 = \frac{4\ a^2\ (\text{sin. OCI})^2}{R^2}$

Putting these two valves of BO^2 equal to each other we have

$$\frac{4\ a^2\ (\text{sin. OCI})^2}{R^2} = \frac{4\ a}{c}\ (s - a)\ (s - c),$$

or, dividing by $4\ a$, and making the denominators to disappear,

$$a\ c\ (\text{sin. OCI})^2 = R^2(s - a)\ (s - c):$$

that is, dividing by ac, putting R equal to 1, and extracting the square root,

$$\text{sin. OCI} = \sqrt{\frac{(s - a)\ (s - c,)}{ac}}$$

which agrees with a formula, in Trig.

3d. We may, from the equation $y^2 + x^2 = a^2$, deduce the following: $y^2 = a^2 - x^2 = (a + x)\ (a - x)$, putting for x its value, as found in the problem we have

$$y^2 = \left(a + \frac{a^2 - b^2 + c^2}{2c}\right)\left(a + \frac{b^2 - a^2 - c^2}{2c}\right)$$

$$= \left(\frac{2\ ac + a^2 + c^2 - b^2}{2c}\right)\left(\frac{2\ ac - a^2 - c^2 + b^2}{2c}\right)$$

$$= \left(\frac{(a + c)^2 - c^2}{2c}\right)\left(\frac{b^2 - (c - a)^2}{2c}\right)$$

$$= \left(\frac{(a + c + b)\ (a + c - b)}{2c}\right)\left(\frac{b + c - a)\ (b - c + a)}{2c}\right);$$

consequently,

$$4\ c^2y^2 = (a + c + b)\ (a + c - b)\ (b + c - a)\ (b - c + a)$$

$$= (a + b + c)\ (a + b + c - 2b)\ (a + b + c - 2a\ (a + b + c - 2c);$$

or, designating the sum of the three sides $a + b + c$ by $2\ s$,

$$4\ c^2y^2 = 2\ s\ (2\ s - 2\ b)\ (2\ s - 2\ a)\ (2\ s - 2\ c)$$

$$= 16\ s\ (s - b)\ (s - c)\ (s - a),$$

or, dividing by 16 and taking the square root,

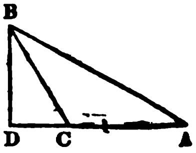
$$\frac{c\,y}{2} = \sqrt{s\ (s - b)\ (s - c)\ s - a)}$$

But $\frac{c\,y}{2}$, or $\frac{\mathrm{AC} \times \mathrm{BD}}{2}$ is the surface of the triangle ABC.

Hence, *to find the surface of a triangle by means of the three sides, we must subtract each side successively from the half sum, multiply the half sum and the three remainders continually together, and take the square root of this product; which agrees with* Prop. XL, B. IV, El. Geom.

4th. The equations $2\ cx - c^2 = a^2 - b^2$ may be resolved as follows,

$$b^2 = a^2 + c^2 - 2\ cx$$



but if the perpendicular fall without the triangle as in the present diagram, AD will then be $c + x$ instead of $c - x$, hence, designating the sides as before, we have $y^2 + x^2 = a^2$, and $y^2 + c^2 + 2\ cx + x^2 = b^2$, the first subtracted from the second gives $c^2 + 2\ cx = b^2 - a^2$, or $c\ (c + 2x) = (b + a)\ (b - a)$;

whence, $c : b + a :: b - a : c + 2x$

Now, $c + 2\ x$, or $x + c + x = \mathrm{CD} + \mathrm{AD}$;

consequently, $\mathrm{AC} : \mathrm{AB} + \mathrm{BC} :: \mathrm{AB} - \mathrm{BC} : \mathrm{CD} + \mathrm{AD}$

5th. The same equation $c^2 + 2\ cx = b^2 - a^2$, may also be put under the following form

$b^2 = a^2 + c^2 + 2\ cx$, which answers to the last figure. comparing this with the equation,

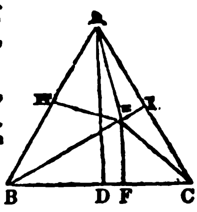
$b^2 = a^2 + {}^2c - 2\ cx$, which answers to the former figure, we observe that b^2 the square of the side AB opposite to the acute angle C, is less than the sum of the squares of the other two sides $a^2 + c^2$ by $2\ cx$; on the contrary, the square of the side AB opposite the obtuse angle, (see last figure,) is equal to $a^2 + c^2 + 2\ cx$, that is, greater than the sum of the squares of the other two sides, by $2cx$, which agrees with propositions XXVI and XXVII, B. IV, *El. Geom*; by these propositions we may determine when the angles of a triangle re to be calculated by means of the sides, whether the angle ought, be acute, or obtuse.

6th. The two equations $b^2 = a^2 + c^2 - 2\,cx$
nd $b^2 = a^2 + c^2 + 2\,cx$
onfirms the theory of positive and negative quantities, for it s plain that the segment CD takes different directions, according as the perpendicular BD falls within the triangle or without it. In these two equations the term $2\,cx$ has, in fact, conrary signs. Hence, whatever result we obtain with regard o one of these triangles, we obtain that which belongs to the nalogous case of the other by merely changing its sign of hat part which takes a different direction on the same line.

Now, since in the above theorem, respecting the surface of triangle the segment CD does not come into consideration; herefore, the proposition is equally applicable to all kinds of lane triangles.

PROBLEM. V.

aving the lengths of the three perpendiculars, EF, EI, EH, *drawn from a certain point* E, *within an equilateral triangle* ABC, *to its three sides, to determine the sides.*



Draw the perpendicular AD, and having oined EA, EB, and EC, put EF$=a$ EI$=b$, EH$=c$, and BD (which is $\frac{1}{2}$BC)$=x$.

Then, since AB, BC, or CA, are each$=2x$, we shall have, Prop. XXIV. B. IV. *El. Geom.*

$$AD=\sqrt{(AB^2-BD^2)}=\sqrt{(4x^2-x^2)}=\sqrt{3x^2}=x\sqrt{3}.$$

And because the area of any plane triangle is equal to half the rectangle of its base and perpendicular, it follows that

$$\begin{aligned} \text{triangle } ABC &= \tfrac{1}{2}BC\times AD = x\times x\sqrt{3} = x^2\sqrt{3},\\ BEC &= \tfrac{1}{2}BC\times EF = x\times a \quad = ax,\\ AEC &= \tfrac{1}{2}AC\times EI = x\times b \quad = bx,\\ AEB &= \tfrac{1}{2}AB\times EH = x\times c \quad = cx. \end{aligned}$$

But the last three triangles BEC, AEC, AEB, are together, equal to the whole triangle ABC, whence

$$x^2\sqrt{3}=ax+bx+cx.$$

And, consequently, if each side of this equation be divided by x, we shall have

$$x\sqrt{3}=a+b+c, \text{ or}$$

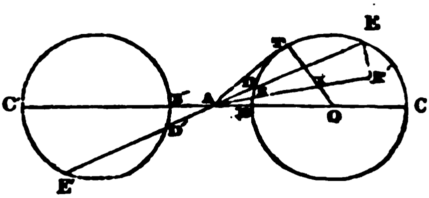
$$x=\frac{a+b+c}{\sqrt{3}}$$

Which is therefore, half the length of either of the three equal sides of the triangle.

Cor. Since, from what is above shown, AD is $=x\sqrt{3}$, it follows, that the sum of all the perpendiculars, drawn from any point in an equilateral triangle to each of the sides, is equal to the whole perpendicular of the triangle.

PROBLEM VI.

From a given point A, *without a circle* BDC, *to draw a straight line* AE *in such a manner that the part* DE, *intercepted in the circle, shall be equal to a given line.*



Since the circle BDEC is given, its diameter is supposed to be known; and, since the point A is given, we draw through the centre O the straight line AOC, the line AB is to be considered as known, and consequently the line AC. In order to know how the line AE is to be drawn, we have only to determine what ought to be the magnitude of AD, that, when produced, the part DE should be equal to the given line. We will designate AD by x, AB by a, AC by b, and the given line, to which DE is to be made equal, by c.

Since the figure BDEC is a circle, the secants AC, AE, must be reciprocally proportional to the parts without the circle; that is,

AC : AE : : AD : AB (Prop. XXXVI. B. IV. *El. Geom.*), or $b : x+c :: x : a$;

whence $x^2+cx=ab$,

an equation of the second degree, which, being resolved, gives

$$x=-\tfrac{1}{2}c\pm\sqrt{\tfrac{1}{4}c^2+ab}.$$

of which the first value only, $-\frac{1}{2}c+\sqrt{\frac{1}{4}c^2+ab}$, satisfies the question under consideration.

In order to finish the solution, it is necessary to construct this quantity, which can be done without employing the transformations made known, art. 2. For this purpose, we draw from the point A the tangent AT, which, being a mean proportional between AB and AC, gives $\mathrm{AT}^2=ab$; the value of x therefore becomes

$$x=-\tfrac{1}{2}c+\sqrt{\tfrac{1}{4}c^2+\mathrm{AT}^2}.$$

The radius TO being drawn, becomes a perpendicular at AT;

if then we take TI equal to $\frac{1}{2}c$, by drawing AI, we shall have $AI=\sqrt{\frac{1}{4}c^2+AT^2}$; therefore, in order to obtain x, we have only to apply TI from I to R, and to describe from the point A, as a centre, and with the radius AR, the arc RD, which will determine D, the point sought; for

$$AD, \text{ or } AR=AI-IR=AI-TI=\sqrt{\tfrac{1}{4}c^2+AT^2}-\tfrac{1}{2}c=x$$

In order now to know what the second value of x signifies, namely,

$$x=-\tfrac{1}{2}c-\sqrt{\tfrac{1}{4}c^2+ab},$$

it must be observed that, as it is wholly negative, it can only fall in the direction opposite to that toward which AD tends. Let us see, then, if there be a question depending upon the same quantities and the same reasoning, which fulfils this condition. If now we suppose a and b negative, the equation $x^2+cx=ab$, undergoes no change; since, therefore, when the circle BDEC becomes B′D′E′C′, situated toward the left in the same manner that BDEC is toward the right, it follows that the solution of this case is contained in the same equation; the second value of x, or $-\frac{1}{2}c-\sqrt{\frac{1}{4}c^2+ab}$, belongs to the same case, and satisfies the same conditions; if, therefore, in the preceding construction, we apply IT from I to R′ on AI produced, and from the point A, as a centre, and with a radius equal to AR′, we describe an arc cutting the circumference B′D′E′C′ in E′, the point E′ will be such that the part intercepted, E′D′, will be equal to c. Indeed,

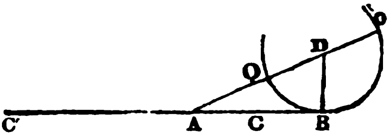
$$AE'=AR'=AI+IR'=\sqrt{\tfrac{1}{4}c^2+AT^2}+\tfrac{1}{2}c,$$

that is, AE′ is equal to the second value of x, the signs being changed. Now, since we apply this quantity in a direction opposite to that in which x extends, it follows that AE′ is in reality the second value of x.

Hence, as the two circles are equal and situated in the same manner, the two solutions may both belong to the same circle, so that if we describe from the point A, as a centre, and with a radius AR′, the arc R′E, the line AE will also resolve the question; indeed, it is evident that the point E, determined in this manner, is in the line AD, (obtained by the first construction,) produced. But of the two solutions, furnished by algebra, the first falls on the right of the point A, and appertains to the point D of the convex circumference, while the second falls on the left, and appertains to the point E′ of the concave part of the circumference.

PROBLEM VII.

Let it be required to find the direction of a given line AB *from a point* C, *such, that its distance from the point* A, *shall be a mean proportional between its distance from the point* B *and the whole line.*



Let the given line AB be designated by a, and the distance AC required by x; then BC will be $a-x$; and, since the proportion required is

$$AB : AC :: AC : CB,$$

or

$$a : x :: x : a-x,$$

we shall have

$$x^2=a^2-ax, \text{ or } x^2+ax=a^2,$$

an equation of the second degree, which, being resolved, gives

$$x=-\tfrac{1}{2}a\pm\sqrt{\tfrac{1}{4}a^2+a^2}$$

In order to construct the first value of x, we must, according to what has been said, (Art. 5,) raise the point B the perpendicular $BD=\frac{1}{2}a$; and, having drawn AD, we shall have

$$AD=\sqrt{BD^2+AB^2}=\sqrt{\tfrac{1}{4}a^2+a^2};$$

we have then only to subtract from this line the quantity $\frac{1}{2}a$, which is done by applying BD from D to O; then we shall have $AO=\sqrt{\frac{1}{4}a^2+a^2}-\frac{1}{2}a$, that is, it will be equal to x. We then apply AO from A to C toward B, and C will be the point sought.

As to the second value of x, namely,

$$x=-\tfrac{1}{2}a-\sqrt{\tfrac{1}{4}a^2+a^2},$$

if we apply BD from D to O' on AD produced, then we shall have

$$AO'=\tfrac{1}{2}a+\sqrt{\tfrac{1}{4}a^2+a^2};$$

and, as the value of x is this quantity taken negatively, we apply AO' from A to C' on AB produced in a direction opposite to that toward which x is supposed in the solution to extend; and we shall have a second point C', which will also be such, that its distance from the point A, will be a mean proportional between its distance from the point B and the whole line AB.

Scholium. 1. We may observe that this question contains that of *dividing a line in extreme and mean ratio*; also the construction which we have obtained, is the same as that given in the *Elements of Geometry*, (Prop. IV. B. IV.) But it will be perceived, that we are made acquainted with this construction by algebra, whereas in the *Elements of Geometry* we supposed the construction, and only demonstrated its truth.

2. With a little attention to the course pursued in the pre- ding questions, it will be evident that we have always taken r the unknown quantity a line, which being once known, rves, by observing the conditions of the question, to deter- ine all the others. This is the course to be pursued in all ses, but there is a choice with regard to the line to be used; ere are often several, each of which has the property of de- rmining all the others, if once known. Among these some ould lead to more simple equations than others. The follow- g rule is given to aid in such cases.

3. *If among the lines or quantities, which would, when taken ch for the unknown quantity, serve to determine all the other antities, there are two which would in the same way answer is purpose, and it would be foreseen that such would lead to the me equation, (the signs + and — excepted); then we ought to nploy neither of these, but take for the unknown quantity one hich depends equally on both;* that is, their half sum, or their alf difference, or a mean proportional between them, or &c., nd we shall always arrive at an equation more simple than employing either the one or the other.

4. The question we have resolved, (Prob. VI,) may be used illustrate what is here said. In this question there is no ason for taking AD rather than AE, for the unknown quanti- ; by taking AD for the unknown quantity x, we have $x+c$ r AE; and, by taking AE for the unknown quantity x, we ould have $x-c$ for AD; and, as to the rest, the mode of oceeding is the same for each case; so that the equations ffer only in the signs. If, therefore, instead of taking either r the unknown quantity, we take their half sum, and desig- ate it by x, since their half difference DE$=c$ is given, we all have

$$\text{AE}=x+\tfrac{1}{2}\,c,\ \text{and}\ \text{AD}=x-\tfrac{1}{2}\,cx,$$

hence, according to the proposition adopted in the first so- ution,

$$(x+\tfrac{1}{2}\,c)\,(x-\tfrac{1}{2}\,c)=ab$$

r

$$x^2-\tfrac{1}{4}\,c^2=ab,$$

more simple equation than the former, and which gives

$$x=\sqrt{\tfrac{1}{4}\,c^2+ab};$$

nd, since AE$=x+\tfrac{1}{2}\,c$, we have immediately

$$\text{AE}=\tfrac{1}{2}\,c+\sqrt{\tfrac{1}{4}\,c^2+ab},$$

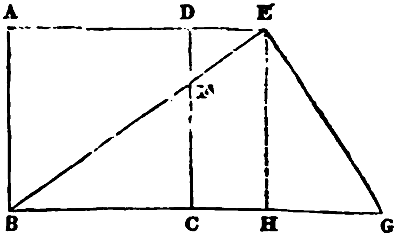
nd

$$\text{AD}=-\tfrac{1}{2}\,c+\sqrt{\tfrac{1}{4}\,c^2+ab},$$

s before found.

PROBLEM VIII.

Let it be required to draw a right line BFE *from one of the angles* B *of a given square* BC, *so that the part* FE *intercepted by* DE *and* DC, *shall be of a given length.*



Draw EG perpendicular to BE to meet BC produced in G, and from the angle E draw EH perpendicular to BG.

Let BC or DC$=a$, FE$=b$, BF$=y$, and CG$=x$.

Since the triangle EHG is similar to the triangle BCF and the side EH=the side BC, hence the hypothenuse EG= the hypothenuse BF.

But $BE^2+EG^2=BG^2$,

or $2y^2+2by+b^2=a^2+2ax+x^2$

and because the triangle BCF and BEG are similar,

BF : BC : : BG : BE

or $y : a :: a+x : y+b$

hence, $y^2+by=a^2+ax$

multiplying this equation by 2, we have

$2y^2+2by=2a^2+2ax$.

Subtracting the last from the former equation, we have

$b^2=-a^2+x^2$,

or $b^2+a^2=x^2$,

hence, $x=\sqrt{b^2+a^2}$

having the value of x, y may be found in the equation

$y^2+by=a^2+ax$

completing the square $y^2+by+\frac{1}{4}b=a^2+ax+\frac{1}{4}b$

hence $y=\sqrt{a^2+ax+\frac{1}{4}b}-\frac{1}{2}b$

Let DE$=z$,

then $y : a :: b : z$

and $yz=ab$

hence, $z=\dfrac{ab}{y}$

Scholium. This problem is susceptible of several modes of solution, but perhaps none more simple than the one here given, for most of the modes of which it is susceptible, involve powers and equations higher than quadratics.

EXAMPLES FOR PRACTICE.

Ex. 1. To find the side of a square, inscribed in a given emicircle, whose diameter is d. Ans. $\frac{1}{5}d\sqrt{5}$

Ex. 2. To find the side of an equilateral triangle inscribed n a circle whose diameter is d; and that of another circumcribed about the same circle. Ans. $\frac{1}{2}d\sqrt{3}$, and $d\sqrt{3}$

Ex. 3. To find the sides of a rectangle, the perimeter of vhich shall be equal to that of a square, whose side is a, and ts area half that of a square. Ans. $a+\frac{1}{2}a\sqrt{2}$ and $a-\frac{1}{2}a\sqrt{2}$

Ex. 4. Having given the perimeter (12) of a rhombus, and he sum (8) of its two diagonals, to find the diagonals.

Ans. $\frac{1}{2}\sqrt{2}+\sqrt{5}$ $4+\sqrt{2}$ and $4-\sqrt{2}$

Ex, 5. Required the area of a right angled triangle, whose ypothenuse is x^{3x} and the base and perpendicular x^{2x} and x^{x},

Ans. 1.029085

Ex. 6. Having given the two contiguous sides (a, b) of a pa-allelogram, and one of its diagonals (d), to find the other digonal. Ans. $\sqrt{(2a^2+2b^2-d^2)}$

Ex. 7. Given the base (194) of a plane triangle, the line at bisects the vertical angle (66), and the diameter (200) of e circumscribing circle, to find the other two sides.

Ans. 81.36587 and 157, 43865

Ex. 8. The lengths of two lines that bisect the acute angles f a right angled plane triangle, being 40 and 50 respectively, is required to determine the three sides of the triangle.

Ans. 35.80737, 47.40728, and 59.41143

Ex. 9. Given the hypothenuse (10) of a right angled trianle, and the difference of two lines drawn from its extremities the centre of the inscribed circle (2), to determine the base nd perpendicular. Ans. 8.08004 and 5.87447

Ex. 10. Having given the lengths (a, b) of two chords, cutng each other at right angles, in a circle, and the distance (c) f their point of intersection from the centre, to determine the iameter of a circle.

Ans. $\frac{1}{2}\sqrt{\left\{\frac{1}{2}(a^2+b^2)+2c^2\right\}}$

Ex. 11. Two trees, standing on a horizontal plane, are 120 et asunder; the height of the highest of which is 100 feet, nd that of the shortest 80; where in the plane must a pern place himself, so that his distance from the top of each ee, and the distance of the tops themselves, shall be all equal each other?

Ans. $20\sqrt{21}$ feet from the bottom of the shortest
and $40\sqrt{3}$ feet from the bottom of the other

Ex. 12. Having given the sides of a trapezium, inscribed in a circle, equal to 6, 4, 5, and 3, respectively, to determine the diameter of the circle. Ans. $\frac{1}{20}\sqrt{(130\times 133)}$ or 6.574572

Ex. 13. Supposing the town A to be 30 miles from B, B 25 miles from C, and C 20 miles from A; where must a house be erected that it shall be at an equal distance from each of them?

Ans. 15.118578 miles from each, viz: in the centre of a circle whose circumference passes through each of the three towns.

Ex. 14. In a plane triangle, having given the perpendicular (p), and the radii (r R) of its inscribed and circumscribing circles, to determine the triangle.

Ans. The base $\dfrac{2r\sqrt{(2p\text{R}-4r\text{R}-r^2)}}{p-2r}$

DETERMINATION OF ALGEBRAICAL EXPRESSIONS FOR SURFACES AND SOLIDS.

7. We have seen in the *Elements of Geometry*, that surfaces depend upon the product of two dimensions, and solids upon the product of three dimensions; so that, if the several dimensions of one or two solids, or two surfaces, which we would compare, have to the several dimensions of the other, each the same ratio, the two surfaces will be to each other as the squares, and the two solids as the cubes, of the homologous dimensions; and more generally still, if any two quantities of the same nature are expressed each by the same number of factors, and if the several factors of the one have to the several factors of the other, each the same ratio, the two quantities will be to each other as their homologous factors, raised to a power whose exponent is equal to the number of factors. If, for example, the two quantities were $a\,b\,c\,d$, $a'\,b'\,c'\,d'$, and we had

$$a : a' :: b : b' :: c : c' :: d : d',$$

then we should have

$$b' = \frac{a'\,b}{a},\ c' = \frac{a'\,c}{a},\ d' = \frac{a'\,d}{a},$$

and consequently,

$$\begin{aligned} a\,b\,c\,d : a'\,b'\,c'\,d' &:: a\,b\,c\,d : \frac{a'^4\,b\,c\,d}{a^3}, \\ &:: \quad a \quad : \frac{a'^4}{a^3}, \\ &:: \quad a^4 \quad : a'^4. \end{aligned}$$

What is here said is true not only of simple quantities; the ame may be shown with respect to compound quantities. Let e quantities whose dimensions are proportional be

$$a\,b + c\,d,\ a'\,b' + c'\,d';$$

nce, by supposition,

$$a : a' :: b : b' :: c : c' :: d : d',$$

re shall have

$$b' = \frac{a'\,b}{a},\ c' = \frac{a'\,c}{a},\ d' = \frac{a'\,d}{a},$$

nd consequently

$$a\,b + c\,d : a'\,b' + c'\,d' :: a\,b + c\,d : \frac{a'^2\,b}{a} + \frac{a'^2\,c\,d}{a^2}$$

$$:: a\,b + c\,d : \frac{a'^2\,ab + a'^2\,cd}{a^2}$$

$$:: a^2\,(a\,b + c\,d) : a'^2\,(a\,b + c\,d),$$

$$:: \quad a^2 \quad : \quad a'^2.$$

It follows, from what is here proved, that the surfaces of nilar figures are as the squares of their homologous dimenons, and that the solidities of similar solids are as the cubes their homologous dimensions; for, whatever these figures d these solids may be, the former may always be considered composed of similar triangles, having their altitudes and ses proportional, (Prop. XXIII. B. IV. *El. Geom.*,) and the ter as composed of similar pyramids, having their three dinsions also proportional. (Prop. XXXII, *Cor.* 5, B. II, *El. L Geom.*)

It will hence be perceived, that quantities may be readily mpared, when they are expressed algebraically; and this y be done, whether the quantities be of the same or of a ferent species, as a cone and a sphere, a prism and a cylinr, provided only that they are of the same nature, that is, th solids, or both surfaces.

t it be required to investigate the properties of a pyramid and and also of a frustum of a pyramid.

Let h = the altitude, s = the greater base and s' the smaller se, and h' = the altitude of the vertical pyramid taken from e top of the frustum,

then we shall have $\sqrt{s'} : \sqrt{s} :: h' : h + h'$ or the altitude of e whole pyramid, and consequently,

$$(h + h')\,\sqrt{s'} = h'\,\sqrt{s} = h\,\sqrt{s'} + h'\,\sqrt{s'}$$

d $$h'\sqrt{s} - h'\,\sqrt{s'} = h\,\sqrt{s'}$$

dividing by $\sqrt{s} - \sqrt{s'}$

we have $$h' = \frac{h\sqrt{s'}}{\sqrt{s} - \sqrt{s'}}$$

whence h' becomes known.

Let now $h + h'$ be represented by k, and we may have from the first equation $\frac{h'\sqrt{s}}{\sqrt{s'}} = k$

then we shall have for the solidity of the whole pyramid $\frac{s\,k}{3}$ (1)

the solidity of the small pyramid $\frac{s'\,h'}{3}$

substituting for k its value, we have for the solidity of the whole pyramid $\frac{h'\,s\sqrt{s}}{3\sqrt{s'}}$

hence the solidity of the frustum will be $\frac{h'\,s\sqrt{s}}{3\sqrt{s'}} - \frac{s'\,h'}{3}$

$$\text{or } \frac{h'}{3}\left(\frac{s\sqrt{s}}{\sqrt{s'}} - s'\right) \text{ or } \frac{h'}{3}\left(\frac{s\sqrt{s} - s'\sqrt{s'}}{\sqrt{s'}}\right) \text{ - - - - } (2)$$

putting for h' its value found above we have,

$$\frac{h\sqrt{s'}}{3(\sqrt{s} - \sqrt{s'})} \times \frac{s\sqrt{s} - s'\sqrt{s'}}{\sqrt{s'}}$$

which being reduced gives,

$$\frac{h}{3}(s + \sqrt{ss'} + s') \text{ - - - - - - - - } (3)$$

that is, the solidity of the frustum is equal to the sum of the greater base, the smaller base, and a mean proportional between the two bases, multiplied by the altitude of the frustum; which agrees with the proposition in geometry.

And if the two bases are equal, viz: if $s = s'$ then the solid becomes a prism, and the expression will become

$$\frac{h}{3}(s + s + s) \text{ or } \frac{1}{3}h(3s) = hs \text{ - - - } (4)$$

that is, the solidity of the prism is equal to its base multiplied by its altitude.

Let the lateral surface of the pyramid be used as an element in its investigation.

To find the lateral surface of a frustum of a regular pyramid, having the two bases and slant height given, as well as the radius of the circle inscribed in the larger base.

Let the larger base be called s, the smaller s', the slant height h, and the radius of the circle inscribed in the larger base, r.

The perimeter of the larger base will be $\frac{s}{\frac{1}{2}r}$ and the perimeter of the smaller base may be found from the following $\sqrt{s} : \sqrt{s'} : : \frac{s}{\frac{1}{2}r} : \frac{s\sqrt{s'}}{\frac{1}{2}r\sqrt{s}}$ the perimeter of the smaller base : whence $\left(\frac{s}{r} + \frac{s\sqrt{s'}}{r\sqrt{s}}\right) h =$ the lateral surface.

Or we may investigate the surface of the frustum in connection with the whole pyramid of which it is a part.

Thus $\frac{s}{\frac{1}{2}r} : \frac{s\sqrt{s'}}{\frac{1}{2}r\sqrt{s}} : : h : \frac{h\sqrt{s'}}{\sqrt{s}}$ the slant height of the whole pyramid, which make $= k$, and the vertical pyramid cut from the frustum will be $k - h$.

Hence, we have $r : k : : s : \frac{sk}{r}$ the lateral surface of the whole pyramid. - - - - - - - - - - - - - (5)

And since the lateral surfaces of similar pyramids are proportional to their bases, we may make $s : s' : : \frac{sk}{r} : \frac{s'k}{r}$ the surface of the vertical pyramid cut from the frustum.

Hence $\frac{sk}{r} - \frac{s'k}{r}$ - - - - - - - - - - (6)

or $(s - s')$ the difference of the two bases, multiplied by $\frac{k}{r}$ the ratio of the slant height of the pyramid to the radius of the base, is equal to the lateral surface of the frustum.

It may be observed that the ratio $\frac{k}{r}$ is constant whether applied to the whole pyramid, to the pyramid cut off, or to the frustum; and is such as would be represented by the sine of the angle formed by its slant side with the plane of the base.

Cor. Whence we have for the lateral surface of the frustum of a pyramid, this rule:

Multiply the difference of the bases by the sine of the angle which the slant side makes with the base, or by the ratio of the whole slant height of a perfect pyramid on the same base to the radius of the base, which will give the lateral surface.

8. If π represent the ratio of the circumference of a circle to the diameter, a ratio which is known with sufficient accuracy for practical purposes (Prop. XIX. B. V. *El. Geom.*,) the circumference of any circle whose radius is r, will be $2\pi r$ - (1,)
and its surface πr^2 - - - - - - - - (2.)
Hence it is evident that the areas of circles increase as the squares of their radii, π being always of the same value, the quantity πr^2 depends on, and is proportional to r^2. - (3.)

If h be the altitude of a cylinder the, radius of whose base is r, for its convex surface we shall have πrh - - (4)
for its solidity πr^2h - - - - - - - - (5)
and for the same reason we shall have $\pi r'^2h'$ for the solidity of another cylinder, whose altidude is h', and the radius of whose base is r', hence the solidities of the two cylinders are to each other as the altitudes multiplied by the squares of the radii of the bases. If their altitudes and radii of their bases are proportional, in which case the cylinders will be similar, we shall have

$$h : h' :: r : r'$$

consequently $=h'\frac{hr'}{r}$

and the ratio $r^2 h : r'^2h$

becomes $r^2h : \frac{r^3h}{r}$

or multiplying by r and dividing by h, $r^3 : r'^3$ - - (6,)
that is the solidities are as the cubes of the radii of their bases, as before shown in geometry.

Also if the altitude of a cone is h, and r the radius of the base, its convex surface may be expressed by

$$\sqrt{r^2 + h^2} \times \frac{2r\pi}{2} = r\pi\sqrt{r^2 \times h^2} \quad - \quad - \quad - \quad (7)$$

Or let $k =$ the slant height of the cone, then will its lateral surface be

$$r\pi \times k = rk\pi \quad - \quad - \quad - \quad - \quad - \quad - \quad (8)$$

which result will be also obtained if we take πr^2 the area of the base, and increase it in the ratio of $r : k$, viz. :

$$r : k :: \pi r^2 : \frac{r^2k\pi}{r} \text{ or } rk\pi$$

The solidity of the cone will be

$$\pi r^2 \times \frac{h}{3} = \frac{\pi r^2h}{3} \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad (9)$$

or if we multiply the convex surface of the cone by one-third of its distance from the centre of the base, (Prop. IX. B. III. *El. S. Geom.*) we shall obtain the same result.

The distance from the centre of the base to the surface may be expressed $k : h :: r : \frac{rh}{k}$ the distance, - - (10.)
hence the solidity will also be

$$rk\pi \times \frac{rh}{3k} = \frac{\pi r^2h}{3} \text{ as before.}$$

If $h' =$ the altitude of a frustum of the cone, then may the slant height of the frustum be $h : k : h' :: \frac{h'k}{h}$ the slant height

required, which call k'; let the radius of the smaller base of the frustum be r', then will the lateral surface be $\frac{2r\pi+2r'\pi}{2}\times k'$ $= rk'\pi+r'k'\pi$ - - - - - - - - - (11)
πr^2 = the greater base, and $\pi r'^2$ = the smaller base (Prop. X, B. III. *El. S. Geom.*) the solidity $=\frac{1}{3}h'(\pi r^2+\pi r'^2+\sqrt{\pi r^2 r'^2})$-(12)

And since $\frac{\pi r^2 h'}{3}$ and $\frac{\pi r'^2 h}{3}$ each expresses the solidity of a cone on one or the other of the bases, and whose altitude is equal to that of the frustum, hence if one of those expressions s taken from that of the frustum, the remainder will express a conesected frustum.

Thus $\frac{1}{3}h\,(\pi r^2+\pi r'^2+\sqrt{\pi r^2 r'^2})-\frac{\pi r^2 h}{3}=\frac{1}{3}h(\pi r'^2+\sqrt{\pi r^2 r'^2})$ - (13)
which is a conesected frustum, having a conical cavity on its larger base and $\frac{1}{3}h\,(\pi r^2+\sqrt{\pi r^2 r'^2})$ - - - - (14)
expresses a conesected frustum the cavity of which is formed on the smaller base.

Or if we multiply the convex surface by $\frac{1}{3}$ its distance from the centre of either base, we shall have the solidity of a conesected frustum, whose cavity is taken from the opposite base (Prop. XI, Cor. B. III, *El. Sol. Geom.*)

Thus $rk\pi + r'k'\pi$, formula (11) the expression of the lateral surface, multiplied $\frac{rh}{k}$ formula (10,) the distance of the surface from the centre of the larger base gives

$$\frac{r^2h\,\pi + rr'hk'\pi}{k} \qquad \text{- - - - -} \qquad (15)$$

When the two bases of the frustum are equal, the coinsected frustum becomes a conesected cylinder, and r and r', k and h' becomes identical. Hence the expression becomes

$$= 2r^2\,k\pi \qquad \text{- - - - - -} \qquad (16)$$

where k represents the altitude.

9. Applying the same notation to express the sphere, we have for the surface of any sphere whose radius is r, $4\pi r^2$; and $4\pi r^2\times\frac{1}{3}r=\frac{4}{3}\pi r^3$, will be its solidity, (Prop. XXI. B. III. *El. Sol. Geom.*) - - - - - - - - - - - - - - - - - (1)

If the surface of a spherical zone is required, it may be expressed by the product of the altitude of the zone multiplied by the circumference of the sphere; let h= the altitude, and we have $2\pi rh$ for the spherical surface of the zone, - - (2.)

The solidity of the sector of which this is the spherical base, is $2\pi rh\times\frac{1}{3}r=\frac{2}{3}\pi r^2 h$, - - - - - - - - - - - (3.)

Now, since the sector CBAD (see the diagram to the Problem on the 187 page,) may be considered to be made up of a cone CBD and a segment BDA; in order to express these portions, it will be necessary to find the area of the circular section BD; for this purpose, since $CP=CA-AP=r-h$, and $CB=r$, we have in the right angled-triangle BPC, $BP=\sqrt{CB^2-PC^2}=\sqrt{r^2-r^2+2rh-h^2}=\sqrt{2rh-h^2}$, - (4.)

The radius of the circular section BPD, hence the area of the circle will be $2\pi rh-\pi h^2$, - - - - - - - - - - (5.)

And the solidity of the cone will be

$$(2\pi rh-\pi h^2)\frac{\times r-h}{3}=\frac{2\pi r^2h-3\pi r^2h+\pi h^3}{3} \quad - \text{ (6.)}$$

Hence the solidity of the segment will be

$$\tfrac{2}{3}\pi r^2h-\frac{2\pi r^2h+3\pi rh^2-\pi h^3}{3}=\pi rh^2-\tfrac{1}{3}\pi h^3, \quad - - \text{ (7.)}$$

Which may be resolved into $\pi h^2(r-h,)$ - - - - (8.)

Hence, *the solidity of the segment is equal to the product of a circle, whose radius is the altitude of the segment multiplied by the radius of the sphere, minus a third of this altitude.*

10. Let r' be the radius of a sphere inscribed in a vertical polyedroid, and r the radius of the circumscribed sphere. Then (Prop. XVIII. B. III, *El. Sol. Geom.*) the surface of the polyedroid will be $2r'\pi\times 2r=4r'r\pi$, - - - - - - - - (1.)

And since the surface of the circumscribed sphere is$=4r^2\pi$, formula (1, Art. 9,) it follows that the surface of the polyedroid and that of its circumscribed sphere, are to each other as $r' : r$, since those are the only variable quantities which enter into their expressions.

If h be the height of any zone of the sphere, its surface, formula (2, Art. 9,) will be $2r'\pi\times h=2rh\pi$, - - - - (2)

The surface of the corresponding zone of the polyedroid, will be $2r'\pi\times h=2r'h\pi$, (P. XVIII. Cor. B. III. *El. S. G.*) - (3.)

And hence the surface of any zone or segments of the polyedroid and sphere made by the same perpendicular to their common axis, will be in the ratio of their whole surface, viz.

$r' : r$, or $\frac{r}{r'}$. - - - - - - - - - - - - - - - - (4.)

The base of any segment of the polyedroid made by a plane passing through a circle common to the polyedroid and sphere, may be found from formula (5, Art. 9.)

Let $2\pi rh-\pi h^2$, be the base of acommon segment of the sphere and polyedroid, and h the common altitude of the polyedroidal and spherical sector.

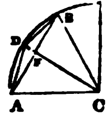
The solidity of the polyedroidal sector will be

$$2r'h\pi\times\frac{r'}{3}=\frac{2}{3}r'^2h\pi \quad - - - - - \text{ (5.)}$$

And since the corresponding spherical sector is $\frac{2}{3}r^2h\pi$, formula (3, Art. 9) it follows that the solidities of the corresponding sector of these solids are as the ratio of $r : r'$, as has been shown with regard to the convex surface. The cone whose base is the base of the segment, and whose vertice is the centre of the polyedroid or sphere, formula (6, Art. 9,) may be expressed $\frac{2\pi r^2h — 3\pi rh^2+\pi h^3}{3}$ hence the segment will be

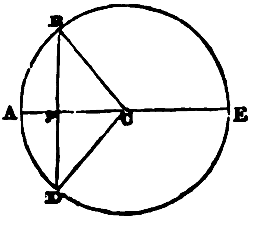
$$\tfrac{2}{3}r'^2h\pi — \tfrac{2}{3}\pi r^2h+\pi rh^2—\pi h^3$$
$$=\pi h(\tfrac{2}{3}r'^2+\tfrac{2}{3}r^2+rh — h^2) \quad - \; - \; - \; - \quad (6.)$$

If the polyedroidal segments consists only of a vertical cone, its solidity will be $2\pi rh — \pi h^2)\times\frac{1}{3}h=\pi rh^2 — \frac{1}{3}\pi h^3$, subtract this from the spherical segment on the same base, formula (7, Art. 9,) $\pi rh^2 — \frac{1}{3}\pi h^3$, and we have $\frac{1}{3}\pi rh^2$. - - - - - - - - (7.) which is the value of that portion of the segment of the sphere not included in that of the inscribed polyedroid, which is such a portion of the sphere as would be generated by the revolutions of a circular segment as BD, about the axis DC, passing through the centre of curvature, and perpendicular to the arc of the segment at the point of contact D.



PROBLEM.

It is required to find when the spherical segment and the cone composing a spherical sector are equal to each other.



Let ABED represent a sphere generated by the revolution of a semicircle ABE about its diameter AE. The sector ABC, by this revolution, generates a spherical sector, which is composed of a spherical segment generated by the revolution of a semisegment ABP, and of a cone generated by the revolution of the right-angled triangle BPC.

The solidity of the sector, formula (3, Art. 9.) will be $\frac{2}{3}\pi r^2h$.

The solidity of the cone, $\frac{2}{3}\pi r^2h — \pi rh^2+\frac{\pi h^3}{3}$, formula (6.)

Now, in order that the cone may be equal to the segment, the sector, which is the sum of both, must be double the cone: hence, $\frac{2}{3}\pi r^2h=\frac{4}{3}\pi r^2h=2\pi rh^2+\frac{2}{3}\pi h^3$,
dividing by 2, transposing, &c.,

$$\pi rh^2=\tfrac{1}{3}\pi r^2h+\tfrac{1}{3}\pi h^3$$

dividing by πh $\quad rh=\frac{1}{3}r^2+\frac{1}{3}h^2$

transposing $\quad \frac{1}{3}h^2-rh=\frac{1}{3}r^2$

$$h^2-3rh=-r^2$$

from which we obtain

$$h=\tfrac{3}{2}r\pm\sqrt{\tfrac{5}{4}r^2}$$

Of these two solutions it is evident that only $h=\frac{3}{2}r-\sqrt{\frac{5}{4}r^2}$ can satisfy the conditions of the question, since $\frac{3}{2}r+\sqrt{\frac{5}{4}r^2}$ is more than $2r$, or more than the diameter of the sphere.

EXAMPLES FOR EXERCISE.

1. What is the solidity of the spherical segments of which the frigid zones are the convex surfaces, the altitude of each segment being 327 miles, and the radius of the base 1575,28 miles? Ans. 1282921583 solid miles nearly.

2. What is the solidity of the spherical segments of which the temperate zones are the convex surface, the radius of the superior base being 1575,28 miles, that of the inferior 3628,86 miles, and the altitude 2053,7 miles?

Ans. 55021192817 solid miles nearly.

3. What is the solidity of the spherical segment of which the torrid zone is the convex surface, the radii of the bases being 3628,86 miles, and the altitude 3150,6?

Ans. 146715018499 solid miles nearly.

4. Having two vats or two tubs in the form of conical frusta, whose dimensions are as follows, viz.: the first has a base whose diameter is 3 feet, its altitude is 3½ feet, and the slant height of its side is 4 feet; the diameter of the base of the second is 3½ feet, its altitude is 5 feet, and the curve surface is 60 square feet, what must be the dimensions of one capable of containing as much as the other two, if the diameter of the bottom and top, and the altitude are in the proportion of 2 2½ and 3.

5. What is the difference in surface of a vertical hexedroid circumscribing a sphere whose diameter is 10, and the whole surface of a conesected frustum of a cone inscribed in the same sphere, and whose wanting base is 6, and perfect base 4?

CONIC SECTIONS.

There are three curves, whose properties are extensively pplied in mathematical investigations, which, being the sec- ons of a cone made by a plane in different positions, as will e shown in another place, are called the *Conic Sections.* 'hese are,

1. The *Parabola.* 2. The *Ellipse.* 3. The *Hyperbola.*

PARABOLA.

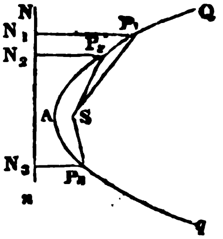
DEFINITIONS.

1. A Parabola is a plane curve, such, that if from any point the curve two straight lines be drawn; one to a given fixed int, the other perpendicular to a straight line given in po- tion: these two straight lines will always be equal to one other.

2. The given fixed point is called the *focus* of the parabola.

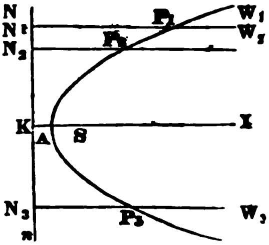
3. The straight line given in position, is called the *directrix* the parabola.

Thus, let QA*q* be a parabola, S the fo- is, N*n* the directrix;
Take any number of points, P_1, P_2, P_3, . . . in the curve;
Join S, P_1; S, P_2; S, P_3; . . . and draw N_1, P_2 N_2, P_3 N_3, ... perpendicular the directrix; then
$SP_1 = P_1 N_1$, $SP_2 = P_2 N_2$, $SP_3 =$ $_3N_3$, ...



4. A straight line drawn perpendicular to the directrix, and tting the curve, is called a *diameter;* and the point in which cuts the curve is called the *vertex of the diameter.*

5. The diameter which passes through the focus is called he *axis,* and the point in which it cuts the curve is called the rincipal *vertex.*



Thus: draw $N_1P_1W_1$, $N_2P_2W_2$, $N_3P_3W_3$, KASX, through the points P_1, P_2, P_3, S, perpendicular to the directrix; each of these lines is a diameter; P_1, P_2, P_3, A, are the vertices of these diameters; ASX is the axis of the parabola, A the principal vertex.

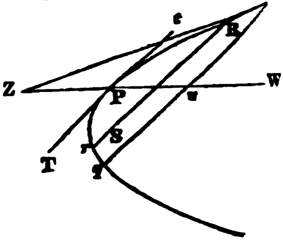
6. A straight line which meets the curve in any point, but which, when produced both ways, does not cut it, is called a *tangent* to the curve at that point.

7. A straight line drawn from any point in the curve, parallel to the tangent at the vertex of any diameter, and terminated both ways by the curve, is called an *ordinate* to that diameter.

8. The ordinate which passes through the focus, is called the *parameter* of that diameter.

9. The part of a diameter intercepted between its vertex and the point in which it is intersected by one of its own ordinates, is called the *abscissa* of the diameter.

10. The part of a diameter intercepted between one of its own ordinates and its intersection with a tangent, at the extremity of the ordinate, is called the *sub-tangent* of the diameter.



Thus: let TPt be the tangent at P, the vertex of the diameter PW.

From any point Q in the curve, draw Qq parallel to Tt and cutting PW in v. Through S draw RSr parallel to Tt.

Let QZ, a tangent at Q, cut WP, produced in Z.

Then Qq is an ordinate to the diameter PW; Rr is the parameter of PW.

Pv is the abscissa of PW, corresponding to the point Q.

vZ is the sub-tangent of PW, corresponding to the point Q.

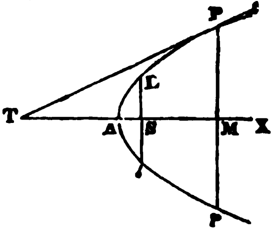
11. A stright line drawn from any point in the curve, perpendicular to the axis, and terminated both ways by the curve, is called an *ordinate to the axis*.

12. The ordinate to the axis which passes through the focus is called the *principal parameter*, or *latus rectum* of the parabola.

13. The part of the axis intercepted between its vertex and the point in which it is intersected by one of its ordinates, is called the *abscissa of the axis*.

14. The part of the axis intercepted between one of its own

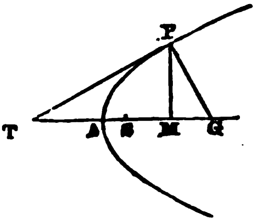
s, and its intersection with a tangent at the extremity
rdinate, is called the *sub-tangent of the axis.*



: from any point P in the
raw Pp perpendicular to AX
ting AX in M. Through S
Sl perpendicular to AX.
PT, a tangent at P, cut XA
d in T.
, Pp is an ordinate to the axis;
e latus rectum of the curve.
s the abscissa of the axis cor-
ing to the point P.
is the subtangent of the axis corresponding to the

l be proved in Prop. III, that the tangent at the princi-
ix is perpendicular to the axis; hence, the four last
ns are in reality included in the four which immedi-
ecede them.

It is manifest from Def. 1, that the parts of the curve
side of the axis are similar and equal, and that every
Pp is bisected by the axis.
f a tangent be drawn at any point, and a straight line
n from the point of contact perpendicular to it, and
ted by the curve, that straight line is called a *normal.*
he part of the axis intercepted between the intersec-
the normal and the ordinate, is called the *sub-normal.*



: Let TP be a tangent at any

P draw PG perpendicular to
gent, and PM perpendicular to

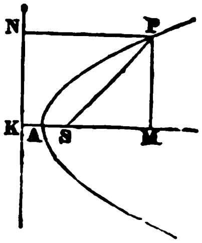
PG is the normal correspond-
he point P; MG is the sub-nor-
responding to the point P.

PROPOSITION I. THEOREM.

The distance of the focus from any point in the curve, is equal to the sum of the abscissa of the axis corresponding to that point, and the distance from the focus to the vertex.

That is, SP=AM+AS.

For

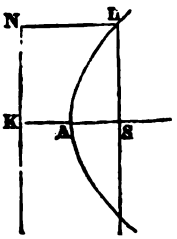


SP=PN by Def. (1,)
=KM ∵ NM is a parallelogram.
=AM+AK
=AM+AS ∵ AK=AS, by Def. (1.)

PROPOSITION II. THEOREM.

The latus rectum is equal to four times the distance from the focus to the vertex.

That is Ll=4 AS.

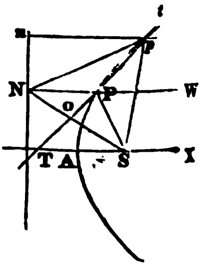


For,

Ll=2 LS, Def. (14.) cor
=2 LN
=2 SK
=4 AS ∵ AS=AK.

PROPOSITION III. PROBLEM.

To draw a tangent to the parabola at any point.



Let P be the given point.

Join S, P; draw PN perpendicular to the directrix.

Bisect the angle SPN by the straight line Tt.

Tt is a tangent at the point P.

For if Tt be not a tangent, let Tt cut the curve in some other point p.

Join S, p; draw pn perpendicular to the directrix; join S, N.

Since SP=PN, PO common to the triangles SPO, NPO, and angle SPO=angle NPO by construction,

∴ SO=NO, and angle SOP=angle NOP.

Again, since SO=NO, O*p* common to the triangles SO*p*, NO*p*, and angle SO*p*=angle NO*p*.

∴ S*p*=N*p*.

But since *p* is a point in the curve, and *pn* is drawn perpendicular to the directrix,

S*p*=*pn*

∴ *p*N=*pn*.

That is, the hypothenuse of a right-angled triangle equal to one of the sides, which is impossible, ∴ *p* is not a point in the curve; and in the same manner it may be proved that no point in the straight line T*t* can be in the curve, except P.

∴ T*t* is a tangent to the curve at P.

Cor. 1. A tangent at the vertex A, is a perpendicular to the axis.

Cor. 2. SP=ST,

For, since NW is parallel to TX

∴ angle STP= angle NPT

= angle SPT by construction,

∴ SP=ST

Cor. 3. Let Q*q* be an ordinate to the diameter PW, cutting SP in *x*.

Then, P*x*=P*v*

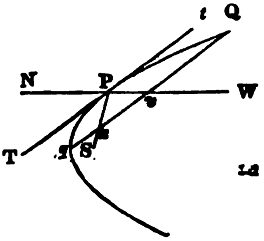
For, since Q*q* is parallel to T*t*

∴ angle P*xv*=angle *x*PT

= NPT by construction,

= P*vx* interior opposite angle,

∴ P*x*=P*v*



Cor. 4. Draw the normal PG, (see diagram Prop. V.)

Then, SP=SG,

For since angle GPT is a right angle,

angle GPT=PGT+PTG=PGT+SPT

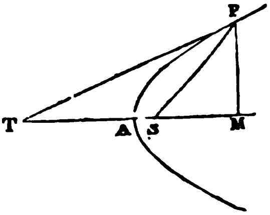
Take away the common angle SPT and there remains angle SPG=angle SGP

∴ SP=SG.

PROPOSITION IV. THEOREM.

The subtangent to the axis is equal to twice the abscissa.

That is, $MT=2\,AM$



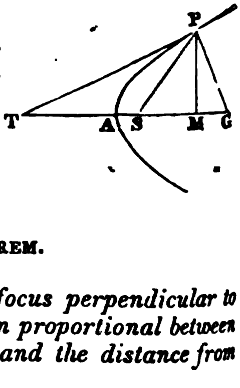
$$\begin{aligned}\text{For, } MT &= MS+ST\\ &= MS+SP. \text{ Prop. III. cor. 2.}\\ &= MS+SA+AM. \text{ Prop. I.}\\ &= 2\,AM.\end{aligned}$$

Cor. MT is bisected in A.

PROPOSITION V. THEOREM.

The subnormal is equal to one half of the latus rectum. That is,

$$MG=\frac{L}{2} \text{ if we denote the latus rectum by } L.$$



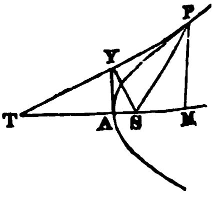
$$\begin{aligned}\text{For, } MG &= SG-SM\\ &= SP-SM. \text{ Prop. III, cor. 4.}\\ &= AS+AM-SM. \text{ Prop. I.}\\ &= AS+AS+SM-SM\\ &= 2\,AS\\ &= \frac{L}{2} \qquad \text{Prop. II.}\end{aligned}$$

PROPOSITION VI. THEOREM.

If a straight line be drawn from the focus perpendicular to the tangent at any point, it will be a mean proportional between the distance from the focus to that point, and the distance from the focus to the vertex.

That is, if SY be a perpendicular let fall from S upon T the tangent at any point P

$$SP : SY :: SY : SA.$$



Join A, Y.

Since SP=ST, and SP is drawn perpendicular to the line PT,

$\therefore$ TY=YP.

Also by Prop. IV.,

TA=AM

$\therefore$ Since AY cuts the sides of a triangle TPM proportionally; AY is parallel to MP,

$\therefore$ AY is perpendicular to AM.

Hence the triangles SYA, SYT, are similar,

$\therefore \quad ST : SY :: SY : SA$

or, $SP : SY :: SY : SA \quad \therefore SP = ST$ by Prop. III, cor. 2.

Cor. 1. Multiplying extremes and means,

$$SY^2 = SP \cdot SA.$$

Cor. 2. $SP : SA :: SP^2 : SY^2$

Cor. 3. By Cor. 1,

$$SP = \frac{SY^2}{SA}.$$

And since SA is constant for the same parabola,

$$SP \text{ proportional to } SY^2.$$

Cor. 4. By Cor. 1.,

$$\begin{aligned} SY^2 &= AS \cdot SP \\ \therefore 4\,SY^2 &= 4\,AS \cdot SP \\ &= L \cdot SP. \quad \text{Prop. II.} \end{aligned}$$

PROPOSITION VII. THEOREM.

The square of any semi-ordinate to the axis is equal to the rectangle under the latus rectum and the abscissa.

That is, if P be any point in the curve

$$PM^2 = L \cdot AM.$$

For,

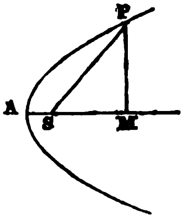
$PM^2 = SP^2 - SM^2$ (Prop. XXIV. B. IV. *El. Geom.*)

$= (AM = AS)^2 - (AM - AS)^2$

$\therefore SP = AM + AS$ (Prop. 1,) & $SM = AM - AS$

$= 4\,AS \cdot AM$. (Prop. X. and XI, B. IV. *El. Geom.*)

$= L \cdot AM$. Prop. I.



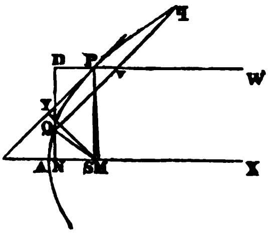
Cor. 1. Since L is constant for the same parabola PM^2 proportional to AM,

That is, *The abscissæ are propotional to the squares of the ordinates.*

PROPOSITION VIII. THEOREM.

If Qq be an ordinate to the diameter PW and Pv, the corresponding abscissa, then,

$$Qv^2 = 4SP \times Pv.$$



Draw PM an ordinate to the axis. Join S, Q; and through Q draw DQN perpendicular to the axis.

From S let fall SY perpendicular on the tangent at P.

The triangles SPY, QDv, are similar.

$$Qv^2 : QD^2 :: SP^2 : SY^2$$
$$:: SP : SA, \text{ Prop, VI. Cor. 2.}$$

The triangles PTM, QDv, are also similar;

$$\therefore\ QD : Dv :: PM : MT$$
$$:: PM^2 : PM . MT$$
$$:: 4AS . AM : 2PM . AM$$
$$:: 4AS : 2PM$$
$$\therefore\ 2PM . QD = 4AS . Dv$$

But,

$$PM^2 - QN^2 = 4AS . AM - 4AS . AN = 4AS(AM - AN)$$
$$= 4AS . MN$$

And, $PM^2 - QN^2 = PM+QN) (PM-QN)$

$$= (PM+QN) . QD$$
$$\therefore (PM+QN) . QD = 4AS . MN = 4AS . DP$$

But, $2\ PM . QD = 4AS . Dv$

$$\therefore (PM-QN) . QD = 4AS . Pv$$

Or, $QD^2 = 4AS . Pv$

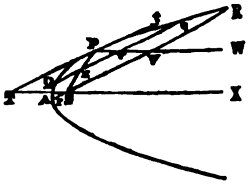
$$\therefore\ Qv^2 : 4AS . Pv :: SP : SA.$$
$$\therefore\ Qv^2 = 4SP . Pv.$$

Cor. 1. In like manner it may be proved, that

$$qv^2 = 4SP \times Pv.$$

Hence, $Qv = qv$; and since the same may be proved for any ordinate, it follows, that

A diameter bisects all its own ordinates.



Cor. 2. Let Rr be the parameter to the diameter PW.

Then, by Prop. III. *Cor.* 3.

$$Px = Pv.$$
$$\therefore\ PS = PV$$

Now, by the Proposition,

$$RV^2 = 4SP . PV$$
$$= 4SP^2$$
$$\therefore 4RV^2 \text{ or } Rr^2 = 16SP^2$$
$$Rr = 4SP.$$

Hence the proposition may be thus enunciated:

The square of the semi-ordinate to any diameter is equal to the rectangle under the parameter and abscissa

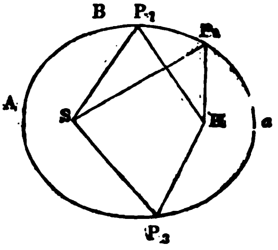
It will be seen, that Prop. VII. is a particular case of the present proposition.

ELLIPSE.

DEFINITIONS.

1. An **ellipse** is a plane curve, such that, if from any point in the curve two straight lines be drawn to two given fixed points, the sum of these straight lines will always be the same.

2. The two given fixed points are called the *foci*.



Thus, let AB*a* be an ellipse, S and H the foci.

Take any number of points in the curve P_1, P_2, P_3, - - - -

Join S,P_1, H,P_1; S,P_2, H,P_2; S,P_3, H,P_3; - - - - then,

$SP_1 + HP_1 = SP_2 + HP_2 = SP_3 + HP_3 =$ - - - -

3. If a straight line be drawn joining the foci and bisected, the point of bisection is called the *centre*.

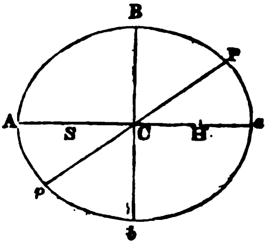
4. The distance from the centre to either focus is called the *eccentricity*.

5. Any straight line drawn through the centre, and terminated both ways by the curve, is called a *diameter*.

6. The points in which any diameter meets the curve are called the *vertices* of that diameter.

7. The diameter which passes through the foci is called the *axis major*, and the points in which it meets the curve are called the *principal vertices*.

8. The diameter at right angles to the axis major is called the *axis minor*.



Thus, let AB*a* be an ellipse, S and H the foci.

Join S,H; bisect the straight line SH in C, and produce it to meet at the curve in A and *a*.

Through C draw any straight line P*p*, terminated by the curve in the points P, *p*.

Through C draw B*b* at right angles to A*a*.

Then, C is the centre, CS or CH the eccentricity, P*p* is a diameter, P and *p* its vertices, A*a* is the major axis, B*b* is the minor axis.

9. A straight line which meets the curve in any point, but which, being produced both ways, does not cut it, is called a *tangent* to the curve at that point.

10. A diameter drawn parallel to the tangent at the vertex of any diameter, is called the *conjugate diameter* to the latter, and the two diameters are called a *pair of conjugate diameters.*

11. Any straight line drawn parallel to the tangent at the vertex of any diameter and terminated both ways by the curve, is called an *ordinate* to that diameter.

12. The segments into which any diameter is divided by one of its own ordinates are called the *abscissæ* of the diameter.

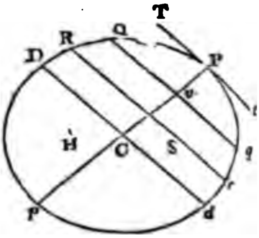
13. The ordinate to any diameter, which passes through the focus, is called the *parameter* of that diameter.

Thus, let Pp be any diameter, and Tt a tangent at P.

Draw the diameter Dd parallel to Tt.

Take any point Q in the curve, draw Qq parallel to Tt, cutting Pp in v.

Through S draw Rr parallel to Tt.



Then, Dd is the conjugate diameter to Pp.

Qq is the ordinate to the diameter Pp, corresponding to the point Q.

Pv, vp are the abscissæ of the diameter Pp, corresponding to the point Q.

Rr is the parameter of the diameter Pp.

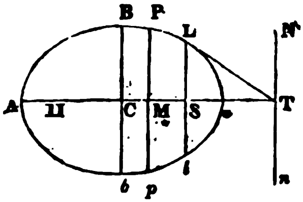
14. Any straight line drawn at right angles to the major axis, and terminated both ways by the curve, is called an *ordinate to the axis.*

15. The segments into which the major axis is divided by one of its own ordinates are called the *abscissæ to the axis.*

16. The ordinate to the axis which passes through either focus is called the *latus rectum.*

(It will be proved in Prop. IV., that the tangents at the principal vertices are perpendicular to the major axis; hence, definitions 14, 15, 16, are in reality included in the three which immediately precede them.)

17. If a tangent be drawn at the extremity of the latus rectum and produced to meet the major axis, and if a straight line be drawn through the point of intersection at right angles to the major axis, the tangent is called the *focal tangent,* and the straight line the *directrix.*



Thus, from P any point in the curve, draw PMp perpendicular to Aa, cutting Aa in M.

Through S draw Ll perpendicular to Aa.

Let LT, a tangent at L, cut Aa produced in T.

Through T draw Nn perpendicular to Aa.

Then, Pp is the ordinate to the axis, corresponding to the point P.

AM, Ma are the abscissæ of the axis, corresponding to the point P.

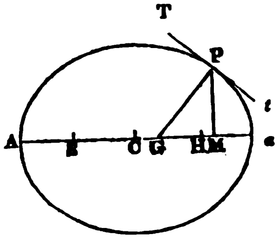
Ll is the latus rectum.

LT is the focal tangent.

Nn is the directrix.

18. A straight line drawn at right angles to a tangent from the point of contact, and terminated by the major axis, is called a *normal*.

The part of the major axis intercepted between the intersections of the normal and the ordinate, is called the *subnormal*.



Let Tt be a tangent at any point P.

From P draw PG perpendicular to Tt meeting Aa in G.

From P draw PM perpendicular to Aa.

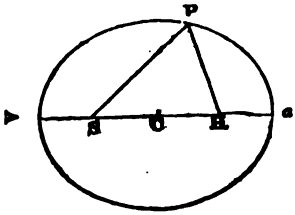
Then PG is the normal corresponding to the point P.

MG is the subnormal corresponding to the point P.

PROPOSITION I. THEOREM.

The sum of two straight lines drawn from the foci to any point in the curve is equal to the major axis. That is, if P be any point in the curve.

$$SP + HP = Aa.$$



For,

$$\left.\begin{aligned} SP + HP &= AS + AH \\ &= AS + SH, \\ &\text{And,} \\ SP + HP &= aS + aH \\ &= 2\,aH + SH, \end{aligned}\right\} \text{Def. 1.}$$

$$\therefore 2\,(SP+HP) = 2\,(AS + SH + Ha)$$

Or,

$SP + HP = Aa.$

Cor. The centre bisects the axis major, for

$2\ AS + SH = 2\ a\ H + SH \therefore AS = a\ H$

And, $SC = CH$ by definition 3. $\therefore AC = aC.$

Cor. 2. $SP + HP = 2\ AC \therefore SP = 2\ AC - HP$

$HP = 2\ AC - SP$

hence $SP - HP = 2\ AC - 2\ HP.$

PROPOSITION II. THEOREM.

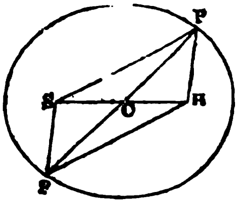
The centre bisects all diameters.

Take any point P in the curve.

Join S,P ; H,P ; S,H ;

Complete the parallelogram SPHp

Join C,p ; CP ;



Then, since the opposite sides of a parallelogram are equal,

$SP = Hp$, $HP = Sp \therefore SP + PH = Sp + pH$

$\therefore$ p is a point in the curve.

Again, since the diagonals of parallelogram bisect each other, and since SH is bisected in C,

$\therefore$ Pp is a straight line, and a diameter, and is bisected in C.

And in like manner, it may be proved that every other diameter is bisected in C.

PROPOSITION III. THEOREM.

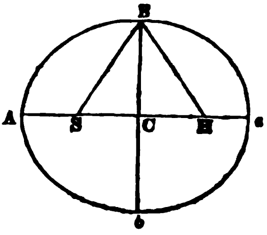
The distance of either focus from the extremity of the axis minor is equal to the semi-axis major.

That is,

$$SB \text{ or } HB = AC.$$

Since $SC = HC$, ard CB is common to the two right-angle triangles SCB, HCB,

$\therefore SB = HB.$



But,

$SB + HB = 2\ AC.$ Prop. I.

$\therefore SB = HB = AC.$

Cor. 1. $BC^2 = AS..Sa.$

For,

$BC^2 = SB^2 - SC^2$

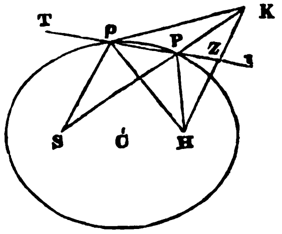
$= AB^2 - SC^2$
$= (AC + SC) . (AS - SC)$
$= AS . Sa.$

Cor. 2. The square of the eccentricity is equal to the differ-
nce of the squares of the semi-axes ;

For, $SC^2 = SB^2 - BC^2$
$= AC^2 - BC^2.$

PROPOSITION IV. PROBLEM.

To draw a tangent to the ellipse at any point.



Let P be the given point.
Join S,P ; H,P produce SP.
Bisect the exterior angle HPK
the straight line Tt.
Tt is a tangent to the curve at P.
For, if Tt be not a tangent, let
cut the curve in some other
int p.
Join S,p ; H,p ; make PK=PH,
n p, K ; H,K cutting Tt in Z.
Since HP=PK, PZ common to the triangles HPZ, KPZ,
d the angle HPZ=angle KPZ by construction,

$\therefore$ HZ=KZ, and the angle HZP=angle KZP.

Again, since HZ=KZ, Zp common to the triangles HZp,
Zp, and angle HZp=angle KZp,

$$\therefore pK = pH.$$

But, since any two sides of a triangle are greater than the
rd side,

$$\begin{aligned} Sp + pK &> SK \\ &> SP + PK \\ &> SP + PH \because PK=PH \text{ by construction,} \\ &> Sp + pH, \text{ by definition 1,} \\ \therefore pK &> pH. \end{aligned}$$

But we have just proved that $pK=pH$, which is absurd,
p is not a point in the curve, and in the same manner it may
proved that no point in the straight line Tt can be in the
rve except P.

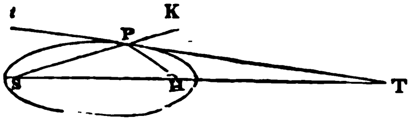
$\therefore$ Tt is a tangent to the curve at P.

Cor. 1. Hence, tangents at A and *a*, are perpendicular to
major axis, and tangents at B and *b* are perpendicular to
minor axis.

Cor. 2. SP and HP make equal angles with every tangent.

Cor. 3. Since HPK, the exterior angle of the triangle SPH, is bisected by the straight line T*t*, cutting the base SH produced in T

$$\therefore \text{ST} : \text{HT} :: \text{SP} : \text{HP}.$$



PROPOSITION V. THEOREM.

Tangents drawn at the vertices of any diameter are parallel.

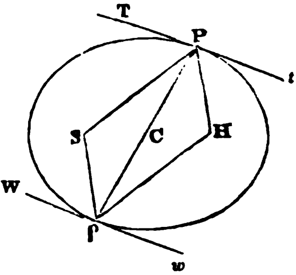
Let T*t*, W*w*, be tangents at P, *p*, the vertices of the diameter PC*p*.

Join S,P ; P,H ; S,*p* ; *p*, H ;

Then, by Prop. II, SH is a parallelogram, and since the opposite angles of parallelograms are equal,

$\therefore$ ang. SPH= angle S*p*H

supplement of ang. SPH = supplement of ang. S*p*H



or,

ang. SPT+ang. HP*t*=ang. S*p*W+ang. H*pw*

$$\left.\begin{array}{r}\text{But ang. SPT} = \text{ang. HP}t\\ \text{And ang. S}p\text{W} = \text{ang. H}pw\end{array}\right\}\text{ by Prop. IV. Cor. 2.}$$

Hence, these four angles are all equal,

$\therefore$ ang. SPT=ang. H*pw*.

And since SP is parallel to H*p*,

ang. SP*p* = ang. P*p*H,

$\therefore$ whole ang. TP*p* = whole ang. *wp*P, and they are alternate angles,

$\therefore$ T*t* is parallel to W*w*.

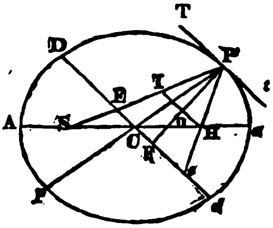
Cor. Hence, if tangents be drawn at the vertices of any two diameters, they will form a parallelogram circumscribing the ellipse.

PROPOSITION VI. THEOREM.

If straight lines be drawn from the foci to a vertex of any diameter, the distance from the vertex to the insertion of the conjugate diameter, with either focal distance, is equal to the semi axis, major.

That is, if D*d* be a diameter conjugate to P*p*, cutting SP in E, and HP in *e*,

$$PE \text{ or } Pe = AC.$$



Draw PF perpendicular to D*d*, and HI parallel to D*d* or T*t*, cutting PF in O,

Then, since the angles at O are right angles, the ang. IPO=ang. HPO, and PO common to the two triangles HPO, IPO,

$$\therefore IP = HP.$$

Also, since SC = HC, and CE is parallel HI, the base of a triangle SHI,

$$\therefore SE = EI.$$

Hence,

$$\begin{aligned} 2\,PE &= 2\,EI + 2\,IP \\ &= SE + EI + IP + HP \\ &= SP + HP \\ &= 2\,AC \end{aligned}$$

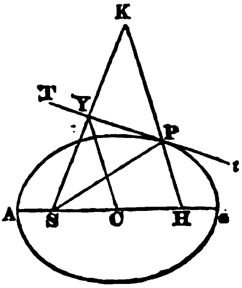
$$\therefore PE = AC.$$

Also, ang. PE*e* = ang. P*e*E. $\therefore PE = Pe$, and

$$Pe = AC.$$

PROPOSITION VII. THEOREM.

Perpendiculars, from the foci upon the tangent at any point intersect the tangent in the circumference of a circle, whose diameter is the major axis.



From S let fall SY perpendicular on T*t* a tangent.

Join S, P; H, P; produce HP to meet SY produced in K.

Join CY;

Then, since angle SPY=angle KPY (Prop. IV.) and the angles at Y are right angles, and PY common to the two triangles, SPY, KPY.

$$\therefore SP=PK$$

And SY=YK.

And, since SY=YK, and HC=CS, CY cuts the sides of the triangle HSK proportionally,

∴ CY is parallel to HK.

Also, since CY is parallel to HK, SY=YK, HC=CS,

$$\begin{aligned} \therefore\ CY &= \tfrac{1}{2}\ HK \\ &= \tfrac{1}{2}\ (HP+PK) \\ &= \tfrac{1}{2}\ (HP+SP) \\ &= \tfrac{1}{2}\ Aa \\ &= AC \end{aligned}$$

Hence, a circle inscribed with Centre C and radius CA will pass through Y.

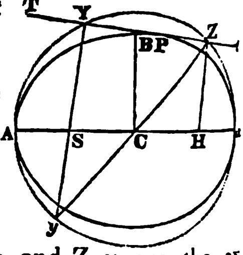
And in like manner, if HZ be drawn perpendicular to T*t*, it may be proved that the same circle will pass through Z also.

PROPOSITION VIII. THEOREM.

The rectangle, contained by the perpendiculars, from the foci upon the tangent at any point, is equal to the square of the semi-axis, minor.

That is, $SY \cdot HZ = BC^2$.

Let T*t* be a tangent at any point P.



On A*a* describe a circle cutting T*t* in Y and Z.

Join S, Y; H, Z;

Then, by the last Prop., SY, HZ are perpendicular to T*t*.

Produce YS to meet the circumference in *y*.

Join C, *y*; C, Z;

Since *y*YZ is a right angle, the segment in which it lies is a semicircle, and Z, *y*, are the extremities of a diameter.

∴ *y*CZ is a straight line and a diameter.

Hence the triangles CS*y*, HCZ, are in every respect equal.

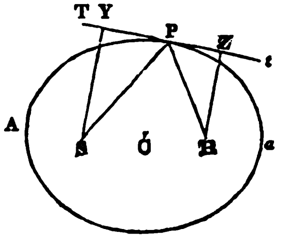
∴ S*y*=HZ

$\therefore\ SY \cdot HZ = YS \cdot Sy = AS \cdot SA = BC^2$ Prop. III. Cor. I.

PROPOSITION IX. THEOREM.

Perpendiculars let fall from the foci upon the tangent at any point are to each other as the focal distance of the point of contact.

That is, $SY : HZ :: SP : HP$.



For the triangles SPY, HPZ, are manifestly similar,

$\therefore SY : HZ :: SP : HP$.

Cor. Hence,

$$SY = HZ . \frac{SP}{HP}$$

$$SY^2 = SY . HZ . \frac{SP}{HP}$$

$$= BC^2 . \frac{SP}{HP} \text{ last Prop.}$$

$$= BC^2 . \frac{SP}{2\,AC - SP}.$$

So also,

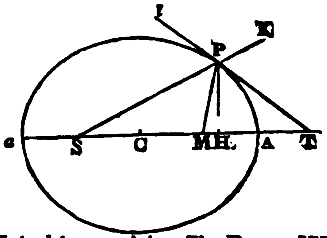
$$HZ^2 = BC^2 . \frac{HP}{SP}$$

$$= BC^2 . \frac{HP}{2\,AC - HP}.$$

PROPOSITION X. THEOREM.

If a tangent be applied at any point, and from the same point an ordinate to the axis be drawn, the semi-axis major is a mean proportional between the distance from the centre to the intersection of the ordinate with the axis, and the distance from the centre to the intersection of the tangent with the axis.

That is, $CT : CA :: CA : CM$.



Since the exterior angle HPK is bisected by T*t*, Prop. IV.

$\therefore ST : HT :: SP : HP$. (B. IV. Prop. XVI. *El. Geom.*)

$\therefore ST + HT : ST - HT :: SP + HP : SP - HP$

$$\begin{array}{llllll} \text{or,} & 2\,CT & : \; SH & :: \; 2\,AC & : SP-HP & \\ \therefore & 2\,CT & : \; 2\,AC & :: \; SH & : SP-HP & \text{- - (1.)} \end{array}$$

But since PM is drawn from the vertex of the triangle SPH perpendicular on base SH,

$$\begin{array}{llllll} \therefore & SM+HM & : SP-HP & :: \; SP+HP & : SM-HM & \\ \text{or,} & SH & : SP-HP & :: \; 2\,AC & : 2\,CM & \text{- - - (2.)} \end{array}$$

Comparing this with the proportion marked (1,) we have,

$$\begin{array}{ll} & 2\,CT : 2\,AC :: 2\,AC : 2\,CM \\ \text{or,} & CT : AC :: CA : CM. \end{array}$$

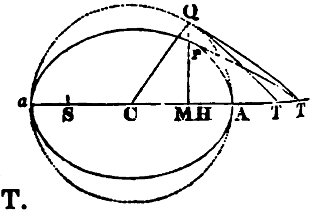
PROPOSITION XI. THEOREM.

If a circle be described on the major axis of an ellipse, and if any ordinate to this axis be produced to meet the circle, tangents drawn to the ellipse and circle, at points in which they are intersected by the ordinate, will cut the major axis in the same point.

Let AQ*a*, be a circle described on A*a*.

Take any point P in the ellipse, draw PM perpendicular to A*a*, and produce MP to meet the circle in Q, join C, Q.

Draw PT a tangent to the ellipse at P cutting CA produced in T.



Join TQ.

Then QT is a tangent to the circle at Q.

For if TQ be not a tangent, draw QT′ a tangent to Q cutting CA in T′.

Then CQT′ is a right angle.

∴ Since QM is drawn from the right angle CQT′ perpendicular on the hypothenuse.

$\therefore\ CT' : CQ :: CQ : CM$. (Prop. XVII, Cor. 2. B. IV. *E. G.*)

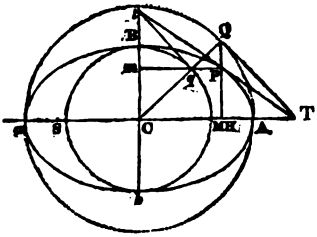
or, $CT' : CA :: CA : CM$, $\because CQ = CA$.

But, by the last proposition,

$$CT : CA :: CA : CM,$$

$$\therefore \quad CT = CT',$$

which is absurd, therefore QT′ is not a tangent at Q; in the same manner it may be proved that no line but QT can be a tangent at Q, ∴ &c.



Cor. 1. Describe a circle on e minor axis.

Draw Pm an ordinate to the mi- or axis cutting the circle in q.

Let a tangent at P cut the mi- or axis produced in t.

Then, since Pm is parallel to AC, nd PM to BC,

$t : Cm : : CT : MT$

$: : CT^2 : QT^2$

$: : Cq^2 : Cm^2$ ∴ the triangles CQT, Cmq are similar.

$: : BC^2 : Cm^2$, ∴ $Ct : CB : : CB : CM$.

Which is analogous to the property proved in the last pro- osition for the major axis.

Cor. 2. Join tq.

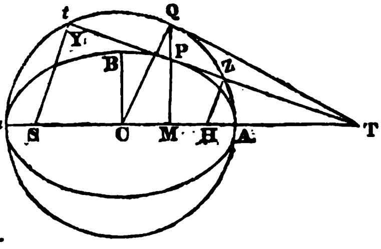
We can prove as above, that tq is a tangent to the circle Bqb.

PROPOSITION XII. THEOREM.

The square of any semi ordinate to the axis, is to the rectangle under the abscissæ, as the square of the semi-axis minor is to the square of the semi-axis major.

That is, if P be any point in the curve,

$$PM^2 : AM . Ma : : BC^2 : AC^2.$$



Describe a circle on Aa, and produce MP to meet it in Q.

At the point P and Q draw the tangent PT, QT, which will intersect the axis in the same point T, (Prop. XI.)

Let the tangent to the el- ipse intersect the circle in Y, Z.

Join S, Y; H, Z; SY and HZ are perpendicular to Tt, (Prop. VII.)

Hence the triangles PMT, SYT, HZT, are similar to each ther.

∴ $PM : SY : : MT : TY$

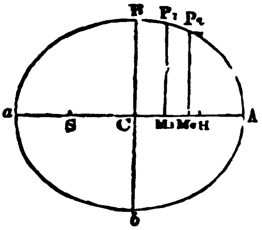
and $PM : HZ : : MT : TZ$

∴ $PM^2 : SY . HZ : : MT^2 : TY . TZ$

or $PM^2 : BC^2 : : MT^2 : TQ^2$ (Prop. XI, and Prop. XVI. B. IV. *El. Geom.*)

$: : QM^2 : CQ^2$ ∵ MQT, MQC are similar triangles.

$$: : AM . Ma : AC^2$$
$$\therefore PM^2 : AM . Ma : : BC^2 : AC^2.$$



Cor. 1

Let P_1M, P_2M_2, ... be ordinates to the axis from any points P_1, P_2 ... Then by Prop.

$$P_1M_1^2 : AM_1 . M_1a : : BC^2 : AC^2$$
$$P_2M_2^2 : AM_2 . M_2a : : BC^2 : AC^2$$
$$\therefore P_1M_1^2 : P_2M_2^2 : : AM_1 . M_1a : AM_2 . M_2a.$$

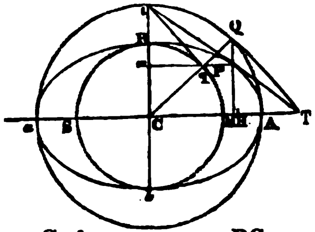
That is, the square of the ordinates to the axis are to each other as the rectangles of their abscissæ.

Cor. 2. By the fifth proportion in Prop.

$$PM : QM : : BC : AC.$$

Cor. 3. By Prop. $PM^2 : AM . Ma : : BC^2 : AC^2$.
But $AM=AC+CM$, $Ma=AC-CM$,

$$\therefore PM^2 ; (AC+CM)(AC-CM) : : BC^2 : AC^2$$
$$PM^2 : AC^2-CM^2 : : BC^2 : AC^2.$$



Cor. 4. Describe a circle on Bb, draw Pm, an ordinate to the minor axis cutting the circle in q.

Then, $Pm=CM$, $PM=Cm$.
Then by Cor. 3.

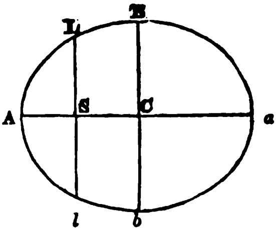
$$AC^2-Pm^2 : AC^2 : : Cm^2 : BC^2$$
$$Pm^2 : AC^2 : : BC^2-Cm^2 : BC^2$$
$$: : (BC^2+Cm)(BC-Cm) : BC^2$$
$$: : Bm . mb : BC^2$$
$$\text{or, } Pm^2 : Bm . mb : : AC^2 : BC^2$$

Which is analogous to the property proved in the proposition for the major axis.

Cor. 5. $Pm : qm : : AC : BC.$

PROPOSITION XIII. THEOREM.

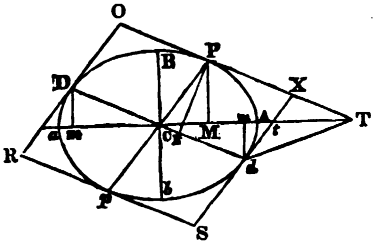
The latus rectum is a third proportional to the axis major and minor.



hat is, $Aa : Bb :: Bb : Ll$
ince LS is a semi-ordinate to the
,
: $BC^2 :: AS . Sa : LS^2$, Prop. XII.
$:: BC^2 : LS^2$, Prop. III.
or. 1.
$C : BC :: BC : LS$
l,
$a : Bb :: Bb : Ll$.

PROPOSITION XIV. THEOREM.

area of all the parallelograms, circumscribing an ellipse, rmed by drawing tangents at the extremities of two conju- ate diameters, is constant, each being equal to the rectangle nder the axes.



et Pp, Dd, be any two
jugate diameters, SROX
rallelogram circumscrib-
the ellipse formed by
wing tangents at P, D, p,
hen Pp, Dd, divide the par-
lograms SROX into four
al parallelograms.

raw PM, dm, ordinates to the axis; PF perpendicular
d.

roduce CA to meet PX in T and Sd in t.

Then, $CT : CA :: CA : CM$
And, $Ct : CA :: CA : Cm$
$\therefore CT : Ct :: Cm : CM$
But, $CT : Ct :: TM : Cm$, by similar triangles.
$\therefore MT : Cm :: Cm : CM$,
$$\therefore CM . MT = Cm^2 \quad - - - - - - - - \quad (1.)$$
Again, $CM : CA :: CA : CT$
$\therefore CM : CA :: MA : AT$, dividendo.
Or, $CM : Ma :: MA : MT$, componendo.
$$\therefore AM . Ma = CM . MT = Cm^2 \quad - - - - \quad (2.)$$
But, $AC^2 : BC^2 :: AM . Ma\ (Cm^2) : PM^2$. Prop. XII.
$\therefore AC : BC :: Cm : PM$
imilarly, $AC : BC :: CM : dm$
Or, $BC : dm :: CA : CM$

But, $CT : CA :: CA : CM$
$\therefore$ $CT : CA :: BC : dm$
But, $PF : CT :: dm : Cd$,
for the triangle $CdT = \frac{1}{2}$ the parallelogram $CPXd$,
$\therefore$ $PF : CA :: BC : Cd$.
$\therefore$ rectangle $PF \,.\, Cd$=rectangle $AC \,.\, BC$
or, parallelogram CX=rectangle $AC \,.\, BC$
$\therefore$ parallelogram $SROX = 4\, AC \,.\, BC$
$= Aa \,.\, Bb$.

Cor. By (2) $Cm^2 = AM \,.\, Ma$
$= (CA+CM)\cdot(CA-CM)$
$= CA^2 - CM^2$
$\therefore CA^2 = CM^2 + Cm^2$
And similarly. $CB^2 = PM^2 + dm^2$.

PROPOSITION XV. THEOREM.

The sum of the squares of any two conjugate diameters, is equal to the same constant quantity, namely, the sum of the squares of the two axis.

That is,
If Pp, Dd, be any two conjugate diameters,

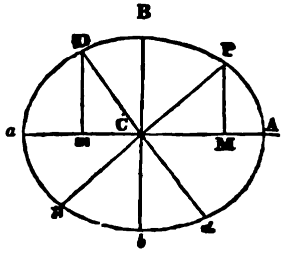
$$Pp^2 + Dd^2 = Aa^2 + Bb^2.$$

Draw PM, Dm, ordinates the axis.

Then, by Cor. to Prop. XIV,

$$AC^2 + BC^2 = CM^2 + Cm^2 + PM^2 + Dm^2$$
$$= CP^2 + CD^2$$
$$\therefore 4AC^2 + 4BC^2 = 4CP^2 + 4CD^2$$

Or, $Aa^2 + Bb^2 = Pp^2 + Dd^2$.



PROPOSITION XVI. THEOREM.

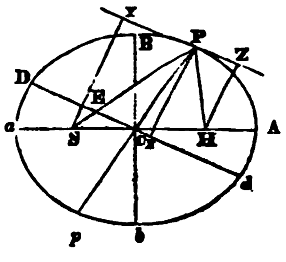
The rectangle under the focal distances of any point is equal to the square of the semi-conjugate.

That is, if CD be conjugate to CP,

$$SP \,.\, HP = CD^2.$$

Draw SY. HZ, perpendiculars to the tangent at P, PF perpendicular on CD.

Then by similar triangles SPY, PEF.



$\mathrm{SP} : \mathrm{SY} :: \mathrm{PE} : \mathrm{PF}$

Or, $\mathrm{SP} : \mathrm{SY} :: \mathrm{AC} : \mathrm{PF} \therefore \mathrm{PE}=\mathrm{AC}$, by Prop. VI

Similarly, $\mathrm{HP} : \mathrm{HZ} :: \mathrm{AC} : \mathrm{PF}$,

$\therefore \mathrm{SP} . \mathrm{HP} : \mathrm{SY} . \mathrm{HZ} :: \mathrm{AC}^2 : \mathrm{PF}^2$,

$:: \mathrm{CD}^2 : \mathrm{BC}^2$, Prop. XIV.

But $\mathrm{SY} . \mathrm{HZ} = \mathrm{BC}^2$, by Prop. VIII.

$\therefore \mathrm{SP} . \mathrm{HP} = \mathrm{CD}^2$.

PROPOSITION XVII. THEOREM.

If two tangents be drawn, one at the principal vertex, the other at the vertex of any other diameter, each meeting the other diameter produced, the two tangential triangles thus formed, will be equal.

That is,

trian. CPT = trian. CAK.

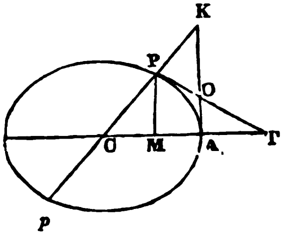
Draw the ordinate PM, then,

CM : CA : : CP : CK, by similar triangles.

But, CM : CA : : CA : CT, Prop. X.

∴ CA : CT : : CP : CK.

The triangles CPT, CAK, have thus the angle C common, and the sides about the angle reciprocally proportional; these triangles are therefore equal.



Cor. 1. From each of the equal triangles CPT, CAK, take the common space CAOP; there remains,

triangle OAT = triangle OKP.

Cor. 2. Also from the equal triangles CPT, CAK, take the common triangle CPM; there remains,

triangle MPT = trapez. AKPM.

PROPOSITION XVIII. THEOREM.

The same being supposed, as in last proposition, then any straight lines, QG, QE, *drawn parallel to the two tangents shall cut off equal spaces.*

That is,

triangle GQE = trapez. AKXG

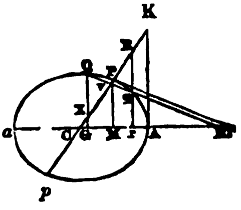
triangle *rq*E = trapez. AKR*r*

Draw the ordinate PM.

The three similar triangles

CAK, CMP, CGX,

are to each other as CA^2, CM^2, CG^2,



∴ trap. AKPM : trap. : AKXG : : $CA^2 - CM^2$: $CA^2 - CG^2$, dividendo,

But, PM^2 : QG : : $CA^2 - CM^2$: $CA^2 - CG^2$,

∴ trap. AKPM : trap. AKXG : : PM^2 : QG^2

But trian. MPT : trian. GQE : : PM^2 : QG^2,

∴ the triangles are similar.

∴ trap. AKPM : trian. MPT : : trap. AKXG : trian. GQE.

But, by Prop. XVII. Cor. 2,

trap. AKPM = triangle MPT

∴ trap. AKXG = triangle GQE

And similarly, trap. AKR*r* = triangle *rq*E

Cor. 1. the three spaces AKXG, TPXG, GQE, are all equal.

Cor. 2. From the equals AKXG, EQG, take the equals AKR*r*, E*qr* ; there remains,

R*r*XG = *rq*QG.

Cor. 3. From the equals R*r*XG, *rq*QG, take the common space *rqv*XG there remains,

triangle *v*QX = triangle *vq*R.

Cor. 4. From the equals EQG, TPXG, take the common space E*v*XG ; there remains,

TP*v*E = triangle *v*QX.

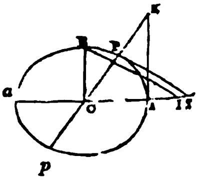
Cor. 5. If we take the particular case in which QG coincides with the minor axis,

The triangle EQG becomes the triangle IBC.

The figure AKXG becomes the triangle AKC,

∴ triangle IBC = triangle AKC

= triangle CPT.



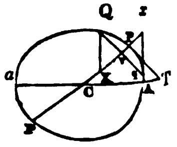
PROPOSITION XIX. THEOREM.

Any diameter bisects all its own ordinates.

That is,

If Q*q* be any ordinate to a diameter CP,

Q*v* = *vq*



Draw QX, *qx*, at right angles to the major axis ;

Then triangle *v*QX = triangle *vqx*; Prop. XVIII., *Cor.* 3.

But these triangles are also equiangular ;

∴ Q*v* = *vq*.

Cor. Hence, any diameter divides the ellipse into two equal parts.

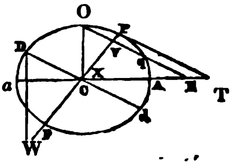
PROPOSITION XX. THEOREM.

he square of the semi-ordinate to any diameter, is to the rectangle under the abscissæ, as the square of the semi-conjugate to the square of the semi-diameter.

That is,

If Qq be an ordinate to any diameter CP,

$Qv^2 : Pv \cdot vp :: CD^2 : CP^2$



Produce Qq to meet the major axis E ;

Draw QX, DW, perpendicular to e major axis, and meeting PC in X id W.

Then, since triangles CPT, CvE, are milar,

trian. CPT : trian. $CvE :: CP^2 : Cv^2$

·, trian. CPT : trap. $TPvE :: CP^2 : CP^2 - Cv^2$

Again, since the triangles CDW, vQX, are similar,

triangle CDW : trangle $vQX :: CD^2 : vQ^2$

ut triangle CDW = triangle CPT ; Prop. XVIII., or. 5.

.nd triangle vQX = trapez. $TPvE$: Prop. XVIII., 'or. 3.

$$\therefore CP^2 : CD^2 :: CP^2 - Cv^2 : vQ^2$$

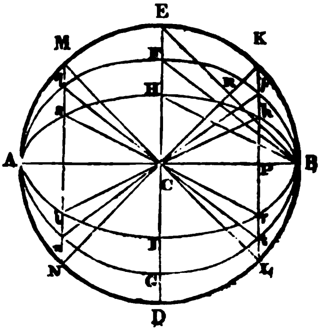
$$\text{Or, } Qv^2 : Pv\, vp :: CD^2 : CP^2$$

Cor. 1. The squares of the ordinates to any diameter, are o each other as the rectangles under their respective ab-cissæ.

Cor. 2. The above proposition is merely an extension of the roperty already proved in Prop. XII, with regard to the re-tion between ordinates to the axis and their abscissæ.

PROPOSITION XXI. THEOREM.

he equal conjugate diameters of all ellipses described on the same axis major, all terminate in the same right lines.



Let any number of ellipses ·J, FG be described on the same tis major, AB, and the right co-rdinates to that axis drawn from he extremities of the equal con-ugate diameters, will all coincide n the same lines KL, MN, or the 'ertices of all their equal diame-ers will all be found in those ines.

Describe on the same axis as a liameter, the circle AEBD, bisect the arcs EB in K and AE in M, from which points through the centre, C, draw the conjugate diameters KN, ML; join KL, MN; draw also the chords BE, BF, BH, also the diameters *fn*, *hi*, and *qt*, *sr*.

The triangles ECB, KPC, being similar, their sides are proportional; and since KC = BC, and KP = BR, the triangle KPC = the triangle BRC = $\frac{1}{2}$ the triangle ECB.

Hence, $CP^2 + PK^2 = \frac{1}{2}\,CB^2 + \frac{1}{2}\,CE^2$
and $KC^2 = \frac{1}{2}EB^2 = EC^2 + CB^2$,
or, $KC^2 + CL^2 = CB^2 + CE^2$
Hence $AB^2 + ED^2 = KN^2 + ML^2$

And since the semi-ordinate KP : *f*P : : EC : FC the triangles FCB, *f*PC, are similar, and *f*PC + $\frac{1}{2}$ FCB and $fC^2 = \frac{1}{2}\,FB^2 = \frac{1}{2}\,FC^2 + \frac{1}{2}\,CB^2$

Hence, $fC^2 + Ct^2 = FC^2 + BC^2$
Therefore, $fn^2 + qt^2 = FG^2 + AB^2$ agreeably to Prop. XV.

Also in the triangles HCB, *h*PC, being for the same reason as before shown, similar, *h*PC = $\frac{1}{2}$ HCB, and $hC^2 = \frac{1}{2}\,HB^2 = \frac{1}{2}\,HC^2 + \frac{1}{2}\,cB^2$, and $HC^2 + rC^2 = HC^2 + CB^2$

Therefore, $hi^2 + sr^2 = HI^2 + AB^2$ agreeably to the property of the ellipse.

Cor. 1. $CP^2 = \frac{1}{2}\,CB$
and $Pf^2 = \frac{1}{2}\,CF^2$
or, $Ph^2 = \frac{1}{2}\,CH^2$

Cor. 2. If AB, instead of being the axis major, should be the minor axis of a series of ellipses, the vertices of their equal conjugate diameters would still all be found in the lines MN, KL, produced.

HYPERBOLA.

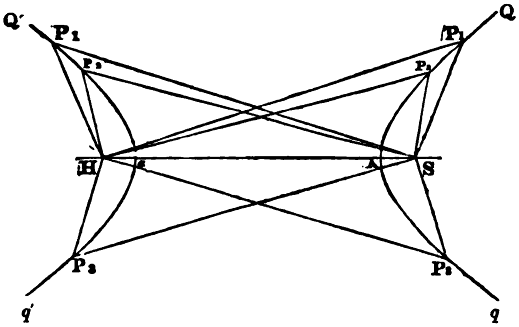
DEFINITIONS.

1. An Hyperbola is a plane curve, such that, if from any point in the curve two straight lines be drawn to two given fixed points, the excess of the straight line drawn to one of the points above the other will always be the same.

2. The two given fixed points are called the *foci*.

Thus, let QAq be an hyperbola, S and H the foci.
Take any number of points in the curve, P_1, P_2, P_3, . . .
Join S,P_1, H,P_1, S,P_2, H,P_2, S,P_3, H,P_3 ; . . . . then.

$$HP_1 - SP_1 = HP_2 - SP_2 = HP_3 - SP_3 = \dots.$$



If $HP_1 - SP_1$ and $SP'_1 - HP'_1$ . . . . be always equal to the same constant quantity, the points P_1 P_2 P_3 . . . and P'_1, P'_2, P'_3, will lie in two opposite and similar hyperbolas QAq, $Q'aq'$, which in this case are called *opposite hyperbolas*.

3. If a straight line be drawn joining the foci, and bisected, the point of bisection is called the *centre*.

4. The distance from the centre to either focus is called the *eccentricity*.

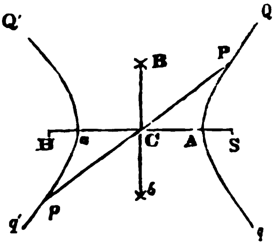
5. Any straight line drawn through the centre, and terminated by two opposite hyperbolas, is called a *diameter*.

6. The points in which any diameter meets the hyperbolas are called the *vertices* of that diameter.

7. The diameter which passes through the foci is called the

axis major, and the points in which it meets the curves the *principal vertices*.

8. If a straight line be drawn through the centre at right angles to the major axis, and with a principal vertex as centre, and radius equal to the eccentricity, a circle be described, cutting the straight line in two points, the distance between these points is called the *axis minor*.



Thus, let Q*q*, Q'*q*' be two opposite hyperbolas, S and H the foci, join S,H;

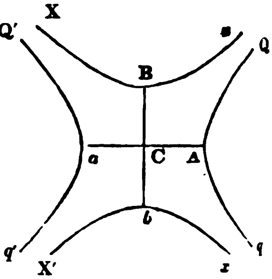
Bisect SH in C, and let SH cut the curves in A,*a*.

Through C draw any straight line P*p*, terminated by the curves in the points P, *p*.

Through C draw any straight line at right angles to A*a*, and with centre A and radius = CS describe a circle cutting the straight line in the points B, *b*.

Then C is the centre, CS or CH the eccentricity, P*p*, is a diameter, P and *p* its *vertices*, A*a* is the major axis, B*b* is the minor axis.

The hpyerbolas X*x*, X'*x*', whose major axis is B*b*, and whose minor axis is A*a*, are called the *conjugate hyperbolas* to Q*q*, Q'*q*'.



9. A straight line, which meets the curve in any point, but which, being produced both ways, does not cut it, is called a *tangent* to the curve at that point.

10. A straight line, drawn through the centre, parallel to the tangent, at the vertex of any diameter, is called the *conjugate diameter* to the latter, and the two diameters are called a *pair of conjugate diameters*.

The vertices of the conjugate diameter are its intersections with the conjugate hyperbolas.

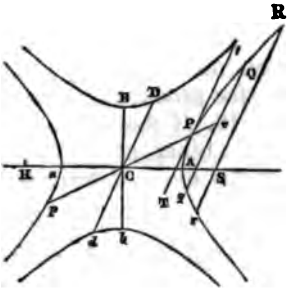
11. Any straight line drawn parallel to the tangent at the vertex of any diameter, and terminated both ways by the curve, is called an *ordinate* to that diameter.

12. The segments into which any diameter produced is di-

ded by one of its own ordinates and its vertices, are called e ***abscissæ*** of the diameter.

13. The ordinate to any diameter, which passes through e focus, is called the *parameter* of that diameter.

Thus, let Pp be any diameter,
ıd Tt a tangent at P ;
Draw the diameter Dd parallel
Tt ;
Take any point Q in the curve,
·aw Qq parallel to Tt and cutting
p produced in v ;
Through S draw Rr parallel to
t ;
Then Dd is the conjugate diame-
r to Pp,



Qq is the ordinate to the diameter Pp corresponding to the ›int Q.

Pv, vp, are the abscissæ of the diameter Pp corresponding the point Q.

Rr is the parameter of the diameter Pp.

14. Any straight line drawn from any point in the curve at ght angles to the major axis produced, and terminated both ays by the curve, is called an *ordinate to the axis.*

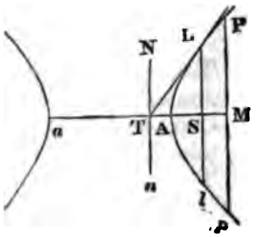
15. The segments into which the major axis produced is di- ıded by one of its own ordinates and its vertices, are called ıe *abscissæ of the axis.*

16. The ordinate to the axis which passes through the fo- ıs, is called the *principal parameter* or *latus rectum.*

(It will be proved in Prop. IV, that the tangents at the prin- ipal vertices are perpendicular to the major axis ; hence de- nitions 14, 15, 16, are in reality included in the three which nmediately precede them.)

17. If a tangent be drawn at the extremity of the latus rec- ım, and produced to meet the major axis ; and if a straight ne be drawn through the point of intersection, at right angles › the major axis ; the tangent is called the *focal tangent,* and ıe straight line the *directrix.*

Thus, form P, any point in the
urve, draw PMp perpendicular to Aa,
utting Aa in M ;
Through S draw Ll perpendicular
› Aa ;
Let LT, a tangent at L, cut Aa in T ;
Through T draw Nn perpendicular
› Aa :
Then, Pp is the ordinate to the axis
orresponding to the point P.



AM, M*a*, are the abscissæ of the axis corresponding to the point P,

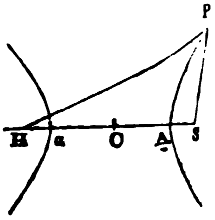
L*l* is the latus rectum,
LT is the focal tangent,
N*n* is the directrix.

18. An *asymptote* is a diameter which approaches the curve continually as they are both produced, but which, though ever so far produced, never meets it.

19. If the asymptotes of four opposite hyperbolas cross each other at right angles, the hyperbolas are called *right angled* or *equilaterial hyperbolas*.

PROPOSITION I. THEOREM.

The difference of two straight lines drawn from the foci to any point in the curve, is equal to the major axis.



That is, if P be any point in the curve,

$$HP - SP = Aa\,;$$

For,

$$\left.\begin{array}{l} HP - SP = AH - AS = Aa + aH - AS \\ \text{And,} \\ HP - SP = aS - aH = Aa - aH + AS \end{array}\right\}\text{Def. 1.}$$

Or,

$$2(HP - SP) = 2Aa$$
$$HP - SP = Aa$$

Cor. 1. The centre bisects the major axis; for, since

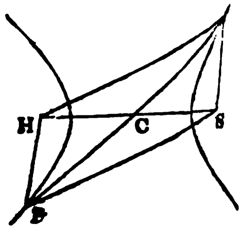
$$\begin{array}{rrcl} & AH - AS &=& aS - aH \\ \text{Or,} & SH - 2AS &=& SH - 2aH \\ \therefore & AS &=& aH \\ \text{And} & CS &=& CH, \text{ by def. 3.} \\ \therefore & AC &=& aC. \end{array}$$

Cor. 2.

$$\begin{array}{rrcl} & HP - SP &=& 2AC \\ \therefore & HP &=& 2AC + SP \\ & SP &=& HP - 2AC \\ & HP + SP &=& 2AC + 2SP. \end{array}$$

PROPOSITION II. THEOREM.

The centre bisects all diameters.



Take any point P in the curve;
Join S, P; H, P; S, H;
Complete the parallelogram SPHp;
join C,p; C, P;
Then, since the opposite sides of parallelograms are equal,

$$HP = Sp,\quad SP = Hp\,;$$
$$\therefore HP - SP = Sp - Hp\,;$$

∴ *p* is a point in the opposite hyperbola by definition 2.

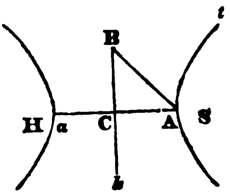
Again, since the diagonals of a parallelogram bisect each other, and since SH is bisected in C, (def. 3,)

∴ P*p* is a straight line and a diameter, and is bisected in C.

In like manner, it may be proved that any other diameter is bisected in C.

PROPOSITION III. THEOREM.

The rectangle under the segments of the major axis produced, made by the focus and its vertices, is equal to the square of the semi-axis, minor.



That is,

$$AS \cdot Sa = BC^2$$

For,

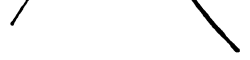
$$\begin{aligned} BC^2 &= AB^2 - AC^2 \\ &= SC^2 - AC^2, \text{ by def. 8,} \\ &= (SC - AC)\,(CS + AC) \\ &= AS \cdot Sa \end{aligned}$$

Cor. The square ef the eccentricity is equal to the sum of the squares of the semi-axes.

$$\begin{aligned} \text{For,}\quad SC^2 &= AB^2, \text{ def. 8,} \\ &= AC^2 + BC^2. \end{aligned}$$

PROPOSITION IV. PROBLEM.

To draw a tangent to the hyperbola at any point.



Let P be the given point ;

Join S, P ; H, P ;

Bisect the angle SPH by the straight line T*t*.

T*t* is a tangent to the curve at P.

For if T*t* be not a tangent, let T*t* cut the curve in some other point *p*.

Join S, *p* ; H, *p* ; draw SYO perpendicular to T*t*, meeting HP in O ; join *p*, O.

Since the angles at Y are right angles, and angle SPY = angle OPY by construction, and side YP common to the two triangles SYP, OYP.

$$\therefore \quad SY = OY$$

$$\text{And}\quad SP = OP.$$

Again, since SY = OY, and Y*p* common to the two triangles SY*p*, OY*p*, and the angles at Y equal ;

$$\therefore \quad Sp = Op\,;$$

$$\begin{aligned}\therefore \mathrm{H}p-\mathrm{O}p &= \mathrm{H}p-\mathrm{S}p\\ &= \mathrm{HP}-\mathrm{SP}\\ &= \mathrm{HP}-\mathrm{OP}\\ &= \mathrm{HO}\\ \therefore \quad \mathrm{H}p &= \mathrm{HO}+\mathrm{O}p\end{aligned}$$

that is, one side of the triangle H$\mathrm{O}p$ is equal to the other two, which is absurd;
∴ p is not a point in the curve: and in the same manner, it may be proved that no point in the straight line Tt can be in the curve, except P;

∴ Tt is a tangent to the curve at P.

Cor. 1. Hence tangents at A and a, are perpendicular to the major axis.

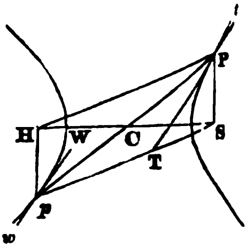
Cor. 2. SP and HP make equal angles with every tangent.

Cor. 3. Since SPH, the vertical angle of the trian. SPH, is bisected by the straight line PT, which cuts the base in T,

∴ HT : TS : : HP : SP.

PROPOSITION V. THEOREM.

Tangents drawn at the vertices of a diameter are parallel.



Let Tt, Ww, be tangents at P, p, the vertices of the diameter PCB.

Join S,P ; H, P ; S, p ; H, p :

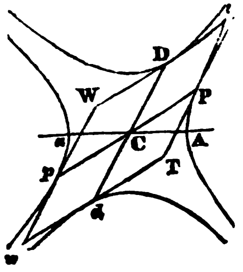
Then, by Prop. II, SH is a parallelogram, and since the opposite angles of parallelograms are equal,

∴ angle SPH = angle SpH.

But the tangents Tt, Ww; bisect the angles SPH, SpH, respectively.

∴ angle WpS = angle HPT
= angle PTS, which is the exterior opposite angle to WpS.

∴ Ww is parallel to Tt.

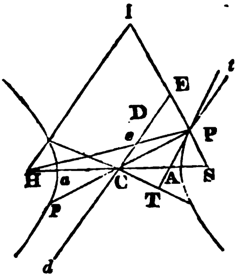


Cor. If Dd be a diameter conjugate to Pp, and terminated by the conjugate hyperbolas, tangents drawn at D and d will be parallel.

Hence tangents drawn at the extremities of conjugate diameters form a parallelogram.

PROPOSITION VI. THEOREM.

traight lines be drawn from the foci to a vertex of a diame-r, the distance from the vertex to the intersection of the con-ıgate diameter with either focal distance, is equal to the mi-axis major.



'hat is, if D*d* be a diameter to conju- to P*p*, cutting SP produced in E, HP in *e*,

PE or P*e* = AC.

raw HI parallel to D*d*, meeting SP luced in I.

'he angle PHI = alternate angle T

= angle TPS

= angle HIP

· HI is parallel to D*d* or T*t*

∴ IP = HP.

lso, since SC = HC, and CE is parallel to HI, the base of trian. SHI,

· SE = EI. Hence, ∵ PE = PI—EI = HP—SE=HP -PE

· 2 PE = HP — SP = 2 AC

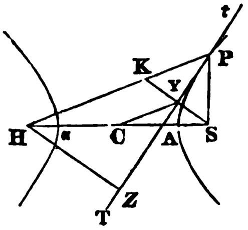
PE = AC.

lso angle PE*e* = angle P*e*E, ∴ P*e* = PE and

P*e* = AC.

PROPOSITION VII. THEOREM.

pendiculars from the foci upon the tangent at any point, in-rsect the tangent in the circumference of a circle whose dia-eter is the major axis.



'rom S let fall SY perpendicular on ı tangent at P.

oin S, P ; H, P ; let HP meet SY ; join C. Y ;

hen, since angle SPY = angle Y, and the angles at Y are right es, and the side PY common to the triangles SPY, KPY,

∴ SY = KY

and KP = SP.

gain, since SY = YK, and SC = CH, CY cuts the sides he trian. HSK proportionally,

∴ CY is parallel to HP.

Also, since CY is parallel to HP, SY = KY, and SC = CH,

$\therefore$ CY $= \frac{1}{2}$ HK $= \frac{1}{2}$ (HP$-$KP) $= \frac{1}{2}$ (HP$-$SP) $= \frac{1}{2}$ A*a* = AC.

Hence, a circle described with centre C and radius = CA, will pass through Y, and in like manner, if HZ be drawn perpendicular to T*t*, it may be proved that the same circle will pass through Z also.

PROPOSITION VIII. THEOREM.

The rectangle contained by perpendiculars from the foci upon the tangent at any point, is equal to the square of the semi-axis, minor.

That is,

$$SY \,.\, HZ = BC^2.$$

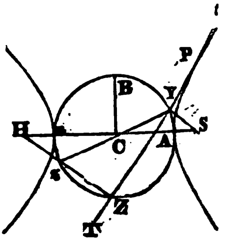
Let T*t* be a tangent at any point P;

On A*a* describe a circle cutting T*t* in Y and Z; join S, Y; H, Z.

Then, by last Prop. SY, HZ are perpendicular to T*t*.

Let HZ meet the circumference in *z*;

Join C, *z*; C, Y;



Since *z*ZY is a right angle, the segment in which it lies is a semicircle, and *z*, Y, are the extremities of a diameter;

$\therefore$ *z*CY is a straight line and a diameter.

Hence the triangles CYS, C*z*H are in every respect equal

$$\therefore SY = Hz$$

$$\begin{aligned} \therefore SY \,.\, HZ &= Hz \,.\, HZ \\ &= HA \,.\, Ha. \\ &= BC^2 \,. \quad \text{Prop. III.} \end{aligned}$$

PROPOSITION IX. THEOREM.

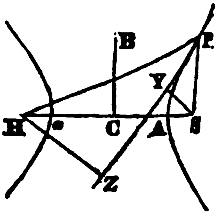
Perpendiculars let fall from the foci upon the tangent at any point, are to each as the focal distance of the point of contact.

That is,

SY : HZ : : SP : HP.

For the triangles SPY, HPZ, are manifestly similar;

$\therefore$ SY : HZ : : SP : HP.



Cor. Hence,

$$SY = HZ\frac{SP}{HP}$$

$$\therefore SY^2 = SY \,.HZ.\frac{SP}{HP}$$

$$= BC^2.\frac{SP}{HP} \text{ last Prop.}$$

$$= BC^2 . \quad \frac{SP}{2\,AC + SP}$$

So also,

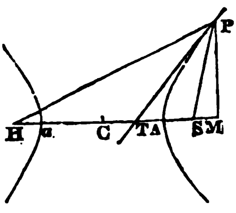
$$HZ^2 = BC^2 \frac{HP}{SP} = BC^2 \frac{HP}{HP - 2\,AC}$$

PROPOSITION X. THEOREM.

ı tangent be applied at any point, and from the same point an ɿrdinate to the axis be drawn, the semi-axis major is a mean ɒroportional between the distance from the centre, to the orlinate with the axis, and the distance from the centre to the intersection of the tangent with the axis.

That is,

CT : CA : : CA : CM.



Since the angle SPH is bisected by Γ, which cuts HS, the base of the angle HPS, in T, ∴

HT : ST : : HP : SP

HT — ST : HT+ST : : HP — SP : P+SP

, 2 CT : SH : : 2AC : HP SP

2 CT : 2 AC : : SH : HP+SP - - - - (1)

But since PM is drawn from the vertex of triangle HPS ɼpendicular to HS produced,

HM—SM : HP+SP : : HP—SP : HM+SM

, SH : HP+SP : : 2 AC : 2 CM - - - - (2)

Comparing this with the proportion marked (1), we have

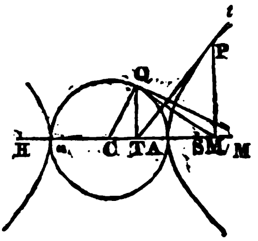
2 CT : 2 AC : : 2 AC : 2 CM

, CT : CA : : CA : CM.

PROPOSITION XI. THEOREM.

t AQ*a be a circle described on the major axis, from the point* T, *draw* TQ *perpendicular to* A*a, meeting the circle in* Q, *join* QM.

Then QM is a tangent to the circle Q



Join C, Q.

For if QM be not a tangent, draw M′ a tangent at Q, cutting AC in ′

Then CQM′ is a right angle.

∴ Since QT is drawn from the ɟht angle CQM′ perpendicular to ɛ hypothenuse,

$\therefore CM' : CQ :: CQ : CT.$

or, $CM' : CA :: CA : CT, \therefore CQ = CA.$

But by the last Prop..

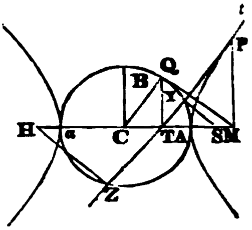
$CM : CA :: CA : CT$

$\therefore CM = CM',$

which is absurd; $\therefore$ QM is not a tangent at Q; and in the same manner it may be proved that no line but QM′ can be a tangent at Q.

PROPOSITION XII. THEOREM.

The square of any semi-ordinate to the axis, is to the rectangle under the abscissæ, as the square of the semi-axis minor, is to the square of the semi-axis major.



That is, if P be any point in the curve,

$PM^2 : AM . Ma :: BC^2 : AC^2.$

Describe a circle on Aa, and draw PT a tangent to the hyperbola at P, intersecting the circle in the points Y, Z, and the major axis in T.

Draw TQ perpendicular to Aa, meeting the circle in Q; join QM

Then QM is a tangent to the circle at Q by Prop. II, and $\therefore$ the angle CQM is a right angle.

Join S, Y; , Z; HSY and HZ are perpendicular to Tt, Prop. VII.

Hence the triangles PMT, SYT, PZT, are similar to each other.

$\therefore PM : SY :: MT : TY$

and, $PM : HZ :: MT : TZ$

$\therefore PM^2 : SY . HZ :: MT^2 : TY . TZ,$

or, $PM^2 : BC :: MT^2 : TQ^2,$

Prop. VII.

$:: QM^2 : CQ^2,$

$\because$ MQT. MCQ are similar triangles.

$:: AM . Ma : AC^2,$

$\because PM^2 : AM . Ma :: BC^2 : AC^2$

Cor. 1.

Let $P_1 M_1$, $P_2 M_2$, ... be ordinates to the axis from any point P_1 P_2 ......

Then by Prop.

$P_1 M_1^2 : AM_1 . M_1 a :: BC^2 : AC^2$

$P_2 M_2^2 : AM_2 . M_2 a :: BC^2 : AC^2$

$\therefore P_1 M_1^2 ; P_2 M_2^2 :: AM_1 . M_1 a : AM_2 . M_2 a,$

That is, the square of the ordinates to the axis are to each her as the rectangles of their abscisa.

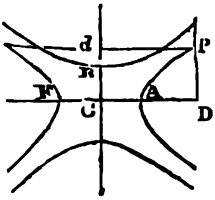
Cor. 2. By Prop.

$$PM^2 : AM . Ma :: BC^2 : AC^2$$

ut $AM = CM - CA$, $Ma = CM + CA$,

$$\therefore PM^2 : CM^2 - CA^2 :: BC^2 : AC^2.$$

Cor. 3. Since by the proposition
$PM^2 : AM : Ma :: BC^2 : AC^2$,
we have in the conjugate hyberbolas
$CR^2 : CA^2 : CR^2 + Cd^2 : dP^2$,
since $dP^2 = CD^2$.



PROPOSITION XIII. THEOREM.

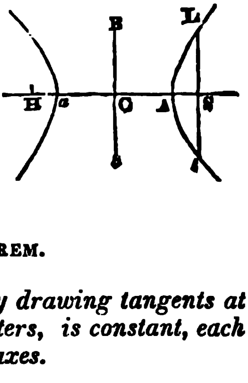
he latus rectum is a third proportional to the axis major and minor.

That is,

$$Aa : Bb :: Bb : Ll.$$

Since LS is a semiordinate to the xis.

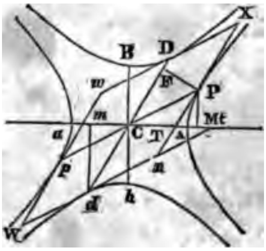
$AC^2 : BC^2 :: AS . Sa : LS^2$,
rop. XII.
$:: BC^2 : LS^2$,
rop. III.
$\therefore AC : BC :: BC : LS$
r, $Aa : Bb :: Bb : Ll$.



PROPOSITION XIV. THEOREM.

he area of all parallelograms, formed by drawing tangents at the extremities of two conjugate diameters, is constant, each being equal to the rectangle under the axes.

Let Pp, Dd, be any two conjugate ameters, $WwXx$, a parallelogram scribed between the opposite and njugate hyperbolas by drawing ngents at P, p, D, d ; then Pp, Dd, vide the parallelogram $WxXw$ to four equal parallelograms.



Draw Pm. dm, ordinates to the is ; PF perpendicular to Dd.

Let CA meet PX in T and Wx in t ;

Then $CT : CA :: CA : CM$

$Ct : CA :: CA : Cm$, Prop XI.
$\therefore CT : Ct :: Cm : CM$
But, $CT : Ct :: MT : Cm$, by similar triangles.
$\therefore MT : Cm :: Cm : CM$
$\therefore CM \,.\, MT = Cm^2$ - - - - - - - - (1)

Again, $CM : CA :: CA : CT$
$\therefore CM : CA :: MA : AT$, dividendo :
Or, $CM : Ma :: MA : MT$, componendo :
$\therefore AM \,.\, Ma = CM \,.\, MT = Cm^2$ - - - - (2)

But, $AC^2 : BC^2 :: AM : Ma : PM^2$
Or, $AC^2 : BC^2 :: Cm^2 : PM^2$
$\therefore AC : BC :: Cm : PM$
Similarly, $AC : BC :: CM : dm$
Or, $BC : dm :: CA : CM$
But, $CT : CA :: CA : CM$
$\therefore CT : CA :: BC : dm$
But, $PF : CT :: dm : Cd$
$\therefore PF : CA :: BC : Cd$

$\therefore$ Rectangle $PF \,.\, CD =$ rectangle $AC \,.\, BC$
or, Parallelogram $CX =$ rectangle $AC \,.\, BC$
$\therefore$ Parallelogram $WwXx = 4\, AC \,.\, BC = Aa \,.\, Bb$

Cor. By (2),

$$Cm^2 = HM \,.\, Ma$$
$$= (CM - CA)\,(CM + CA).$$
$$= CM^2 - CA^2$$
$$\therefore \quad CA^2 = CM^2 - Cm^2$$

And similarly, $CB^2 = dm^2 - PM^2$.

PROPOSITION XV. THEOREM.

The difference of the squares of any two conjugate diameters, is equal to the same constant quantity, namely, the difference of the squares of the two axes.

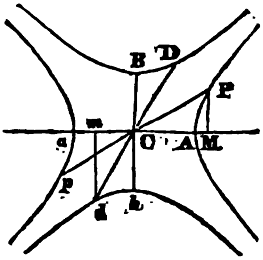
That is, if Pp, Dd, be any two conjugate diameters,

$$Pp^2 - Dd^2 = Aa^2 - Bb^2.$$

Draw PM dm, ordinates to the axis.

Then, by Cor. to last Prop.

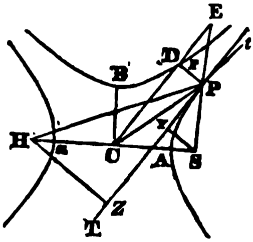
$$AC^2 - BC^2 = CM^2 + PM^2 - (Cm^2 + d^2m)$$
$$= CP^2 - Cd^2$$
$$\therefore Aa^2 - Bb^2 = Pp^2 - Dd^2.$$



PROPOSITION XVI. THEOREM.

angle under the focal distance of any point, is equal to the square of the semi-conjugate.

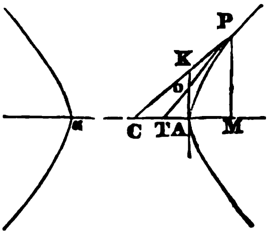
is, if CD be conjugate to CP,
$SP . HP = CD^2$.
' SY, HZ, perpendiculars to
gent at P, and PF perpendicu-
D ;
by similar trian. SPY, PEF
$SP : SY :: PE : PF$
$SP : SY :: DC : PF$
$\because PE = AC$, by Prop. VI.
y, $HP : HZ :: AC : PF$
$P . HP : SY . HZ :: AC^2 : PF^2$
$:: CD^2 : CB^2$, by Prop. XIV.
t $SY . HZ = CB^2$, by Prop. VIII.
$\therefore SP . HP = CD^2$.



PROPOSITION XVII. THEOREM.

ngents be drawn, one at the principal vertex, the other vertex of any other diameter, each meeting the other's ter produced, the two tangential triangles thus formed e equal

is,
le $CPT =$ triangle CAK.
' the ordinate PM ; then
$: CA :: CP : CK$, by similar
s.
$: CA :: CA : CT$
$: CT :: CP : CK$.
wo triangles CPT, CAK, have
e angle C common and the
out that angle reciprocally
onal ; these triangles are ∴ equal.

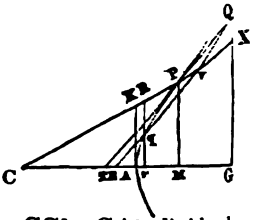


1. Take each of the equal triangle CPT, CAK, from mon space CAOP ; there remains
triangle $OAT = OKP$.

2. Also take the equal triangles CPT, CAK, from the triangle CPM ; there remains
triangle $MPT =$ trapez. AKPM.

PROPOSITION XVIII. THEOREM.

The same being supposed, as in last proposition, then any straight lines QG, QE, *drawn parallel to the two tangents shall cut off equal spaces.*



That is,
triangle GQE = trapez. AKXG
triangle rqE = trapez. AKRr .

Draw the ordinate PM.

The three similar triangles CAK, CMP, CGX, are to each other as CA^2, CM^2, CG^2,

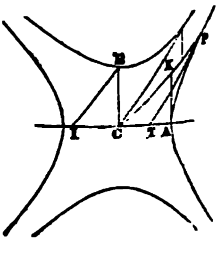
$\therefore$ AKPM : trap. AKXG : : $CM^2 - CA^2 : CG^2 - CA^2$, dividendo.
But, $PM^2 : QG^2 : : CM^2 - CA^2 : CG^2 - CA^2$,
$\therefore$ trap. AKPM : trap. AKXG : : $PM^2 : QG^2$
But, trian. MPT : trian. GQF : : $PM^2 : QG^2$,
$\because$ the triangles are similar.
$\therefore$ trap. AKPM : trian. MPT : : trap. AKXG : trian. GQE,
But, by Prop. XVII, Cor. 2.
trap. AKPM = triangle MPT ;
$\therefore$ trap. AKXG = triangle GQE.
And similarly, trap. AKRr = triangle rqE.

Cor. 1. The three spaces AKXG, TPXG, GQE, are all equal.

Cor. 2. From the equals, AKXG, EQG, take the equals AKRr, Eqr ; there remains,
RrXG, = rqQG.

Cor. 3. From the equals RrXG, rqQG, take the common space rqvXG ; there remains,
triangle vQX = triangle vqR.

Cor. 4. From the equals EQG, TPXG, take the common space EvXG ; there remains,
TPvE = triangle vQX.



Cor. 5. If we take the particular case in which QG coincides with the minor axis,

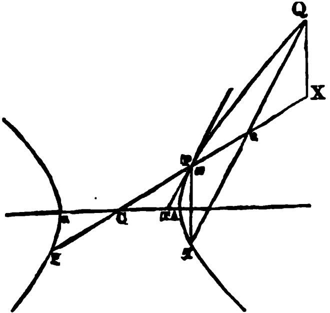
The triangle EQG becomes the triangle IBC,

The figure AKXG becomes the triangle AKC,

$\therefore$ triangle IBC = triangle AKC
= triangle CPT.

PROPOSITION XIX. THEOREM.

Any diameter bisects all its own ordinates.



That is,
Qq be any ordinate to a dia-
r CP,

$$Qv = vq$$

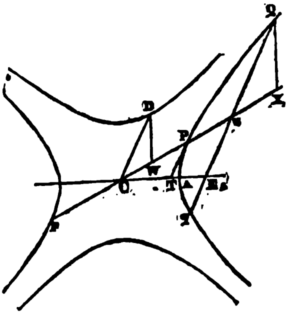
w QX, qx, at right angles to major axis ;
hen triangle vQX = trian- vqx ; Prop. XVIII., Cor. 3.
ut these triangles are also angular ;

$$\therefore Qv = vq.$$

'or. Hence, any diameter divides the hyperbola into two ıl parts.

PROPOSITION XX. THEOREM.

square of the semi-ordinate to any diameter, is to the rec- ngle under the abscissæ, as the square of the semi-conjugate the square of the semi-diameter.



That is,
Qq be an ordinate to any dia-
ːr CP,

$$Qv^2 : Pv \cdot vp :: CD^2 : CP^2.$$

et Qq meet the major axis in E ;
raw QX, DW, perpendicular to najor axis, and meeting PC in nd W.
hen, since the triangles CPT, ı, are similar,

ian. CPT : trian. CvE :: CP^2 Cv^2
:rian. CPT : trap. TPvE :: CP^2 : $-CP^2$

gain, since the triangles CDW, vQX, are similar,
triangle CDW : triangle vQX :: CD^2 : vQ^2 ;
, triangle CDW = triangle CPT ; Prop. XVIII, Cor. 5,
l triangle vQX = trapez. TPvE ; Prop. XVIII., Cor. 3.

$$\therefore CP^2 : CD^2 :: Cv^2 - CP^2 : vQ^2$$

$$\text{Or, } Qv^2 : Pv \cdot vp :: CD^2 : CP^2.$$

Cor. 1. The squares of the ordinates to any diameter, are ːach other as the rectangles under their respective abscissæ.

Cor. 2. The above proposition is merely an extension of property already proved in Prop. XII, with regard to the ıtion between ordinates to the axis and their abscissæ.

PROPOSITION XXI. THEOREM.

If tangents be drawn at the vertices of the axes, the diagonals of the rectangle so formed are asymptotes to the four curves.

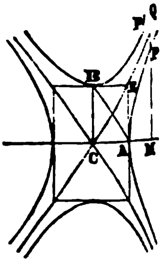
Let MP meet CE in Q;

Then, $MQ^2 : CM^2 :: AE^2 : AC^2$

$:: BC^2 : AC^2$

$:: MP^2 : CM^2 - CA^2$.

Now, as CM increases, the ratio of CM^2 to $CM^2 - CA^2$ continually approaches to a ratio of equality; but $CM^2 - CA^2$ can never become actually equal to CM^2, however much CM may be increased. Hence, MP is always less than MQ, but approaches continually nearer to an equality with it.



In the same manner it may be proved, that CQ is an asymptote to the conjugate hyperbola BP'.

Cor. 1. The two asymptotes make equal angles with the axis major and with the axis minor. .

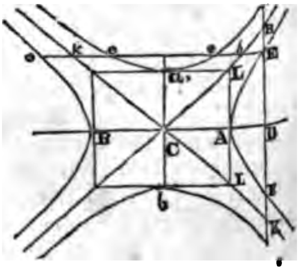
Cor. 2. The line AB joining the vertices of the conjugate axes is bisected by one asymptote and is parallel to the other.

Cor. 3. All lines perpendicular to either axis and is terminated by the asymptotes are bisected by the axis.

PROPOSITION XXII. THEOREM.

If a line be drawn through any point of the curves, parallel to either of the axes, and terminated at the asymptotes, the rectangle of its segments, measured from that point, will be equal to the square of the semi-axis to which it is parallel.

That is,
the rect. HEK or HeK = CA^2,
and rect. hEk or hek = CA^2.



For, draw AL parallel to Ca, and aL to CA. Then by the parallels. $CA^2 : Ca^2$ or $AL^2 :: CD^2 : DH^2$; and by Prop. XII, $CA^2 : Ca^2 :: CD^2 - CA^2 : DE^2$; ∴ by subtr. $CA^2 : Ca^2 :: CA^2 : DH^2 - DE^2$ or HEK. But the antecedents CA^2, CA^2 are equal, herefore, the consequents Ca^2, HEK must also be equal.

In like manner it is again,

y the parallels, $CA^2 : Ca^2$ or $AL^2 :: CD^2 : DH^2$;
$CA^2 : Ca :: CD^2+CA^2 : De^2$;
by subtr. $CA^2 : Ca^2 :: CA^2 : De^2-DH^2$ or HeK.
the antecedents CA^2, CA^2 are the same,
ie conseq. Ca^2, HeK must be equal.
ke manner, by changing the axes, is hEk or $hek = CA^2$.

or. 1. Because the rectangle HEK = the rectangle HeK,
$\therefore EH : eH :: eK : EK$
And consequently HE : is always greater than He.

or. 2. The rectangle hEK = the rectangle HEk.
For, by similar triangles $Eh : EH :: Ek : EK$.

holium. It is evident that this proposition is general for line oblique to the axis also, namely, that the rectangle of egment of any line, cut by the curve, and terminated by symptotes, is equal to the square of the semi-diameter to h the line is parallel—since the demonstration is drawn properties that are common to all diameters.

r. 3. Hence it is evident that all the rectangles are equal h are made of the segments of any parallel lines, cut by urves, and limited by the asymptotes; and therefore, that rectangle of any two lines drawn from any point in the e, parallel to two given lines, and limited by the asymp-, is a constant quantity.

GENERAL REMARK.

ving thus discussed at length, the properties of the para- ellipse and hyperbola, in the relations of their local and liar constructions, it may be observed, that there are many erties common to each—especially in the ellipse and rbola. These curves have many striking similarities in determinations, although there is but little similarity in construction; for the axes, of the hyperbola are thrown ut the curve, while in the ellipse they are within; hence it d not be surprising, that the same or corresponding tities, thus differently associated, should propagate by a what similar condition of their mutations, curves so apparently dissimilar in their developments. The ellipse is a curve of limited extent returning into itself as the circle, but the hyperbola is unlimited in its construction, or its determination, for its branches may be extended indefinitely ; the same may be observed in relation to the parabola, which may be indefinitely extended, and the branches of the curve become at length parallel to its axis ; this, however, is only in their infinite extension. But the hyperbolic curve never approaches toward or even to a parallelism with the axis, but approaches infinitely toward its asymptotes ; but without ever touching them, except in their infinite extension. The manner in which these curves are derived from the sections of a cone, as their name indicates, their origin will be shown in another volume, and their quadratures, and some other properties in relation to them, will be there discussed.

HIGHER GEOMETRY

AND

MENSURATION:

BEING THE

FOURTH PART OF A SERIES

ON

ELEMENTARY AND HIGHER

OMETRY, TRIGONOMETRY, AND MENSURATION;

TAINING MANY VALUABLE DISCOVERIES AND IMPROVEMENTS IN MATHEMATICAL
IENCE, ESPECIALLY IN RELATION TO THE QUADRATURE OF THE CIRCLE,
AND SOME OTHER CURVES, AS WELL AS THE CUBATURE OF CERTAIN
CURVILINEAR SOLIDS; DESIGNED AS A TEXT-BOOK FOR COLLEGIATE
AND ACADEMIC INSTRUCTION, AND AS A PRACTICAL
COMPENDIUM ON MENSURATION.

BY NATHAN SCHOLFIELD.

LONDON:
WILEY AND PUTNAM, 6, WATERLOO PLACE.
1845.

PREFACE.

ving, in the former parts of this series, treated of the nts of geometry, trigonometry, conic sections, &c., it emains for us, in accordance with our original design, ke such application of the former principles, as to elicit other truths or principles as depend on their various nations, and to investigate the relations of such subjects tain to the higher geometry. And, without attempting e a full and perfect treatise on the subject, which would e volumes, we shall endeavor to present some portions ancient subject in a new dress; hoping, thereby, to r its beauties more plainly visible, and its oracles more gible.

ne new solids are introduced into this volume, the most tant of which is a class termed *revoloids;* which, from rganization, seem to serve as a connecting link between near and curvelinear solids. The properties of those are discussed, and their surfaces and solidities are de- ied. Some new curves are also introduced and inves- d, among which is the *revoloidal curve,* whose quadrature ermined; and, from its relation to the circle, and also to inear figures, we are enabled to approximate to the cir- uadrature to an indefinite extent. During the investiga- f this subject, other important properties of the circle e developed, by which the area of the segment of a whose arc and sine are known, may be computed with tle labor as that of the area of a triangle whose base erpendicular are given.

have also introduced into this work a mode of construction for variable quantities, or such magnitudes as depen on variable factors; and have adapted a notation, embracing some of the principles of the calculus, by which variable magnitudes may be algebraically discussed, and their conditions rendered intelligible. By this notation, some of the more difficult geometrical subjects are susceptible of the most elegant solution; and we are also enabled to get a definite algebraic expression for the circle's quadrature, in terms of the diameter. The mensuration of such superficies and solids as depend on the higher geometry, follows at the close of the work.

From the hasty manner with which a considerable portion of this work has been prepared, it can hardly be presumed to be entirely free from errors; but it is believed that if any errors exist, they are such as involve no important principle.

The author, with these remarks, submits the work to the consideration of an intelligent public.

CONTENTS.

BOOK I.

PAGE.
ecies and Quadrature of the Sections of Elementary Solids,
ng the Parabola, Ellipse, and Hyperbola, ... 7
, ... 7
ns of a Cone, ... 8
ns of a Polyedroid, ... 10
ns of a Prism, &c., ... 11
e of the Parabola, ... 16
ipse, its Quadrature, &c., ... 20
perbola, its Quadrature, &c., ... 23
to the Conic Sections, ... 31

BOOK II.

ons, or Segments of Solids of Revolution, Ungulas, &c., . 34
, ... 34
Ungulas and Segments, ... 36
gments and Ungulas, ... 44
nd Ungulas of an Elliptical Cylinder, ... 47
Jngulas and Segments, ... 49
Prisms and Ungulas, ... 51

BOOK III.

ds and Solids formed by the Revolution of the Conic Sections, 54
, ... 54
of the Surface of a Revoloid determined, . . . 60
and Ungulas equivalent to a Sphere or Spheroid, . . 63
ure of the Revolvoid determined, ... 68
of Revolvoid, ... 75
Revoloid, its Cubature determined, ... 84
Solids formed by the Revolution of the Conic Sections, . 87
l Formulæ, in relation to Cylindric and Conical Ungulas, . 91

BOOK IV.

volvoidal Curve, the Rectification of the Ellipse, and other
nd on the Quadrature of the Circle, ... 98
, ... 98
of a Revolvoidal Curve, ... 100
the Curve, ... 102
, its Rectification, &c., ... 103
n of the Revoloidal Curve, ... 108
le's Circumference, and its Quadrature, ... 109
e Circle's Quadrature, ... 129
drature of the Segment of a Circle, ... 131
—their Quadrature, &c., ... 139
loid, ... 143

BOOK V.

PAGE.
On the Production and Resolution of Geometrical Magnitudes, . . 145
CHAP. I.—Definitions and Principles, ... 145
  Production of Surfaces and Solids, ... 146
CHAP. II.—On the Construction of Quantities whose Elements are a series either of constant or variable quantities, . . 157
  Explanation of Principles, and Notation, ... 157
  Construction of Variables, ... 159
  Construction of Curves from their Equations, . . 159
  Quadrature of the Circle expressed algebraically, in terms of known functions of the diameter, ... 160
  Equivalent Constructions for the Equations of Curvelinear Solids, ... 161
CHAP. III.—The Differential and Integral Calculus, ... 166
  Differential Calculus, ... 166
  Integral Calculus, ... 169
CHAP. IV.—On the Centre of Surfaces and Solids, the Virtual Centre and Centre of Magnitude, &c., ... 173
  Definitions, ... 173
  Virtual Centre of a System of Points, ... 174
  Virtual Centre of Surfaces, ... 175
  The Virtual Centre of Solids, ... 179
  The Virtual Centre of a Circular Arc, ... 180
  The Virtual Centre of the Surface of a Solid, . . 181
CHAP. V.—On the Relations of Lines, Surfaces, and Solids, generated by the motion of the Virtual Centre, ... 182
  General Proposition, ... 182

MENSURATION OF SUPERFICIES.

Circular Segments and Zones, ... 186
To find the Circumference of a Circle, or any Arc, ... 192
Mensuration of the Ellipse, ... 195
Mensuration of the Parabola, ... 202
Mensuration of the Hyperbola, ... 207
To find the Area of any Plane Surface by Equi-distant Ordinates, . 210

MENSURATION OF SOLIDS.

Mensuration of the Sphere and Revoloid, ... 212
Segments of a Sphere or Revoloid, ... 214
Segments of a Spheroid, ... 216
Mensuration of the Paraboloid, &c., ... 218
Mensuration of the Hyperboloid, &c., ... , . 219
Circular Elliptical and Parabolic Spindles, ... 221
Ungulas, ... 224
Rings, ... 227
Gauging of Casks, ... 228

SPECIFIC GRAVITY OF BODIES.

Table of Specific Gravities, ... 233
Magnitude of Bodies, by their Weight and Specific Gravity, . . 234
QUESTIONS FOR EXERCISE, ... 236
Description of an Instrument for Measuring Distances and Heights by a single observation, ... 242
NOTES, ... 248

HIGHER GEOMETRY.

PART II.—BOOK I.

SPECIES AND QUADRATURE OF SUPERFICIAL SECTIONS OF ELEMENTARY SOLIDS.

DEFINITIONS.

1. *Superficial sections* are surfaces formed when solids are cut by *plane* or *curved surfaces*.

2. If the cutting surface is a *plane*, the section is a *plane section.*

3. Superficial sections of solids take different names, according to the form of the solid in the plane of the section.

4. From the cylinder, we have the *rectangle*, the *circle*, and, as will be shown, (Prop. VIII. *Cor.*) the *ellipse*.

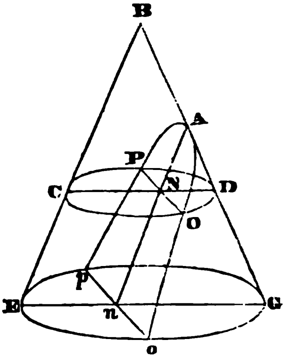
5. From the cone, we have five different figures, viz: a triangle, a circle, and, as will be shown in Propositions I., II., and III., a *parabola*, an *ellipse*, and a *hyperbola*.

6. From the sphere, we have only the circle.

Scholium. The *parabola*, *ellipse*, and *hyperbola*, will be more specially the subjects of this book.

PROPOSITION I. THEOREM.

If a right cone BEG *be cut by a plane* Ap o, *which is parallel to a plane touching the cone along the slant side* BE, *the section* A p o *is a parabola.*



Let BEG be that position of the generating triangle which is perpendicular to the cutting plane Ap o; An their common section, which is parallel to BE. Then, since the plane BEG passes through the axis, it is perpendicular to the base EoG, and to every circular section CPD parallel to the base; it is also perpendicular to Ap o. Hence, the common section PO of the planes Ap o, CPD, is perpendicular to BEG and therefore to An and CD.

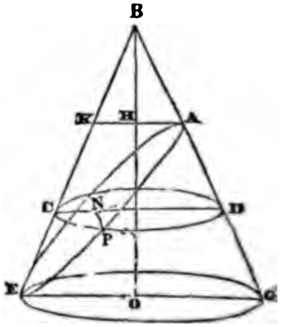
But, AN : ND : : BE : EG, which is a constant ratio; therefore, by the properties of the circle, AN : ND : : CN$\times$ND : NO^2, since CN is equal and parallel to En, and constant. Hence the curve is a parabola whose axis is An.

Cor. If L be the latus rectum of the parabola pAo, L$\times$AN $=$NP^2$=$CN$\times$ND.

$$L=CN\times\frac{ND}{AD}$$

PROPOSITION II. THEOREM.

If a cone BEG *be cut by a plane* EAP *through both slant sides the section is an ellipse.*



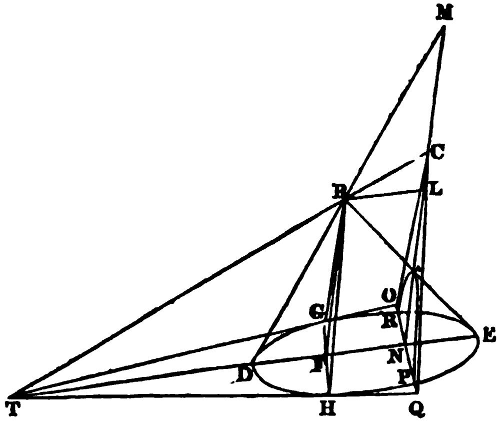
Let BEG be that position of the generating triangle which is perpendicular to the cutting plane: CPD any circular section. Draw AHK parallel to EG, and therefore bisected by the axis BO.

Then EN : CN : : EA : AK
NA : ND : : EA : EG;
$\therefore$ EN$\times$NA : CN$\times$ND (NP^2)
: : EA^2 : EG$\times$AK

which is the property of an ellipse, one of whose axes is EA and the other a mean proportional between EG and AK. (Conic Sections, Ellipse, Prop. XII.)

PROPOSITION III. THEOREM.

a right cone BED *be cut through one side* BE *by a plane* RAP *which being produced backwards, cuts the other side* DB *produced, the section is an hyperbola.*



Let DGEH be any circular section, BGH a triangular sec-
n through the vertex B of the cone parallel to the plane
AP.

$$\text{Then, } AN : EN :: BF : EF$$
$$NM : ND :: BF : FD$$
$$AN \times NM : EN \times ND\ (NP^2) :: BF^2 : EF \times FD\ (FH^2)$$

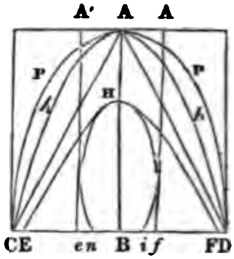
hich is the property of an hyperbola, whose axis major is
M, and whose conjugate axis is to AM as FH to BF.

Cor. If GT, HT, be tangents to the circle at G, H; and
ines passing through GT, HT, respectively, touch the cone
ng the lines BG, BH; also, if TB, the common section of
e planes, meet AM in C, then the common section CO, CQ,
the plane RAP, extended to meet the tangent planes, are
e asymptotes of the hyperbola.

Draw BL parallel to DE, meeting AM in L: then the axes
the hyperbola being in the proportion of BF to FH, the an-
e GBH, or the equal angle OCQ is the angle between the
ymptotes.

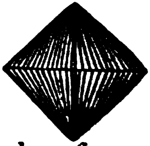
Now, by similar triangles ALB, BFE, and CLB, BFT;
L : CL :: TF : FE, and therefore AC : CL :: TE : FE. In
e manner, by similar triangles MLB, BFD, and CLB, BFT;
L : CL :: TF : DF, and therefore CM : CL :: TD : DF.
t, by the property of the circle, TE : FE :: TD : DF.

Therefore, CA=CM. Hence C is the centre of the hyperbola, and CO, CQ, are the asymptotes. (Conic Sections, Hyperbola, Proposition XII.)



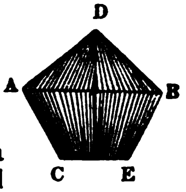
Scholium 1. Let EHF represent an hyperbola, and AC, AD the asymptotes ; let the two branches HE and HF be brought into the position H*e* and H*f*, so that the asymptotes become A′*e* and A′*f*, or till they become parallel to each other, and the curve becomes a parabola ; the parabola then, may be regarded as an hyperbola, whose asymptotes are parallel, and infinitely extended in each direction. If the extremities *e*,*f* of the curve are brought into the positions *n*, *i*, so as to incline toward the axis AB, so that the curve may again return into itself as its axis is extended, it then becomes an ellipse or portion of an ellipse. These different figures are the result of the position of the plane forming the section through the cone.

Scholium 2. A section of a polyedroid by a plane oblique to its axis may assume a combination of one, two, or three, of the varieties of surfaces. Thus, a section through a regular vertical quadredroid by a plane, parallel to a plane touching one of its sides, making an angle of 45° with the axis, toward the vertex, will consist of two parabolas on opposite sides of the same base, these will evidently be parabolas of equal type or similar parabolas, when the section passes through the centre of the solid, and as it approaches toward one of its sides, the parabolas become dissimilar.



But if the plane cuts the solid so as to make angles of more than 45° with the axis, the section will consist of the segments of two ellipses on opposite sides of the same ordinate, which ordinate is the line formed by the intersection of the cutting plane, with the plane through the centre of the solid perpendicular to its axis ; and if the cutting plane passes through the centre of the solid the two segments will be of similar type, but they will vary as the section recedes from the centre toward either side.

If the plane should be passed through so as to make an angle with the axis less than 45°, then we should have hyperbolas on the same base, these would be similar hyperbolas when the plane should pass through the centre, but would become dissimilar as it recedes from the centre.

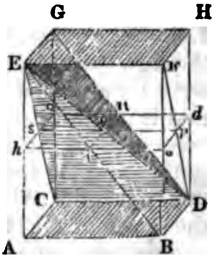


In like manner, if a plane should be passed rough the pentadroid AB parallel to one of s sides AD, the section would consist of two pecies, viz., that part which passes through e portion ABD would be a parabola, that assing through the part ABEC would be a egment of an ellipse. But if the section should e passed so near to BE as to cut the base of the solid, the ection would consist of a parabola, and a middle segment of n ellipse, and would have a rectilinear base. If the plane hould be parallel to the side EB, then the section through the wer part of the solid would be a parabola, or a segment of a arabola, and the section through the upper part an hyperola, and in fine, if the plane should make an angle with the xis less than that of the side EB, then the section through oth parts of the solid would consist of the dissimilar hyperbo- s or segments of hyperbolas.

Let a plane be passed through a polyedroid of a greater mber of sides oblique to the axis, and the section may be so ade as to consist of segments of all the varieties of the conic ctions, and may also, under certain conditions, have one, o, three, and at most four rectilinear sides, but it can have more than two rectilinear sides, except where the section parallel to the axis.

PROPOSITION IV. LEMMA.

rom a rectangular prism there may be taken two pyramids of equal base and altitude with the prism; when there will remain two other pyramids, each equal to half one of the lateral sides as a base, multiplied by one third of the distance of such base to the opposite side.



For let the rectangular prism AH be vided into two parts, by passing the anes EGDB through the opposite edges, d the portion ABDCGE will consist the pyramid, whose base is ABDC, and rtex E, plus the triangular pyramid, hose base may be taken as GEC and ver- x D, or DCG may be regarded as the se, and E the vertex. And since the her portion of the prism is similar to this, it can be divided in similar manner. Hence the whole prism consists of two ual pyramids erected on the upper and lower bases of the ism + two other pyramids erected on the lateral sides as bases, and whose vertices will be in an angle formed by the intersection of its opposite side with one of the bases. Hence as in the proposition.

Cor. 1. If a prism have a square base and a section be made through the prism parallel to the base, the sections through the pyramids erected on the upper and lower bases will be squares, and the sections through the pyramids, whose bases are the lateral faces of the prism, will be rectangles whose factors are the sides of the squares composing the sections through the former pyramids.

That the sections through the pyramids, erected on the square bases will be squares, is sufficiently manifest; and since each of the other pyramids coincides with one of these along one of its slant sides, and with the other along another of its sides, it follows that the measure of its section through any parallel portion of the solid will be the rectangle of the edges of the section formed by the same plane through the two former pyramids. And since, if the section is taken in the middle, equidistant between the two bases, the sides of the sections through the pyramids with square bases are = half the sides of their bases, the sections at such place will each be $=\frac{1}{4}$ the section of the whole prism; the sections of the two quadrangular pyramids will be 2 squares = 2 quarters of the whole section; hence the sections through the two triangular pyramids are squares = to the former, and equal to each other, since the four pyramids fill the space, and constitute the whole prism.

Cor. 2. Hence, also any section of a prism with a square base, made by a plane parallel to such base may be expressed by the square of a binomial, whose terms are composed of the sides of the sections through the quadrangular pyramids. Let *hi* or *hs*, the side of the square forming a section through the pyramid erected on the lower base, be represented by a; and let *pd* or *dn*, the side of the square forming a section through the pyramid erected on the upper base, be represented by b; then will $a+b=$ the line *ho*, the side of the whole section *hodc*, and a^2 will represent the section through the pyramid ABDCE, b^2 will represent the section through the pyramid EFHGD, and ab will represent a section through each of the other pyramids, hence $2ab$ will represent the sections through both, and $a^2+2ab+b^2$ will represent the whole section in whatever parallel the section is taken, always observing that the values of a and b vary according to the sides of the respective sections, which they represent.

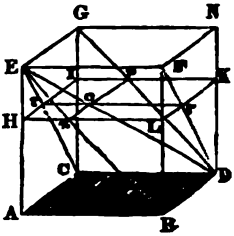
Cor. 3. If there be a series of numbers in arithmetical progression, whose first term is 0 and last term z, and if the number of terms is infinite between these extremes, then the sum of the squares of the series of numbers will be equal to one third of the square of the last term drawn into the series. For if an infinite number of planes be passed through the pyramid, parallel to its base, and equidistant from each other through the sides of the sections made by those planes, they will be a series of numbers in arithmetical progression, whose first term is 0, and the last term may be called z. Now the sum of the sections drawn into their distance will represent the solidity of the pyramid, but the solidity of the pyramid is equal to one third of an equal series of z, the last term or base of the pyramid drawn into the distance or ratio ; or= to one third z drawn into the series.

Cor. 4. If there be a series of numbers in arithmetical progression increasing from 0 up to z, drawn into a similar series, decreasing from z down to 0, then will the sum of their products be equal to half the sum of the squares of one of the series. For the sections through the pyramid HDGE, as we have seen, represent the rectangles of the sides of the corresponding sections through the two pyramids, formed on the two bases, and the sides of these sections are evidently, in each pyramid, a series in arithmetical progression ; one increasing from 0 to z, while the other decreases from z to 0. Moreover, the sections through the pyramid CDGE represent the solidity of that body, as the corresponding sections through the pyramid ABDCE, represents the solidity of that body. But the solidity of the pyramid CDGE, is equal to half the pyramid ABDCE, which as we have shown, may be represented by the sum of the squares of a series of arithmeticals, &c.

Cor. 5. If in the expression $a^2+2ab+b^2$, a be made to pass successively through all the values from 0 up to z ; and b at the same time pass through all the changes from z to 0; then the sum of all the a^2 will be equal to the sum of all the $2ab$, = the sum of all the b^2.

Cor. 6. If a in the expression above, be made to increase from h successively to z, and at the same time b be made to pass through all the values from z to h, then the series represented by $2ab$, will be a mean proportional between those represented by a^2 and b^2, since in this condition the two pyramids represented by the series of a^2 and b^2, would be such as pertain to pyramids inscribed in a frustum of a pyramid, and erected on the two bases, and the portions represented by the series of $2ab$, would be such as pertain to the pyramids erected on the lateral sides as bases, and since the sum of these pyramids is equal to either of the others, when the two bases are equal, and because ab is a mean proportional between a^2 and b^2, it follows that the series represented by $2ab$ is a mean proportional between those represented by a^2 and b^2, which agrees with the property of the frustum of a pyramid found in the Elements of Geometry.

Cor. 7. Let any plane KIHL be passed through a prism parallel to its base, cutting the pyramids ABDCE, EFNGD, DCGE, and EFBD in the sections H*not* K*rop*, I*pot*, and L*ron*. Then since I*t* is equal to L*n* or *ro*, and H*n*=*on*, it follows that as the section H*not* : I*pot* : : I*pot* : K*rop* : : H*not* : L*nor*.

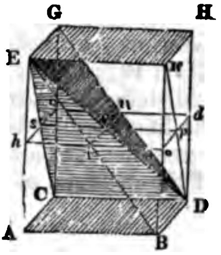


And by addition : : (K*not*+I*pot*)=HI*pn* : KL*np*.
Hence H*not* : HI*pn* : : HI*pn* : KIHL.

Cor. 8. Hence, of two quantities, the square of the first is to the rectangle of the first and second, as the rectangle of the first and second to the square of the second, and as the sum of the first and second × by the first, is to the sum of the first and second × by the second. Also as the square of the first is to to the sum of the first and second, × by the first, so is the sum of the first and second × by the first, to the first and second × by the first and second. Thus let a and b be two given quantities, then will $a^2 : ab :: ab : b^2, :: a^2+ab : ab+b^2$,
and $a^2 : a^2+ab :: a^2+ab : a^2+2ab+b^2$.

Scholium. It has been shown (Prop. XXXIV. B II. *El. Sol. Geom.*) that the solidity of a prismoid is equal to the product of the sum of the areas of the two ends+4 times AD and EH a middle section, equidistant between them × $\frac{1}{6}$ of the altitude; we may easily infer that this is also true of all prisms and pyramids and pyramidal frusta. Then since the prism AH is equal to the sum of the areas of the two bases+4 times the middle section *hodc* × $\frac{1}{6}$ of the altitude AE, it follows that whatever parts make up this product are equal to the whole prism.

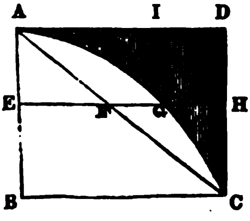
First, then let us take the pyramid ABDCE = (the base ABDC+4 *higs*, *a* middle section) × $\frac{1}{6}$ AE, and also the pyramid



IGD=(the base EFHG+4*gpdn*) $\times \frac{1}{6}$; if from the expression for the whole n we take the expressions for the two mids erected on the bases we have, $c \times \frac{1}{6}$AE+4*iopg*$\times \frac{1}{6}$AE, that is the two aining portions, are equal to four times middle sections, multiplied by $\frac{1}{6}$ of altitude. Hence we may infer that if portions of either of the pyramids into ch the prism has been conceived to be led, be cut off by a plane, or plancs parallel to the base, portions so cut off will be equal to the product of the sum heir two bases, + four times a middle section between , $\times$ by $\frac{1}{6}$ of the altitude of such portions. It may be obed that since a regular pyramid has but one base, its solis hence equal to the sum of this base + four times a secmidway between the base and vertice$\times \frac{1}{6}$ the altitude; and in the two pyramids, whose bases are on the lateral sides e prism, since they have no bases parallel to the middle on, their solidities for that reason are equal to four times middle sections $\times$ $\frac{1}{6}$ of their height. It may be further rved that each portion *sgnc*, GE into which these pyraare divided by the plane *hodc* is a wedge, whose base is ection forming the division.

PROPOSITION V. THEOREM.

m the extremity of an ordinate to the axis of a parabola d perpendicular thereto, a line be drawn meeting another e, drawn from the vertex perpendicular to the axis, form- with the ordinate and abscissa a rectangle ABCD, *and a diagonal be drawn from the vertex* A *to the extremity of ordinate, forming a right angled triangle* ABC *of the e base* BC *and altitude* AB, *then any line or ordinate wn from the axis across the triangle, the parabolic area, d the rectangle, parallel to the ordinate, will be cut in conued proportion by the sides of those figures.*



at is, EF : EG : ; EG : EH.

EF, EG, EH are in continued ortion.

r by (Prop. VII. Cor. of Parabola)

3 : AE : : $BC^2 : EG^2$.

d since AB : AE : : BC : EF

e EF : BC : : $EG^2 : BC^2$.

F : EH : : $EG^2 : EH^2$,

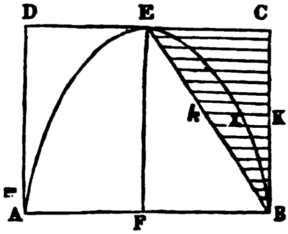
fore (Prop. XXIV. B. I. *El. Geom.*) EF, EG, EH, roportionals, or EF : EG : : EG : EH.

Cor. Let any number of ordinates GI be drawn across the exterior parabolic space parallel to the axis, and since $EG^2 : BC^2 :: AE : AB$, by equality we have $AI^2 : AD^2 :: IG : DC$, and since this is true from whatever position on the line AD, the line IG may be drawn, it follows that any line IG drawn across the exterior parabolic space is proportional to the square of the distance of such line from the vertice A.

PROPOSITION VI. THEOREM.

If a parallelogram be circumscribed about a parabola, the area of the space exterior to the parabola will be equal to one third of the parallelogram, and the interior space will be two thirds of the parallelogram.

Let ABCD be a parallelogram circumscribed about the parabola AEB, and the space ECB, EDA exterior to the curve will be $= \frac{1}{3}$ the parallelogram ABCD, and the space AEBA within the curve will be $= \frac{2}{3}$ ABCD.



From the vertice E draw EF the axis of the parabola, which will divide the parabola and rectangle into two equal parts; it is to be proved that the exterior space ECB is $= \frac{1}{3}$ EFBC.

Draw the diagonal EB, and let an indefinite number of equidistant ordinates K*k* be drawn across the triangle EBC, and exterior parabolic space EIBC, and those ordinates will represent their respective surfaces in the relation of their magnitudes respectively. Then since the distances of those ordinates on the line BC, estimated from the vertice B, are a series of numbers in arithmetical progression, the ordinates K*k* drawn across the triangle, are also a series in arithmetical progression, for K*k* is proportional to BK in whatever position the ordinate K*k* is drawn; and since it has been shown (Prop. V. Cor.,) that any ordinate IK drawn across the exterior parabolic space is proportional to the square of its distance BK from the vertex B, it follows that the ordinate is proportional also to the square of K*k*, the corresponding ordinate drawn across the triangle. But the ordinate EC or base of the triangle is equal and identical with the base of the parabolic exterior space; hence each of the ordinates IK terminated by the parabolic curve is equal to the square of K*k*, the corresponding ordinate drawn across the triangle.

Hence all the ordinates Kk may be represented by a series of numbers in arithmetical progression, whose first term beginning at B, is infinitely small or 0, and last term EC, and all the IK will be a similar series of squares of those arithmeticals; but (Prop. IV. Cor. 3,) the sum of an infinite series of the squares of a series of numbers in arithmetical progression, whose first term is 0, and last term z, or EC, is equal to $\frac{1}{3}$ EC×CB, which is $\frac{1}{3}$ of the rectangle EFBC.

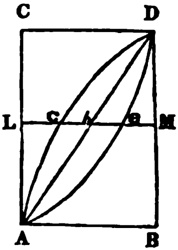
Cor. 1. Hence the parabolic segment EIBE, cut off by the diagonal, is equal to $\frac{1}{6}$ of the rectangle, = $\frac{1}{4}$ of the interior parabolic space EIBF, = $\frac{1}{2}$ the exterior parabolic space EIBC.

Scholium. It has been shown in the argument above that the ordinates drawn across the exterior parabolic space are severally = the squares of the same ordinates drawn across the triangle EBC, this is true where EC=1, and for all other values of EC, they are in the same proportion.

PROPOSITION VII. THEOREM.

The area of the exterior parabolic space included in its circumscribing rectangle is equal to the sum of its base + four times the line or ordinate drawn parallel to the base, and equidistant from the base to the vertex, multiplied by $\frac{1}{6}$ of the altitude.

And the area of the interior space of a parabola is equal to the sum of its base, + four times the ordinate equidistant from this base to the vertex, multiplied by one sixth of the altitude.



Let ABDC be a rectangle circumscribing the semi-parabola AcDB, and let Lc be an ordinate drawn cross the exterior space ACDcA equidistant from A to C, and let M be an ordinate drawn across the interior space AcDB in a similar manner, then will the area ACDcA = (CD+4Lc) $\frac{1}{6}$ AC. And the area AcDB = (AB+4cM) $\frac{1}{6}$ BD. For draw the diagonal AD, then, since we have shown in the argument to Prop. VI, that Lc=Lh^2, when CD=CD^2, and since Lh=$\frac{1}{2}$ CD: then if we call Lh, a, and CD, $2a$, we shall have Lc = a^2, and CD = $4a^2$ that is Lc = $\frac{1}{4}$ CD or CD=4Lc.

Hence CD+4Lc=2CD, then by the proposition 2CD×$\frac{1}{6}$AC = CD× $\frac{1}{3}$ AC = the surface AcCDA, as was also found in Proposition VI.

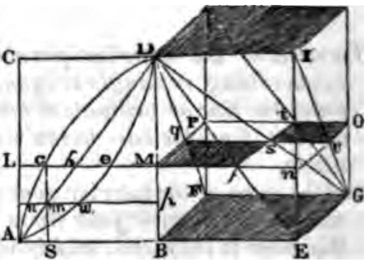
Again since LM = CD or AB, and L*c* = ¼ CD, *c*M = ¾ AB. Then by the proposition (4*c*M+AB) × ⅙ BD=4AB×⅙BD= ⅔ (AB×BD)=the area, as found in the preceding proposition.

Cor. 1. Hence the parabolic segment AD*c*A is equal to four times the middle ordinate *ch* × ⅙ the altitude AB. For since L*h* = ½ LM, *ch*=L*c* = ¼ AB, and ¼ AB×4=AB, and AB×⅙ BD = the surface which agrees with Cor. 1 Prop. VI.

Cor. 2. Also if another similar parabolic curve D*e*A be described on the opposite side of the diagonal, the space between the two curves would in like manner be found, = 4 times the ordinate *ce* × ⅙ of the altitude BD.

Cor. 3. The same may be shown in reference to the area of the triangles ACD, and also of the rectangle ABDC, viz., their areas are equal to the sum of their upper and lower bases, + 4 times a middle ordinate, × ⅙ of their respective altitudes.

Scholium. Let there be a rectangle CB, and a prism BK of the same altitude, and let the base of the prism be a square, one of whose sides is = AB the base of the rectangle.



Let the prism be divided into the pyramids BEGFD, DIKHG, DHFG, EGID, as in Prop. IV. Also let the parabolic curves A*c*D A*e*D and the diagonal AD be drawn as in the proposition above. Then if a plane M*n*OP be passed through the prism parallel to its base, and if an ordinate LM be made to pass through the rectangle in the same plane, the plane will cut the pyramidal portions of the prism in the same relation, as the ordinate cuts the parabolic portions of the rectangle; viz., the section M*fsq* through the pyramid BEGCD will be to the section M*n*OP through the prism, as the ordinate *e*M through the exterior parabolic space ABD*e*A, to the ordinate LM through the rectangle ABDC; and the section *sv*O*t* through the pyramid DIKHG is to the whole section through the prism, as the ordinate L*c* through the exterior space ACD*c*A, to the whole ordinate LM through the rectangle, and so for the sections through the other pyramids, and ordinates through the interior parabolic segments, and this is true in whatever parallel position the plane forming the section and ordinate is drawn.

For it has been shown (Prop. VI. Sch.) that the ordinate *e*M is equal to $h\mathrm{M}^2$; the section M*fsq* is evidently=the square of M*f*; but M*f* is = *h*M, hence M*fsq* =$h\mathrm{M}^2$; therefore M*fsq* rill be expressed by the same numbers as eM, and this is true whatever parallel position the plane is passed.

The same may also be shown in reference to the sections Ot through the pyramid DIKHG, and the ordinates Lc and h, viz., the section $svOt=Ot^2=fn^2=Lk^2$ and $Lc=Lh^2$. And nce every ordinate LM through the rectangle, has the same lation to the rectangle, as every section through the prism, as to the prism, and each section $Mfsq$, $svOt$, has the same elation to the solid as each ordinate Lc, cM, has to the surface; follows that the remaining sections, $fnvs$, $stPq$, have the ame relations to their respective figures, and since the secions $Mfsq$ $svOt$ are expressed by the same powers of the ame factors as the ordinates Lc and eM are expressed, ch and e must also be expressed by the same terms as the sections ws, $stPq$, viz., $he=Lh\times hM$, ch also$=Lh\times hM$, and $ce=$ $(Lh\times hM.)$ Hence if $Lh = a$ and $hM=b$ then will $Lc=a^2$, $e=2ab$ and $cM=b^2$, and the whole ordinate may also be exressed by $a^2+2ab+b^2$, which is the square of a binomial, the ame as has been shown, (Prop. IV. Cor. 2) in reference to he pyramids composing a prism, and the expression is true n whatever parallel position the ordinate is drawn. It folows therefore that the ordinates drawn across any of the paabolic portions of the rectangle, have the same power of deermining the area of the exterior or interior parabolic spaces, s the corresponding sections through the pyramidal portions f the prism have, in determining the solidities of those pyraids. But we have shown (Prop. IV. Sch.) that each ortion of the prism divided as above, however selected or ompounded, is equal to the sum of its two bases+4 times a iddle section $\times \frac{1}{6}$ of the altitude; hence also the area of any arabolic portion or portions of the rectangle is equal to the m of its bases + 4 times a middle ordinate $\times \frac{1}{6}$ the altitude.

Cor. 4. The parabolic area included between two parallel rdinates is=the product of the sum of the two ordinates + 4 mes an ordinate equidistant between them $\times \frac{1}{6}$ of the altitude the parabolic segment. For let $ABMc$ be a segment of e parabola included between the ordinates cM, AB, the axis B and the curve Ac; draw cS perpendicular to AB, and the rabolic segment ScA will be equal to $(AS + 4\,um) \times \frac{1}{6}\,Sc$, d the rectangle $SBMc$ is evidently$=(Sb+cM+4mp)\ \frac{1}{6}\,cS$; nce the whole space $ABMc=(AS+SB+cM+4um+4mp,)$ $\frac{1}{6}Sc=(AB+cM+4up)\ \frac{1}{6}\,Sc$.

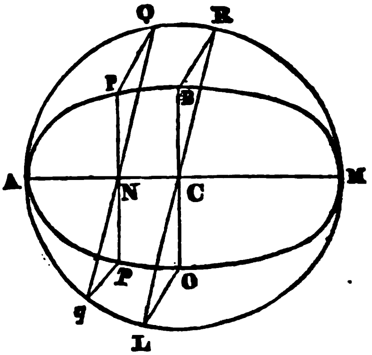
Cor. 5. In like manner it may be shown that the space BA is$=(AB+eM+4wp)\ \frac{1}{6}MB$, and hence also that any paralic portion $AuchmA$ or $uchm$ may be determined in the same nner.

Cor. 6. Hence generally, if a space be terminated by a parabolic curve on one side, and either a parabolic curve or a right line on the other, the space included between two parallel lines, drawn across the figure cutting those sides, will be = to the product of the sum of those lines + 4 times another line drawn across equidistant between the two multiplied by $\frac{1}{6}$ the perpendicular distance of the two parallel lines.

PROPOSITION VIII. THEOREM.

If a circle be cut by a plane through its axis, and perpendiculars be drawn from every point in the circumference to the plane, the orthographic projection of the circle so drawn will be an ellipse.

Let the circle ARML be inclined to the plane of this paper in such a manner, that the semicircle ARM may be above the paper, and the semicircle ALM below it, and let AM be the common intersection of the two planes. Let the semicircle ARM be projected downwards upon the plane of the paper, by drawing perpendiculars QP, RB, from each point of the circle, and let the semicircle A L M be projected upwards, by drawing the perpendiculars *qp*, LO, &c.; then the curve ABMO, marked out by this projection, will be an ellipse. For draw QN, RC, at right angles to AM, and join PN, BC; then the angles QNP, RCB, will measure the inclination of the planes, and PN, BC will be perpendicular to their common intersection AM. Now QN : : PN : *rad.* : cos. QNP, and RC : BC : : *rad.* : cos. (RCB = QNP); ∴ QN : PN : : RC or AC : BC. which agree with the properties of the ellipse (Prop. XII. Cor. *Conic Sec.*) In a similar manner it may be shown that the semicircle ALM is projected into a semi-ellipse AOM; and thus the whole circle ARML is projected into an ellipse ABMO, whose major axis is AM.



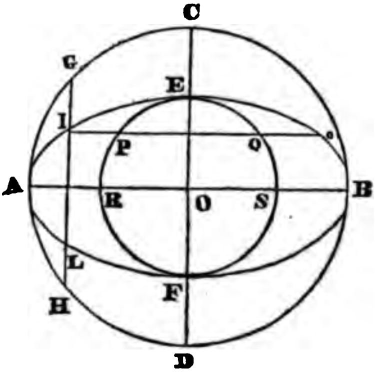
Scholium. This proposition is manifestly true, when the plane of the projection does not cut the circle, or cuts it unequally.

Cor. Hence any section of a cylinder by a plane not perendicular or parallel to its axis, is an ellipse, or portion of an llipse.

PROPOSITION IX. THEOREM.

f on the major axis of an ellipse a circle be described, the area of the ellipse will be to that of the circle, as the minor axis of the ellipse to the major axis.

Let ACBD be a circle described on the major axis AB of be ellipse AEBF, then will the area of the ellipse be to that f the circle as EF to EB.



For it has been shown (Prop. XII. Cor. *Conic Sec.*) that any ordinate IL of the axis AB of the ellipse is to the corresponding ordinate GH of the circle, as EF to CD or AB, as the minor axis to the major axis. And since this is true in whatever position the ordinates to this axis are drawn; and because if we suppose an indefinite number of equidistant ordinates to be drawn across the ellipse and circle, the sum of the ordinates in each will represent those figures in the relation of their areas, the sum of the ordinates in the ellipse will be to the sum of those in the circle, as EF to CD, and hence the area of the ellipse will be to that of the circle in the same ratio.

Scholium. As the area of the ellipse bears this given ratio to that of its circumscribing circle, the quadrature of the ellipse must therefore depend on the quadrature of the circle. If $\pi\times$(AO . AO, or CO,) or πAO^2=the area of the circle, then π . AO . EO=the area of the ellipse. Hence the area of the ellipse is found by multiplying the rectangle under its semiaxis by the constant number π, the ratio of the diameter to the circumference of a circle, which in the Elements of Geometry we have found, developed to a certain order of decimals to be 3,1415926. But 3,1416 may be regarded as the value of π which is a convenient number to use, and is sufficient where extreme accuracy is not required.

From this it also appears, that the area of an ellipse is equal to the area of a circle whose radius is a mean proportional between its semi-axis; for the area of that circle is $= \pi R^2 = \pi \times$ the square of $\sqrt{AO \times EO} = \pi . AO . EO$.

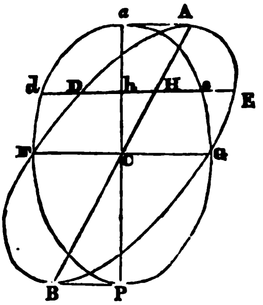
Cor. 1. The area of an ellipse has the same ratio to the area of its circumscribed parallogram as the area of a circle has to its circumscribed square. For the area of the parallelogram circumscribing the ellipse is $= 4\,AO \times EO$; hence the area of the ellipse : area of the parallelogram : : $\pi . AO . EO$: $4AO . EO$: : 3,1416 : 4 : : 7854 : 1.

Cor. 2. If a circle be described on the minor axis EF of the ellipse, then any ordinate PQ of the circle will be to the corresponding ordinate I*e* of the ellipse as EF to AB. Hence also the area of the ellipse is to that of the circle described on its minor axis, as the major axis of the ellipse to its minor axis.

Cor. 3. Hence any segment of the ellipse cut off by ordinates, either of the major or minor axis is to a similar segment of the circle, described on such axis, as its conjugate is to such axis. Thus the segment IAL of the ellipse AEBF is to the segment GAH of the circle described on the axis AB, as the ordinate IL is to the ordinate GH or as the minor axis EF of the ellipse is to the axis CD of the circle, or to AB the transverse axis of the ellipse. And the segment I*e*E is to the segment PQE as the axis AB as conjugate to EF of the ellipse, to the axis RS of the inscribed circle, or EF of the ellipse.

Cor. 4. Since the area of a circle BESF is equal to its circumference multiplied by half the radius EO, and since the area of the ellipse AEBF is = to that product increased in the ratio of EO to AO or CO, it follows that the area of the ellipse is equal to the circumference of its inscribed circle multiplied by ½ the radius of its circumscribed circle, and that it is also equal to the circumference of the circumscribed circle multiplied by ½ the radius of the inscribed circle.

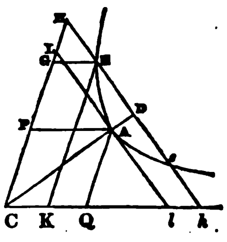
Cor. 5. As a circle is to the square of its diameter so is any ellipse to the rectangle of its two axes, or the rectangle of any two conjugate diameters drawn into the sine of their included angle. Any two like segments or zones of the circle and ellipse, are also in the same proportion.



Cor. 6. Hence we have the following construction; let ADE be an oblique segment of the ellipse AFBGA, cut off by an ordinate to the diameter AB, FG being the conjugate. Through the centre C draw aP perpendicular to FG meeting Aa and BP, in a and P, both parallel to FG; then about the axes a P, FG, describe the ellipse $a$FPG$a$, meeting the ordinate produced in e and d. Then will the right elliptical segment dae, be equal to the oblique segment DAE, and so will the whole ellipse $a$FPG$a$=AFBGA; moreover their corresponding ordinates de DE parallel to the common diameter FG, are everywhere equal, as are the like parts or zones contained between any two of such ordinates. And the same may be said of all ellipses of the same diameter FG, contained between the parallels aA, BP infinitely produced.

PROPOSITION X. THEOREM.

All the parallelograms are equal, which are formed between the asymptotes and curve, by lines drawn parallel to the a symptotes of an hyperbola.



That is, the lines GE, EK, AP, AQ, being parallel to the asymptotes CH, Ch, then the parallelogram CGEK = parallelogram CPAQ.

For, let A be the vertex of the curve, or extremity of the semi-transverse axis AC, perpendicular to which draw AL or Al, which will be equal to the semi-conjugate, by definition XIX. *Hyp*. Also, draw HEDeh parallel to Ll.

Then, $CA^2 : AL^2 :: CD^2 - CA^2 : DE^2$,

and by parallels, $CA^2 : AL^2 :: CD^2 : DH^2$;

therefore, by substract. $CA^2 : AL^2 :: CA^2 : DH^2 - DE^2$

or rectangle $HE \,.\, Eh$;

consequently, the square AL^2=the rectangle $HE \,.\, Eh$.

But, by similar trian. $PA : AL :: GE : EH$,

and, by the same, $QA : Al :: EK : Eh$;

therefore, by comp. $PA \,.\, AQ : AL^2 :: GE \,.\, EK : HE \,.\, Eh$;

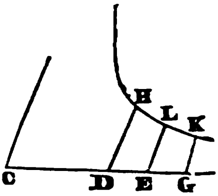
and, because $AL^2 = HE \,.\, Eh$, therefore $PA \,.\, AQ = GE \,.\, EK$.

But the parallelograms CGEK, CPAQ, being equiangular, are as the rectangles GE.EK and PA . AQ.

And therefore the parallelogram GK = the parallelogram PQ.

That is, all the inscribed parallelograms are equal to one another.

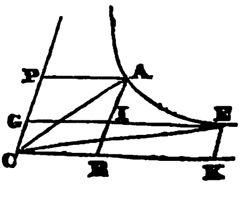
Cor. 1. Because the rectangle GEK or CGE is constant, therefore GE is reciprocally as CG, or CG : CP :: PA : GE. And hence the asymptote continually approaches towards the curve, but never meets it; for GE decreases continually as CG increases; and it is always of *some* magnitude, except when CG is supposed to be infinitely great, for then GE is infinitely small or nothing. So that the asymptote CG may be considered as a tangent to the curve at a point infinitely distant from C.



Cor. 2. If the abscissas CD, CE, CG, &c., taken on the one asymptote, be in geometrical progression increasing; then shall the ordinates DH, EI, GK, &c., parallel to the other asymptote, be a decreasing geometrical progression, having the same ratio. For, all the rectangles CDH, CEI, CGK, &c., being equal, the ordinates DH, EI, GK, &c., are reciprocally as the abscissas CD, CE, CG, &c., which are geometricals. And the reciprocals of geometricals are also geometricals, and in the same ratio, but decreasing, or in converse order.

PROPOSITION XI. THEOREM.

The three following spaces, between the asymptotes and the curve, are equal; namely, the sector or trilinear space contained by an arc of the curve and two radii, or lines drawn from its extremities to the centre; and each of the two quadrilaterals, contained by the said arc, and two lines drawn from its extremities parallel to one asymptote, and the intercepted part of the other asymptote.



That is,

The sector CAE = PAEG = BAEK, all standing on the same arc AE

For, as has been already shown, CPAB = CGEK;

Subtract the common space CGIB,

So shall the parallel PI = the parallel IK;

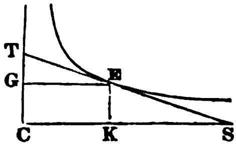
To each add the trilineal IAE,

Then is the quadril. PAEG=BAEK.
Again, from the quadrilateral CAEK, take the equal triangle AB, CEK, and there remains the sector CAE=BAEK.
Therefore, CAE=BAEK=PAEG.

PROPOSITION XII. THEOREM.

Every inscribed triangle, formed by any tangent and the two intercepted parts of the asymptotes, is equal to a constant quantity ; namely double the inscribed parallelogram.

hat is, the triangle CTS=2 parallelogram GK.

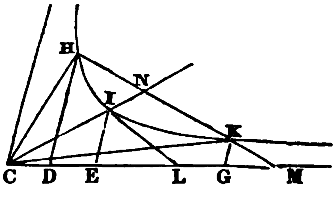


For, since the tangent TS is bisected y the point of contact E, and EK is parallel to TC, and GE to CK ; therefore K, KS, GE are all equal, as are also G, GT, KE. Consequently the triangle TE = the triangle KES, and each qual to half the constant inscribed parallelogram CK. And herefore the whole triangle CTS, which is composed of the wo smaller triangles and the parallelogram, is equal to ouble the constant inscribed parallogram GK.

PROPOSITION XIII. THEOREM.

f from the point of contact of any tangent, and the two intersections of the curve with a line parallel to the tangent, three parallel lines be drawn in any direction, and terminated by either asymptote ; those three lines shall be in continued proportion.

That is, if HKM and the ngent IL be parallel, then are he parallels DH, EI, GK in ontinued proportion.



For, by the parallels, EI : IL : : DH : HM ;
and, the same EI : IL : : GK : KM ;
therefore by compos. $EI^2 : IL^2$: : DH . GK : HMK ;
t, the rect. $HMK=IL^2$;
d therefore the rectangle DH . $GK=EI^2$,
r DH : EI : : EI : GK.

PROPOSITION XIV. THEOREM.

Draw the semi-diameters CH, CIN, CK ;
(see last diagram.)
Then shall the sector CHI = *the sector* CIK,

For, because HK and all its parallels are bisected by CIN,
therefore the triangle CNH=trian. CNK,
and the segment INH=seg. INK ;
consequently the sector, CIH=sec. CIK.

Cor. If the geometricals DH, EI, GK be parallel to the other asymptote, the spaces DHIE, EIKG will be equal ; for they are equal to the equal sectors CHI, CIK.

So that by taking any geometricals CD, CE, CG, &c., and drawing DH, EI, GK, &c., parallel to the other asymptote, as also the radii CH, CI, CK ;
then the sectors CHI, CIK, &c.
or the spaces DHIE, EIKG, &c.
will be all equal among themselves.
Or the sectors CHI, CHK, &c.
or the spaces DHIE, DHK, &c.
will be in arithmetical progression.

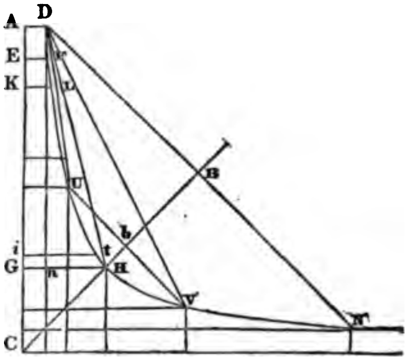
And therefore these sectors, or spaces, will be analogous to the logarithms of the lines or bases CD, CE, CG, &c. ; namely CHI or DHIE the log. of the ratio of CD to CE, or of CE to CG, &c. ; or of EI to DH, or of CK to EI, &c. ; and CHK or DHKG the log. of the ratio of CD to CG, &c. or of CK to DH, &c.

Scholium. If the common logarithms are multiplied by 2,302585093 their products will be the hyperbolic logarithms.

PROPOSITION XV. THEOREM.

If any number of ordinates AD, EF, *&c. to* GH, *between the hyperbola and the asymptote, are taken in geometrical progression increasing, then will their distances* AE, *&c., on the asymptote be a series of geometricals decreasing, and* AG *will represent the sum of the decreasing series.*

hat the geometrical nates and their dis- es on the asymptote reciprocal geometri- is manifest from (Prop. Cor.,) and since the nces AE, EK &c., the asymptote, inter- ted by those ordinates pose the whole series lecreasing geometri- , the sum of that series be represented by , the part of the asymptote taken up by the series.



or. 1. The last term G*i*, is=AE×EF÷GH. For the area FE = the area HG*it*, and if the distances on the asymp- are taken indefinitely small, *ti* may be regarded as =GH, AD may be regarded as = EF, and if the hyperbola is lateral, then the space ADFE will be=AE×AD=G*i*×GH erting this equation into a proportion,

we have AE : G*i* : : GH : AD,
or GH : AD : : AE : G*i*.
Hence, G*i*=AE×(AD or EF)÷GH

cholium. It is evident that this proposition is true, however the series may extend on the asymptote CA.

or. 2. If the series commence at any point A on the asymp- s, and decrease to infinity, CA will be the sum of the es; hence the sum of an infinite series of decreasing geo- ricals is a finite number.

PROPOSITION XVI. THEOREM.

ny number of portions AE, EK, *&c., to* G, (see diagram bove) *are the reciprocals of a series of ordinates* AD, EF, L, *&c., to* GH, *in geometrical progression; then will the rea or space intercepted by* AD *and* GH, *between the curve d asymptote be = the space* ADFE, *multiplied by the mber of terms of the series.*

or since the space intercepted by each two consecutive nates of the series is equal to ADFE, it follows that the le space intercepted by all the ordinates, or the space IG is = ADFE × by the number of terms in the series.

Cor. Hence the area of the interior hyperbolic space HDB may be found, if the exterior space is known. For the interior space is equal the triangle HBD + triangle HnD + the rectangle nDAG—the exterior hyperbolic space GHDA.

Scholium. Let AD$=b$, EF $= e$, GH $= g$, and AE the first term of a geometrical series$=a$; the last term $= z$; and AG the sum $= s$. Then we shall have a the first term, and $z = \frac{ae}{g}$ the last term, and s the sum of all the terms of a geometrical series to find the ratio and number of terms.
Then by geometrical progression will

$$s = a + ar + ar^2 + ar^3 \quad - \; - \; - \quad + ar^{n-1}$$

multiply by r

$$sr = ar + ar^2 + ar^3 \; - \; - \; - \; - \; - \; - \; - \; + ar^{n-1} + ar^n$$

Subtracting the first from the second,

$$s\,(r-1) = ar^n - a$$

$$\therefore\; s = \frac{a\,(r^n - 1)}{r-1}$$

And since,

$$z = ar^{n-1}$$

$$s = \frac{rz-a}{r-1}$$

multiplying by $r-1$, $sr - s = rz - a$

By transposing and dividing; $r = \frac{s-a}{s-z}$

and in the equation $z = ar^{n-1}$
multiplying by r and dividing by a we have

$$r^n = \frac{zr}{a} \quad \text{which equation}$$

is irreducible by the ordinary methods when n is the unknown quantity, it being an exponential equation, and can be solved only by logarithms, which see. (Chap. V· Trigonometry) ; but if we proceed to raise r to some power, whose value is $\frac{zr}{a}$ the index of that power will be the value of n ; this can always be done to approximate exactness.

Let the series of geometricals commence at the vertex H, making Gi the first term $= a$, and let n be a given number, then assuming any value to s, z, r, each of the others may be found by the formula above, and hence the area of the exterior and interior hyperbolic spaces may be found, as well as that of any section or segment.

holium 2. If the portion of the hyperbolic curve is small, rdinates AD, EF, &c., and consequently AE, EK, &c., be very nearly in arithmetical progression as may readily een by the rapid manner in which the ordinates DN, DV, DU, &c., approximate toward the curve as the hyperbo- rc is decreased, hence the hyperbolic area may be approx- tely obtained by first dividing it into portions having wn ratios to each other, and finding the value of one when whole will become known.

PROPOSITION XVII. PROBLEM.

it be required to find how far from G *on the asymptote* AC *ordinate must be drawn parallel to* GH *in order to inter- pt with* GH *an area* =A

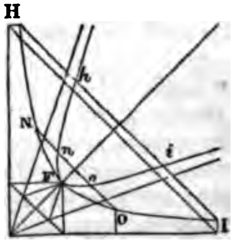
ake any distance G*i* (see fig. to Prop. XV,) and since AG : GH : G*i* we have the two parallel sides GH, *it*, and their ance G*i* of the quadrilateral GH*ti* to find its area, which *f*, then $\frac{A}{f} = n$, let *a* represent the distance, G*i* the first term e geometric series, and GH ÷ *ti* = *r*, the ratio.

Vhence we have $s = a\left(\frac{r^n - 1}{r - 1}\right)$ = the distance from G he asymptote, that the ordinate must be drawn to intercept GH, the area A.

cholium. Since the spaces intercepted by the ordinate GH D, with any base IN, are the hyperbolic logarithms of ratios of AD to GH (P. XIII. Cor.) hence the area may be d by taking the logarithms of the ratio, and multiplying it the base.

PROPOSITION XVIII. THEOREM.

ll the ordinates to the diameter of an equilateral hyperbola e reduced in any ratio r, then the area of the hyperbola ill be reduced in the same ratio.

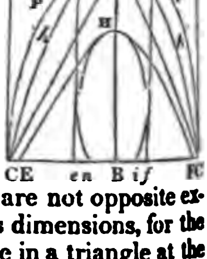


et all the ordinates HI, NO, &c., of equilateral hyperbola HFI be re- ed to *hi*, *no*, &c., forming a hyperbola , then will the area of the equilate- hyperbola HFI be to the area of the r hyperbola *h*F*i* as HI to *hi*, or as to *no*. For, if an indefinite num- of equidistant parallel ordinates be wn across the two hyperbolas, the areas of each will be as the sum of the lengths of all the ordinates in each, but by hypothesis, each of the ordinates HI, NO, are to their corresponding ordinates *ni*, *no* in the given ratio *r*; hence, their sums in each, must also be in the same ratio.

Cor. 1. Since the areas of similar figures, are as the squares of the lines similarly drawn in each; similar segments of similar hyperbolas, are as the squares of their axes, or of their like diameters.

Cor. 2. Hyperbolas of the same transverse axis and abscissa are to each other as their conjugate axes; but if their bases, or ordinates, and conjugate axes be the same, they will be as their transverse axes; and generally, hyperbolas having the same abscissa, are as the rectangle of their major and minor axes; and, consequently, having the quadrature of any one hyperbola, we may from it find that of any other. Thus knowing the area answering to any abscissa in any one hyperbola, we can find a similar abscissa in the other; then as the rectangle of the axes of the squared hyperbola, is to the rectangle of the proposed one, so is the area of the former, to that of the latter.

Scholium. 1. It may be shown that a parabola is always greater than a triangle of equal base and altitude, and that the hyperbola is always between the two; since from the nature of the section through the cone by which the hyperbola is produced, it may vary from a vertical section through the cone forming a triangle, to the section parallel to one of its sides forming a parabola, each of which extremes it can never reach, but may approach infinitely near.



When the section passes through the vertice and base of a cone, the section is evidently a triangle, whatever angle it may make with the axis. Let the section be removed from the vertice by any quantity however small, and it becomes one of the curves described above; it may be an hyperbola or a parabola, either of which may agree infinitely near with the triangle. Hence, the triangle and parabola are not opposite extremes of the hyperbola as the limits of its dimensions, for the hyperbola and parabola may both terminate in a triangle at the same moment, or the two curves may become assimilated when they are infinitely removed from a triangular form; nevertheless, when the hyperbola *hh*, parabola PP, and the triangle ACC are described on the same base C C, and of equal altitude B A, the hyperbola may always be described between the triangle and parabola; this arises from the nature of the sections from which they are formed. Let the axis be indefinitely extended as when the section becomes parallel to one of the sides of the cone, in which case the two asymptotes AC, AC of the hyperbola CHC, become parallel to each other, as A′ *e*, A′ *f*, in which case the asymptotes may be conceived to vanish, and the curve, at this point, assumes other properties, as we have seen in our investigations; the axis, after being increased to infinity in the direction, BA, vanishes or becomes changed, and is infinitely extended in the direction B; but the same cause that brings the asympotes to a parallel position, may, by continuing, cause them to assume a greater distance toward A′ A′ than in the opposite direction, and the axis in such case becomes a definite magnitude, in which case the curve will be inclined inward, as at *n i*, when, by extending the axis, it will return into itself and become an ellipse.

Scholium 2. As it has been shown (Prop. IX Cor.6) in reference to ellipses, it may be also shown that all hyperbolas having the same centre and equal bases, and described between the same parallels, although infinitely produced, are equal to each other, as are also their corresponding sections parallel to the bases, and likewise any frustum or segments intercepted by the parallels.

The same may also be shown of parabolas of the same base and altitude, or those with the same base, and between the same parallels, that they are equivalent in area, &c.

EQUATIONS TO THE CURVES FORMED BY SECTIONS OF THE CONE.

1. *For the Ellipse.*

Let d denote AC, the semi-axis major or semi-diameter;
$c =$ CM its conjugate;
$x =$ AK, any abscissa, from the extremity of the diam.
$y =$ DK the correspondent ordinate.

Then, (Prop. XII Ellipse,) $AC^2 : CM^2 :: AK \,.\, KB : DK^2$, that is, $d^2 : c^2 :: x\,(d-x) : y^2$, hence $d^2\,y^2 = c^2\,(d-x^2)$ or $dy = c\surd(dx-x^2)$, the equation of the curve.

And from these equations, any of the four letters or quantities, d, c, x, y, may easily be found, by the reduction of equations, when the other three are given.

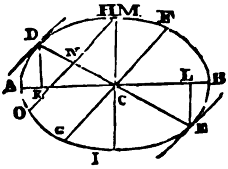
Or, if p denote the parameter. $= c^2 \div d$ by its definition; then, (Prop. XII. Cor. 1) $d : p :: x\,(d-x) : y^2$, or $dy^2 = p\,(dx-x^2)$ which is another form of the equation of the curve.

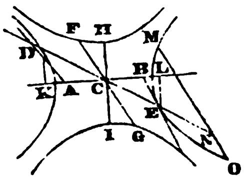
Otherwise. Or, if d = AC the semi-axis; c= CH the semi-conjugate; $p = c^2 \div d$ the semi-parameter; x = CK the abscissa counted from the centre; and y = DK the ordinate as before.

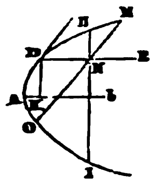
Then is AK = $d—x$, and KB = $d+x$, and AK . KB = $(d—x) \times (d+x) = d^2—x^2$.

Then, $d^2 : c^2 :: d^2 — x^2 : y^2$, and $d^2y^2 = c^2(d^2—x^2)$, or $dy = c\surd(d^2—x^2)$, the equation of the curve.

Or, $d : p :: d^2-x^2 : y^2$, and $dy^2=p\ (d^2-x^2)$ another form of the equation to the curve; from which any one of the quantities may be found, when the rest are given.







2. *For the Hyperbola.*

Because the general property of the opposite hyperbolas, with respect to their abscissæ and ordinates is the same as that of the ellipse, therefore the process here is the same as in the former case for the ellipse; and the equation to the curve must come out the same also, with the exception of the signs which are sometimes changed from + to —, or from — to +, because the abscissæ on the axis, lie beyond or without the curve, whereas they lie within it, in the ellipse. Thus making the same notation for the whole diameter, conjugate, abscissa, and ordinate, as at first in the ellipse; then, the one abscissa AK being x, the other BK will be $d + x$, which in the ellipse was $d — x$; so the sign of x must be changed in the general property and equation, by which it becomes $d^2 : c^2 :: x\ (d+x) : y^2$; hence $d^2\ y^2 = c^2\ (dx+x^2)$ and $dy = c\surd(dx+x^2)$, the equation of the curve.

Or, using p the parameter as before, it is, $d : p :: x\ (d+x) : y^2$, or $dy^2=p\ (dx+x^2)$, another form of the equation of the curve.

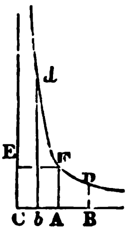
Otherwise, by using the same letters d, c, p, for the semi-axis, semi-parameter, and parameter, and x for the abscissa CK counted from the centre; then AK=$x-d$, and BK=$x+d$ and the property $d^2 : c^2 :: (x—d) \times (x+d) : y^2$, gives $d^2\ y^2 = c^2\ (x^2—d^2)$ or $dy = c\surd(x^2—d^2)$, where the signs of d^2 and x^2 are changed from what they were in the ellipse.

Or, again using the semi-parameter, $d : y : x^2-d^2 : y^2$, and dy $p(x^2-d^2)$ the equation of the curve.

But for the conjugate hyperbola, (Prop. XII, Cor. 3, Hyp.) s the signs of both x^2 and d^2 will be positive ; for the proerty in that case being $\mathrm{CA}^2 : \mathrm{CB}^2 :: \mathrm{CD}^2+\mathrm{CA}^2 : \mathrm{D}p^2$, it is d^2 : $:: x^2+d^2 : y^2=\mathrm{D}p^2$, or $d^2y^2=c^2(x^2+d^2)$ and $dy=c\sqrt{(x^2+d^2)}$, he equation to the conjugate hyperbola.

Or, as $d : p :: x^2+d^2 : y^2$, and $dy^2=p(x^2+d^2)$, also the equaion to the same curve.

On the Equation to the Hyperbola between the Asymptotes.



Let CE and CB be the two asymptotes to the yperbola dFD, its vertex being F; and EF, bd, F, BD, ordinates parallel to the asymptotes. Put F or EF$=a$, CB$=x$, and BD$=y$. Then, (Prop. II Hyp.,) AF . EF = CB . BD, or $a^2=xy$, the quation to the hyperbola, when the abscissæ and rdinates are taken parallel to the asymptotes.

3. *For the Parabola.*

If x denote any absciss beginning at the vertex, and y its rdinate, also p the parameter. Then K : KD : : KD : p : or $x : y :: y : p$; hence $px=y^2$ is the equaon to the parabola.

4. *For the Circle.*

Because the circle is only a species of the ellipse, in which e two axes are equal to each other ; therefore making the vo diameters d and c equal in the foregoing equations to the lipse, they become $y^2=dx-x^2$ y being the mean proportional tween x and $d-x$, when the abscissa x begins at the rtex of the diameter : and $y^2 = d^2 - x^2$, when the abscissa gins at the centre.

Scholium. In each of these equations, we percieve that they e to the 2d or quadratic degree, or to two dimensions ; hich is also the number of points in which every one of these rves may be cut by a right line. Hence it is that these four rves are said to be lines of the 2d order. And these four e all the lines that are of that order, every other curve being some higher, or having some higher equation, or may be t in more points by a right line.

BOOK II.

SOLID SECTIONS OR SEGMENTS OF SOLIDS OF REVOLUTION, CYLINDROIDS, AND PARABOLIC PRISMOIDS AND UNGULAS.

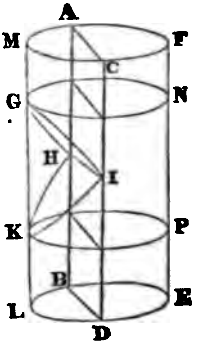
DEFINITIONS.

1. *Solid Sections* or segments are the portions of a solid cut off, or out, from another solid by one or more plane or curve surfaces.

2. If any portion of a cylinder or a cone, is cut off by a plane which is not parallel to the base, the solid section, so cut off, is called an *ungula*.

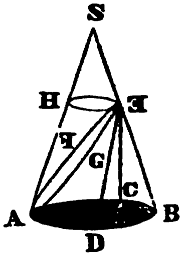
3. Ungulas cut from a cylinder are called *cylindric ungulas*.

Thus the section ACDBEF cut off by the plane ACDB ; as also the several sections ACIHGM, GHIK and HIKLBD, are *cylindric ungulas*.



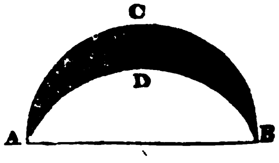
4. Ungulas cut from a cone or conic frustum are called *conical ungulas*.

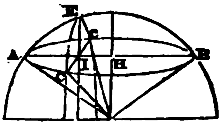
Thus the sections ECDB and AFEGAB are *conical ungulas*.



5. Portions cut from a sphere by the intersection of two planes, are called *spherical wedges or ungulas*.

Thus the spherical section ACBDAB is a *spherical wedge* or *ungula*.





A portion cut from a segment of a re by a plane perpendicular to the of the segment is called a second nent.

he portion CceA is a second segment of a sphere.

Conical ungulas take particular names according to figure of the superficial section, viz., *parabolic elliptical yperbolic.*

The portion of a cylinder or cone remaining after an un- is taken, is called the *complement* of the *ungula,* or ungu- complement, and its altitude is equal to that of the un-

hus GNPKIHG (Fig. at Def. 3) is the complement of the ula GKIH, and ADCEHA (Def. 4) is the complement of ungula ECDB.

An *ungulical supplement* is what remains of the whole l after an ungula is taken therefrom.

0. An *elliptical cylinder* is a cylindrical solid, every super- l section of which by planes perpendicular to the axis are l ellipses.

1. An *elliptical cone* is one, every section of which per- licular to its axis are similar ellipses, and is the solid in- ed between an elliptical base perpendicular to its axis, and int as its vertex.

2. A *cylindroid* is a solid included between two bases of l perimeter, one of which is an ellipse, and the other an se of a different excentricity or a circle.

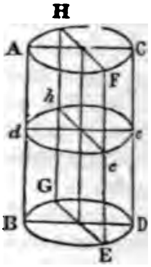
3. If a solid have two parallel bases, consisting of dissimi- ellipses of different perimeters, or one elliptical and one lar base, of different perimeters. The solid may be d a *conoidal frustum.*

4. A parabolic prism is a prismatic solid, whose base is a bola, and each of whose sections parallel to the base is l and similar to the base.

5. A parabolic pyramid is a pyramidal solid, whose base parabola; and is the solid included between such base, and int above as its vertex.

PROPOSITION I. THEOREM.

If a cylinder be cut by a plane parallel to its axis, the solid section so cut off will be equal to the area of its base multiplied by its altitude, and its curve surface will be equal to the arc of its base multiplied by its altitude.

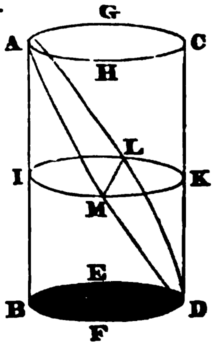


Let HFCGED, be a section of the cylinder, AD, cut off by a plane HFEG, parallel to the axis of the cylinder; then will the solidity of the section be equal to the area GD of the base multiplied by the altitude DC, and its convex surface will be equal to the arc EDG of the base multiplied by the altitude DC.

For, if a plane be passed through a cylinder parallel to the axis, it will divide the two bases proportionally, and every section *dhec* parallel to the base will be divided in the same ratio; hence the curve surface of the cylinder will be divided in the ratio that the circumference of the base is divided. But the convex surface is equal to the circumference of its base multiplied by its altitude, (Prop. I. B. III. *El. S. Geom.*,) and its solidity is equal to the area of its base multiplied by its altitude, (Prop. II. B. III. *El. S. Geom.*) Hence the curve surface of the portion so cut off, proportional to the section of the base, is also equal to the arc of its base multiplied by its altitude; and its solidity, for the same reason, is equal to the area of its base multiplied by its altitude.

PROPOSITION II. THEOREM.

If a plane cut a cylinder diagonally, passing through the opposite edges of the two bases, then the cylinder will be divided into two equal ungulas, and the curve surface of each ungula will be equal to the perimeter of its base multiplied by half its altitude; and the solidity of each will be equal to the area of the base multiplied by half the altitude.

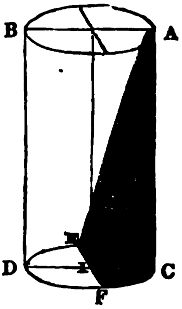


Let ALDMA be a plane passing diagonally through the cylinder BC, cuttingthe opposite edges of the two bases at A and D, and the two ungulas ADB, ADC will be equal in surface and solidity, and the curve surface of each is equal to the circumference of its base multiplied by half its altitude, and their solidities are equal each, to the area of its base multiplied by half its altitude.

For, the two ungulas are symmetrical (Def. 19, B. II. *El. S. Geom.*,) being solids, similarly formed, in opposite sides of the plane ALDMA as a base, and hence equal each to each. Moreover, let a plane be passed ugh the cylinder, parallel to, and at equal distances be- :n the two bases, forming the superficial section ILKMI, the two ungulas will be cut by that plane in the same ı, since they are similar solids, and since they are cut by ıne parallel to and at equal distances from their bases: :e the solid section, or partial ungula ALMI, is equal to partial ungula DMLK, both in surface and solidity; and the same reason, their complemental ungulas ALMKC, [DB are equal, each to each, both in surface and solidity. refore, the cylindric section KB between the two parallels M and BEDF, is equal to the ungula ABEFDA, both in ıce and solidity; but the cylindric surface of KB is equal ıe circumference BEDFB of the base multiplied by IB, l to half the altitude of the ungula ABD; and the solidity B is equal to the area BEDF multiplied by IB, equal to the altitude of the ungula. Hence, &c.

PROPOSITION III. THEOREM.

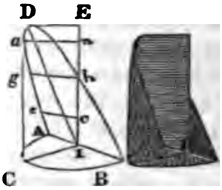
cylinder be cut diagonally by a plane which bissects the se, the ungula cut off by such plane will be equal to its lindric surface multiplied by one-third of the radius of the cle of the base.



:t AFEC be an ungula, cut off from the der ABDC by the plane AFE, bisecting ıase CFDE in FE, then will the ungula ;C be equal in solidity to its cylindric ce multiplied by one-third of the radius [the base.

ır, conceive the ungula to be divided in- 'lindrical elementary pyramidals (Prop. *Schol.* 1 and 4, B. III. *El. S. Geom.*) by :s parallel to the axis of the cylinder, and assing through the centre I of the base, these will all be perfect pyramidals, since drawn from every point in the cylindrical base to the e I, lie wholly in the solid, or in the plane surfaces of the ; and since all parts of the solid are included between ylindrical base of the ungula and centre I, in right lines. these pyramidals (Prop. XIV. B. III. *El. S. Geom.*) are equal to its cylindrical base multiplied by one-third of its de, viz., one-third of CI. Therefore, the whole ungula, ; made up of all the pyramidals, is equal to the sum of all bases multiplied by one-third of the common altitude CI.

Cor. 1. If a line *ec* be passed along the axis of the cylinder, and the edge of the ungula, through its whole extent, being always perpendicular to the axis, in every position *ec* or *gh*, or *an* the solid included within the surface described by the motion of this line, together with the ungula, is equal to the cylindric surface of the ungula multiplied by half the radius of the base.

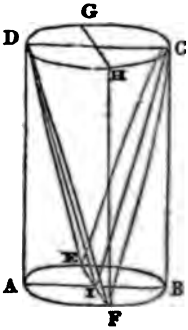


For let the cylindric surface be divided at pleasure, by planes passing through the axis, and the several divisions will all be elementary portions of the cylinder, and (Prop. III, *Cor.* B. III, *El. S. Geom.*) each will be equal to its cylindrical base multiplied by half its altitude, or half the radius of curvature of the cylinder.

Cor. 2. Hence the section included between the ungula and the axis of the cylinder is equal to half the ungula; since the ungula, is equal by the proposition to its cylindrical surface, multiplied by one-third of the radius of the base, and the two sections together, are equal to the same surface multiplied by $\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$ the same radius.

PROPOSITION IV. THEOREM.

From the complement of two similar ungulas whose bases are together equal to the base of the cylinder, a cone may be taken of equal base and altitude to that of the cylinder, when there will be left a residual portion equal to its cylindric surface multiplied by one-third of the radius of the base.



Let DEFC be the complement of the two similar ungulas DEFA, and CEFB whose bases AEF and BEF are together equal to the base of the cylinder, and there may be taken a cone DGCHI of equal base and altitude with the cylinder, and the portion of the cylinder remaining will be equal to its cylindric surface multiplied by one-third of the radius IF of the base

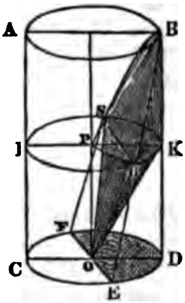
For since each of the ungulas are cut off by planes passing from the centre I, of the lower base of the cylinder, and through opposite edges of the upper base, those planes just pass along the sides, of the cone touching it on the two opposite sides, through its whole length, and be cause the cone terminates in a point I in the centre of the lower base, there exists a portion FIDHC on one side of the cone, and a similar portion EIDGC on the other side, included between the surface of the cone, and that of the cylinder; and these two portions consist of regular pyramidals with cylindric bases and their vertices all centre in I, at the vertex of the cone; for if lines be drawn from every point in their cylindric surface to the centre I, those lines will pass through every point in the solid portions to which those surfaces belong. And because the cone DGCHI is supposed to be taken from the complement, these pyrimidals are elementary portions of a conesected cylinder, (Prop. XIII. Sch. B. III. *El. & Geom.* and Def. 2,). Hence (Prop. XIII. Sch. 5, B. III., *El. & Geom.*) they are equal to their cylindric bases multiplied by one-third of the radius IF.

Cor. The opposite ungulas DEFA, CEFB and the complements IFDHC, IEDGC of a conesected cylinder are segments which are in the same proportion to each other in their cylindric surfaces, as in their solidities.

PROPOSITION V. THEOREM.

The solidity of an ungula, whose base is less than half the base of the cylinder, is equal to its cylindric surface, multiplied by one-third of the radius of the base of the cylinder, minus the pyramidal segment of a cone, whose base is the base of the ungula, and whose vertex, is the point where the diagonal plane produced forming the ungula, would cut the axis of the cylinder or its axis produced.

Let BSHK be an ungula whose base SHK is less than half the base of the cylinder, and its solidity will be equal to its cylindric surface, multiplied by one-third of the radius of the base of the cylinder, minus the pyramidal segment KSHO of a cone whose base is KSH, the base of the ungula; and whose vertex is O, the point where the plane BSH produced to FE cuts the axis of the cylinder.



For the solidity of the ungula BFED, is equal to its cylindric surface multiplied by one-third of the radius of the base of the cylinder (Prop. III,) and the section KSHFED minus the conical section KSHO is a portion of a conesected cylinder, which (Prop. XIII, B. III, *El. S. Geom.*) is equal to the product of its cylindric surface multiplied by one-third of the radius of the base, and because this section of the conesected cylinder is formed by a plane passing through the centre O, cutting it in such manner that every point in its cylindric surface may be connected by right lines with the vertice O, these lines being included in the same solid; this section is also equal to its cylindric surface multiplied by one-third of the distance OD of this vertice from the cylindric surface; hence, the other two portions of the ungula, viz., the conical KSHO, and the ungula BSHK are equal to the remainder of the cylindric surface of the ungula BFED, viz., the cylindric surface of the small ungula BSHK multiplied by one-third of the radius OD of the base of the cylinder. Now, if from this product we take away the conical segment SHKO we shall have the ungula BSHK. Hence, &c.

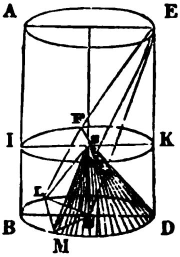
Scholium. If on the diagonal plane BSH of the ungula as a base, the pyramidal BSHP be described, P being the vertice of such pyramidal situated in the centre of the circular section of the cylinder, and in the plane KSH of the base of the ungula produced, the sum of this pyramidal and ungula is equal to their cylindric surface multiplied by one-third of the radius of the base of the cylinder; and the pyramidal BSHP is equal to the conical portion, KSHO.

For the pyramidal BSHP together with the ungula BSHK, constitutes the regular elementary pyramidal, BSHKP, with a cylindrical base, which (Prop. XIV. B. III., *El. S. Geom.*) is equal to its cylindrical surface multiplied by one-third of the radius, PK of the base, and because the sum of the ungula, BSHK, and pyramidal section of a cone KSHO, is equal to the same product (Prop. V.) it follows that the pyramidal section BSHP is equal to the conical portion KSHO.

PROPOSITION VI. THEOREM.

The solidity of an ungula ELMD *whose base is greater than half the base of the cylinder, is equal to its cylindrical surface multiplied by one-third of the radius of the base of the cylinder, plus a conical segment, whose base is the base of the ungula, and whose vertice is the point* S, *where the plane* ELM *cuts the axis of the cylinder.*

the solidity of the ungula is equal to the same product plus the pyramidal ELMO, *whose base is the elliptical sections* ELM *and whose vertice is* O, *the centre of the base of the cylinder.*



For through the point S pass the plane KC, and the ungula will be divided into o portions, one of which, EFCK, is equal rop. III.) to its cylindrical surface multied by one third of the radius of the base the cylinder; and the portion FCK MD, from which, if we take a conical secn LMDS, we shall have a portion of a nesected cylinder remaining which is equal its cylindric surface multiplied by one third of its radius.

Hence, the whole ungula ELMD is equal to its cylindric rface multiplied by one third of the radius of the base of the linder plus a segment of a cone, whose base is the base the ungula and vertice, the point where the plane ELM uts the axis of the cylinder.

Again, if we pass the two surfaces MOE and LOE meetg each other in the right line OE, those surfaces will ut out the pyramidal ELMO, whose base is the elliptical ection ELM, and whose vertice is the centre of the base of e cylinder, which if we take from the ungula, will leave a ortion which is equal to its cylindrical surface multiplied by ne-third of the radius of the base of the cylinder, since it conists of a portion which may be divided indefinitely by lanes passing through the vertice O, and each of those portions o divided will be perfect pyramidals, and will remain elementry to the whole section. (Prop. XIII, Schls. B. III, *El. S. eom.*) Hence, &c.

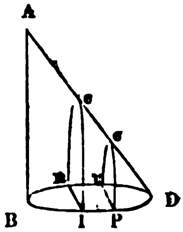
Cor. 1. Hence the pyramidal ELMO, is equal to the conial segment LMDS.

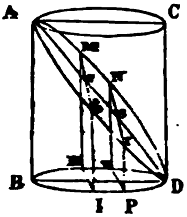
Cor. 2. As the plane section ELM is an ellipse, the surface of e pyramidal ELMO contiguous to the other portions of the ngula so divided is a conical surface; for if the several oints in the elliptical curve be connected with the vertice O, ose lines must include a conical surface, since the section of cone by a plane passing through both sides is an ellipse.

Cor. 3. If the ungula is so cut as to include the whole of e base BLDM, the circumstances will still be the same, and e ellipse of the section ELM becomes perfect, and the pyraidal ELMO becomes a perfect oblique cone, and is equal to e conical body LMDS, which, in such case, becomes a perct right cone, whose base is the whole base BLDM of the ylinder, and whose altitude is half the altitude of the cylinder.

PROPOSITION VII. THEOREM.

If in a cylindric ungula ADB, *whose base is equal to that of the cylinder, a cone be described on the same base and of an equal altitude with that of the ungula, the cone will be equal to two-thirds of the ungula ; and each solid section of the cone made by planes parallel to the axis of the cylinder, and perpendicular to a plane passing through the axis of the cone and cylinder, will be equal to two-thirds of the corresponding section of the ungula cut off by the same plane.*





Let the oblique cone ABD of equal base and altitude to that of the ungula A*b*DNB, be supposed to be inscribed in the ungula, and the solidity of the cone will be equal to two-thirds of the ungula, and each solid section IR*c*D, P*ve*D of the cone made by planes parallel to the axis of the cylinder, and perpendicular to a plane passing through the axis of the cone and cylinder, are equal to $\frac{2}{3}$ their corresponding sections IRM*b*D, P*v*N*t*D of the ungula cut by the same planes.

For, conceive an indefinite number of parallel planes IRM*b*, N*t*P*v*, to be passed through the cone and ungula indefinitely near to each other, and the two bodies will be divided into an indefinite number of strata, the sum of which, in each solid, will be in the relation of the magnitudes of the two bodies ; and the corresponding *strata* in each solid will be in the relation of the superficial sections contiguous to such *strata*, in each, since by hypothesis the *strata* are indefinitely thin, so that no appreciable space intervenes between them ; hence, if the solidities of any *stratum*, or number of contiguous *strata* in each solid be represented by the superficial sections passing through such *stratum* or associated *strata*, they will exist in each in the relation of the superficial sections in each, made by the same planes.

Now the superficial sections IR*c*, P*ve* &c., made by planes parallel to the side AB of the cone, are all parabolas, (by def.) and all the corresponding sections of the ungula are rectangles circumscribing the several parabolas, since each side of those sections M*b*, N*r* of the ungula are parallel to the opposite sides RI, *v*P, passing through the base. But the area of the parabola (Proposition VI, Book I,) is equal to two-thirds of its circumscribing rectangle or parallelogram, and as each of the several rectangles circumscribe their respective parabolas, throughout the whole extent of the two solids, it follows that not only is the cone equal to two-thirds of the ungula, but also that each solid section, or segment of the cone, cut by the planes RM*b*I, &c., is equal to two-thirds of the corresponding segment of the ungula cut by the same plane.

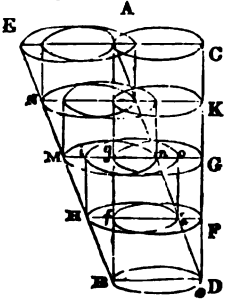
Cor. Hence, as in Prop. VIII, Cor. B. III, *El. S. Geom.* a cone is equal to one-third of its circumscribing cylinder of the same base and altitude.

Scholium. If the cone and ungula are cut by planes parallel to the plane A*b*DN, the superficial sections of the ungula will be ellipses, or portions of ellipses, and the corresponding sections of the cone will be parabolas.

PROPOSITION VIII. THEOREM.

Two cylinders erected on the same base, and between the same parallel planes are equivalent or equal in solidity.

Let ACDB and AEBD be two cylinders erected on the same base BD, and included between the same parallel planes BD and EAC, and the two cylinders will be equivalent.



For let the cylinder ABDC be divided into sections BDF*f*, F*fg*G by planes parallel to the base, and let each of the sections so divided, be removed from their positions in the cylinder ABDC, so that their several bases shall agree with, and become sections of the oblique cylinder AEBD; viz., let the solid sections *f*FG*g*, be removed to the position H*hoi*, and let the other sections be similarly posited in reference to the two cylinders, so that the bases H*h*, M*n*, &c. of the several solid sections may be included in the oblique cylinder AEBD, and these several sections will still be equal in their altitude to AB, the altitude of the cylinder from whence they are severally derived. Let, now, the number of these solid sections be indefinitely increased, and the altitude of each will be indefinitely small; and hence the right cylinder will become identical with the oblique cylinder AEBD, and will still have the same altitude AB.

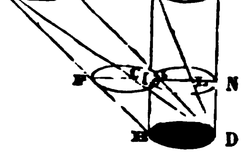
Cor. 1. Since the base of the oblique cylinder AEBD is a circle, every section parallel to the base is likewise a circle, and equal to the base.

Cor. 2. Hence every section of the oblique cylinder made by the planes perpendicular to its axis, is an ellipse. Therefore, if referred to its axis, it becomes an elliptical cylinder, but when referred to its base, it is circular.

PROPOSITION IX. THEOREM.

Cylindrical ungulas of the same base and equal altitude are equivalent.

Let ABD and EBD, be two ungulas described on the same base BD ; and let their common altitude be AB, and the two ungulas will be equivalent, or equal in solidity.



For each section FC or IL of the two ungulas parallel to the base BD, have the same relation to the respective sections of their cylinders ; viz., the ungulical section FC has the same relation to the cylindric section F*h*, as the ungulical section IL has to the cylindrical section IN ; and the equality of this relation remains through the whole altitude AB of the ungulas, and by Prop. VIII, it appears that two cylinders erected on the same base, and between the same parallels are equal in solidity. Therefore, the ungulas erected on the same base, and of the same altitude, are equivalent.

Scholium. The above proposition is manifestly true, whether the base of the ungulas is equal to that of the cylinder in which it is erected, or less than that base.

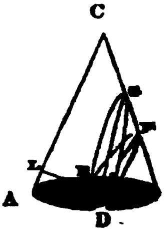
Cor. 1. As the same arguments would apply to conical ungulas, it may be inferred that conical ungulas of the same base and equal altitude are equivalent, if the cones from which they are taken are similar.

Cor. 2. Since cylinders with equal bases are proportional to their altitudes, cylindric ungulas on the same base are proportional to their altitudes.

PROPOSITION X. THEOREM.

If a cone be cut by a plane passing through the centre of the base, the solidity of the ungula formed by such plane will be equal to its curve surface multiplied by one-third of its distance from the centre of the base.

Let the cone ABC be cut by the plane DEF, cutting off the ungula EFDB ; also by the plane EDG cutting off the ungula GEDB, or the ungula GEDF, and the solidities of the several ungulas will be equal to their curve surfaces multiplied by $\frac{1}{3}$ of the distance IL of the surface of the cone to the centre of the base.



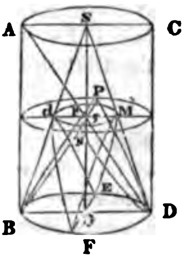
'or the sections may each be divided into an indifinite nber of pyramidals or elements (Prop. XIII and XIV, B. *El. S. Geom.*) by dividing their curve surfaces as bases, by passing the planes of division through the centre I of cone's base, at which point the several vertices of the py- idals all centre ; and because those pyramidals are all fect ; viz., as they include all the space intercepted between ir bases and vertices in right lines, each one is equal to its e, multiplied by one-third of its altitude, and since the alti- e is equal in each, and equal to the distance of the curve face of the cone from the centre of its base, IL is that itude, and hence IL is the common altitude of each of e pyramidals; hence the sum of the pyramidals con- uting any section, is equal to the sum of their bases, con- uting the curve surface of such section multiplied by one- rd of IL.

Cor. Hence, if a cone be cut by a plane passing through centre of the base, the solidity of the cone, and its convex face is divided in the same ratio, since the cone (Prop. IX. III., *El. S. Geom.*) is equal to its convex surface multiplied $\frac{1}{3}$ of its distance from the centre of the base.

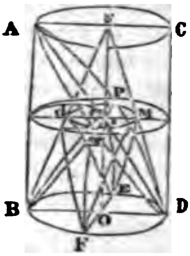
PROPOSITION XI. THEOREM.

a cone be described in a cylinder, and two parabolic ungulas e cut by planes passing through the centre of the cone's base, he ungulas so cut, will be equal to two thirds of the unguli- al complement of the cylinder, of equal base and altitude to he parabolical ungulas.

et BDS be a cone described in the cy- ler ABDC, and let EFB*d*, and EFDM be equal parabolic ungulas, described on including the whole base of the cone, the two ungulas will be equal to two- ds of the ungulical complement BEDFNP he cylinder, of equal base and altitude.



'or let the cone BDS be brought in the position BDA. and parabolic ungula IEFD of the cone BDA will be equal to -thirds of the semi-ungulical complement NPEFD of the inder, (Prop. VII ;) and because conical ungulas on the e base and equal altitudes are equivalent (Prop. IX *Cor.*,) ungula IEFD of the oblique cone BDA is equal to the ungula EFDM of the right cone BDS; hence the parabolic ungula EFMD is equal to two-thirds of the semi-ungulical complement NPEFD of the cylinder; and the two similar ungulas *d*EFB, MEFD are, together, equal to $\frac{2}{3}$ of the ungulical complement BEDFPN of the cylinder.

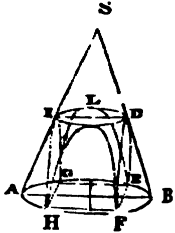


Cor. 1. Hence the conical complement EF*d*MS of the parabolic ungulas=the complement PMN*d* EF+the cone PMN*d*S, is equal to two-thirds of the cylindrical ungula APNB. For if the whole cone BDA or BDS is=$\frac{2}{3}$ of the ungula BDA, (Prop. VIII,) and if the two conical ungulas *d*EFB, MEFD, as shown above, is = $\frac{2}{3}$ of the complement PNBD, then must the complement EF*d*MS= $\frac{2}{3}$ the cylindrical ungula APNB

Cor. 2. The small cone *d*MS, whose base passes through the vertices of the ungulas, is = to $\frac{1}{8}$ of the cone BDS, since the diameter of its base is necessarily = $\frac{1}{2}$ that of the larger, and since they are similar solids; for similar solids are to each other as the cubes of their like sides, (Prop. XXXV, B. II, *El. S. Geom.*) and cube of 1 is 1, cube of 2 is 8; hence 1 : 8 :: cone *d*MS : cone BDS.

PROPOSITION XII. THEOREM.

If a right cone be cut by a plane perpendicular to the plane of its base, the convex surface of the cone, and the plane of the base, will be divided in the same ratio by the cutting plane.



Let DEF be a plane cutting the cone ABS, perpendicular through the base AEBF, and the convex surface of the cone will be divided by the cutting plane, in the same ratio that the surface of the base is divided.

For since it is shown (Prop. XXII, B. II, *El. S. Geom.*,) that the convex surface of a pyramid and its base is divided in the same ratio, by a plane perpendicular to its base, and because a cone may be considered as a pyramid, whose convex surface consists of an indefinite number of planes, which are indefinitely narrow, it follows that if the convex surface and base of a right cone are cut by a plane perpendicular to the base, those surfaces will be each divided in the same ratio.

Cor. 1. Hence if any portion of the base of a right cone is taken., the portion of the curve surface perpendicular above it, will be to the whole curve surface of the cone, as such portion of the base is to the whole base.

Cor. 2. Hence, also, if a regular polygon, for instance a quare EFHG, be described in the base of the cone, and if on ach side of this square, a plane be raised perpendicular to the se, the portion of the conical surface, cut off toward the tis, is to that of the rectilineal polygon EFHG, which cor- sponds to it perpendicularly below, as the surface of the ne is to the area of its base; or as the slant side AS of the ne is to the radius of the base; and, in fact, whatever ure be inscribed in the base, if we conceive a right prisma- surface raised perpendicular from the perimeter of the ure, it will cut off from the conical surface a portion which ll be to it in the same ratio.

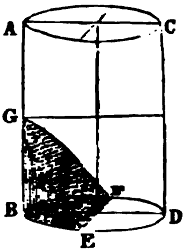
Scholium. The solid sections DEFB, LGE, &c., are hyper- lic ungulas, (Def. 7.) And if ungulas DEFB, LGE, &c., e taken from the cone, the remaining portion or compliment ll be equal to its curve surface multiplied by one-third of e distance of the curve surface of the cone from the centre the base, + the surface of the plane hyperbolic sections iltiplied by one-third of their respective distance from the me centre of the base.

Cor. Hence the portion of the cone included between the ntre of the base, and that portion of the convex surface left the ungulas, since it is equal to its convex surface, multi- ied by one-third of its perpendicular distance from the centre, e quadrature of which we have shown to be attainable, be- mes known in absolute terms, or its cubature is attained ithout regard to the circle's quadrature.

PROPOSITION XIII. THEOREM.

an elliptical cylinder, and a circular cylinder, have equivalent bases and equal altitudes, they are equal in solidity; and any ungulas similarly cut from each, with equivalent bases and altitudes, are equivalent.

For it has been shown (Prop. II. B. III. *El. Geom.*) that the idity of a cylinder, with circular base, is equal to its base iltiplied by its altitude; and because this is true of any prism, atever be the form of its base, (Prop. XVI. B. II. *El. S. om.*) it must be true of a cylinder with an elliptical base. erefore, an elliptical and a circular cylinder of equivalent es and equal altitudes, are equivalent.



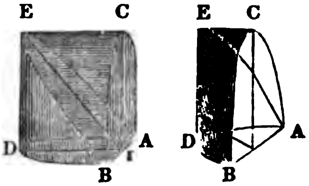
Again, let GEFB be an ungula cut from a circular cylinder ABDC; and if the base of this cylinder, including that of the ungula, is drawn out or elongated in the direction from BA toward DC, so as to become elliptical; and if every section of the cylinder parallel to its base should become equal and similar ellipses, then the cylinder becomes an elliptical cylinder.

Now, if planes should be passed through the cylinder parallel to its axis, and in the direction AC or BD of its elongation, in passing from a circular to an elliptical cylinder, this transformation will have been effected by an elongation of these planes proportional to the elliptical elongation of the base, or of the solid; and, as this would be true in every parallel plane, it follows that the elongation of those planes may be regarded as a measure of the ratio of enlargement of the solidity, by the same means. And as every such parallel section becomes enlarged in the same ratio, any specific portion of such section must suffer the same specific enlargement.

And as the increase of any solid sections through which any portions of the plane sections pass, may be measured by the increment of those planes, it follows, that both the ungula and its complement are each increased in the ratio of the increments of the parallel sections passing through each in their enlargement. And the ungula which was cut from a circular cylinder, becomes the ungula of an elliptical cylinder, which ungula has become enlarged in the ratio of the enlargement of its base; and the solidity of the ungula from the elliptical cylinder, is to the solidity of the ungula from the circular cylinder, as the base of the former to the base of the latter. And the same would be true, if instead of an enlargement of the circular cylinder to form the elliptical cylinder, it should be contracted in the direction AC, so as to give it eccentricity in the other direction; but, by hypothesis, the cylinder and an elliptical cylinder have equivalent bases; hence, ungulas similarly cut from each, are equivalent.

Cor. 1. Hence, also, if a cone with a circular base, and one with an equivalent elliptical base, have equal altitudes, their solidities will be equivalent; and ungulas with equivalent bases and equal altitudes cut from each, are equivalent.

Cor. 2. Ungulas, whose bases are the like parts of circular or elliptical cylinders, are as their altitudes; and it having been shown that they are also as their bases when their altiles are equal, it follows that they are generally as the rectgle of their bases into their altitudes.



Let ABEC and ABED be two gulas cut from any cylinders, cirlar or elliptical, such that AB ill be the same in each, and such it ID, the altitude of the base in first, shall be equal CE. the altiude of the second, and IC the alude of the base of the second shall be equal to the altitude D of the first, and the two ungulas so described will be uivalent.

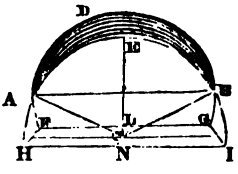
PROPOSITION XIV. THEOREM.

he solidity of a cylindroid is equal to the product of the sum of the areas of the two ends, and four times the area of a parallel section, equally distant between the two ends multiplied by $\frac{1}{6}$ of the height.

Demonstration same as for the prismoid, Prop. XXXV, r. B. II, *El. S. Geom.*, which see.

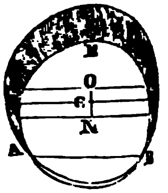
PROPOSITION XV. THEOREM.

n ungula is cut from a sphere by two planes which interect each other, not in the centre of the sphere, then the solidity of he ungula will be equal to its spherical base multiplied by oneird of the radius of the sphere, plus the products of the supercial sections of the ungula multiplied by one-third of the perendicular distances of the planes of those sections from the ntre of the sphere, estimated from the sides of those planes oposite the ungula.

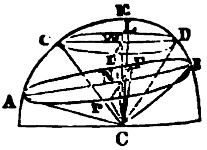


et ABDE be an ungula cut by the nes HEI and FDG intersecting h other in the line AB, not passing ugh the centre of the sphere, then ll the solidity of the ungula ABDE be ual to the spherical surface ADBEA, ltiplied by one-third of the radius of e sphere, — (the plane surface ABE, multiplied by one third the distance CN) — (the plane surface ABD multiplied by e-third the distance CL) when those planes include the ntre on opposite sides of each, from that of the ungula.

For let each point in the perimeter, ABEA, ABDA, be conted by lines to the centre C, and the solid included by such lines will be a spherical pyramid, whose solidity is equal to its spherical base multiplied by $\frac{1}{3}$ the radius of the sphere; from which if we take away those portions included between the planes ABE, ABD and the centre C we shall have the ungula. But the pyramidal ABEC is equal to the surface ABE $\times \frac{1}{3}$ CF, and the pyramidal ABDC is equal the surface ABD $\times \frac{1}{3}$ CL. Hence as enunciated above.



Scholium. If an ungula is cut by two planes which pass the centre before their intersection, so as to include the centre of the sphere within the ungula, then will its solidity be equal to the spherical surface multiplied by $\frac{1}{3}$ the radius, plus the two pyramidals erected on those plane sections, and whose vertices are in the centre of the sphere, which is agreeable to our proposition; for the sign becomes changed from minus to plus according to the conditions.

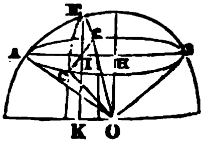


2. Also, if the two intersecting planes pass the centre on one side before their intersection, so as to cut out an oblique ungula ABDG, then since the pyramidal erected on the plane AFBP, and whose vertice is the centre, C, is considered negative, and the pyramid erected on the plane GIDL is considered positive, by the proposition, then will the ungula be equal to its (spherical surface $\times \frac{1}{3}$ the radius) — the plane APBF $\times$ ($\frac{1}{3}$ CN) + the plane GLDF $\times \frac{1}{3}$ CW.

3. And, generally, if the planes do not intersect each other within the sphere, the same proposition will still hold true, even though the planes may be parallel; in which case, the portion cut out will be a segment or zone of the sphere.

PROPOSITION XVI. THEOREM.

The solidity of the second segment of a sphere is equal to its spherical surface, multiplied by $\frac{1}{3}$ of the radius of its sphere minus each of the plane surfaces whose planes pass between the segment and centre of the sphere multiplied by $\frac{1}{3}$ of their respective perpendicular distances from the centre, and plus each of the plane surfaces which include the centre of the sphere on the same side with the segment, multiplied by $\frac{1}{3}$ the perpendicular distances of such planes to the centre.



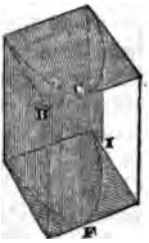
Let ABD be a segment of a sphere cut off by the plane ABCE, and if this segment is cut by the plane CE*e* perpendicular to the former plane, the two portions into which the segment is divided will be second segments. (Def. 6.)

AO, EO, CO, and *e*O, and the spherical pyramidal AE /ill be equal to its spherical base $\times\frac{1}{3}$ AO or EO. Now, his pyramidal may be taken the pyramidal whose base section C*e*E, and whose altitude is KO, and also the iidal whose base is the section AC*e* and altitude HO, and will remain the second segment AE*e*C.

) the second segment may be shown to consist of the al pyramid, whose base is the spherical surface, plus a iidal CE*e*O, and minus the pyramid C*e*BO. Hence the iition is true, as enunciated.

PROPOSITION XVII. THEOREM.

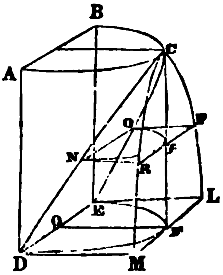
lidity of a parabolic prism is equal to two-thirds of its circumscribing quadrangular prism.



the solidity of any prism or solid, all of sections parallel to the base are equal to se, is equal to the base multiplied by the e: and hence, prisms of the same altire as their bases; but the base of the pa- : prism, which is a parabola, is equal VI, B. I).$\frac{2}{3}$ its circumscribing rectangle, e rectangle circumscribing the base of .rabolic prism, is the base of the rectan- prism.

PROPOSITION XVIII. THEOREM.

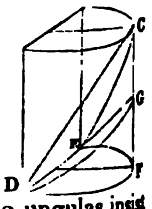
:ngula be cut from a parabolic prism by a plane passthrough the vertical line of the parabolic surface, and ie intersection forms an ordinate to the axis of the parala of the base; this ungula is equal to two-thirds of ingulical complement of the prism of the same base and ude.



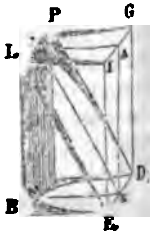
DECF be an ungula cut from the lic prism DEFCAB, by the plane rom the vertical line CF of the pasurface. to the base DEF forming linate ED to the axis FQ of the y its intersection with the plane of ie, and the ungula so cut will be :o $\frac{2}{3}$ of the ungulical complement B.

let a rectangle DELM be describit the base of the ungula, and let any number of planes

NO*f* parallel to the base be passed through the ungula; and each of those sections made by those planes are parabolas; and if about each of those parallel sections, rectangles NRPO are described, those parabolic sections will each be equal to two-thirds of their respective circumscribing rectangles, (Prop. VI. B. I.) Now, if these parallel sections through the ungula are infinite in number and equidistant from each other, they will represent the whole ungula, and their sum may be taken as a function of the solidity of the ungula, and the sum of their several circumscribing rectangles may be taken as a similar function of the solid DECLMC; hence the ungula DEFC is equal to two-thirds of the solid CDELMC. Now, if we take the altitude FC of the ungula=QF, the axis of the parabola of the base. the side CLMC of the new solid will be a parabola equal and similar to the base DEF of the ungula, or ABC of the complement, since perpendiculars from every point in the perimeter of the base DEF, trace out the parabolic section DEC; and since perpendiculars from each point in the .perimeter DNCOE to the plane CLM, includes and traces out the parabola CLM. And because QF or EL would be equal to EB, the rectangle DELM would be equal to the rectangle DEBA; hence, the two solids CABDEC, CLMDEC being similar figures on opposite sides of the same base, CDE are symmetrical, (Def. 19, B. II. *El. Sol. Geom.*); and hence they are equivalent. But the ungula CDEF has been shown to be equal to two-thirds of the solid CLMDEC, it is therefore equal to two-thirds of the ungulical complement CEDABC.



Cor. 1. The above properties are true in whatever part of the base the plane CDE may cut, or whatever be the altitude of the ungula, provided the ungula and its complement have equal bases and altitudes; and because ungulas on the same base are as their altitudes, an ungula CGED is equal to the ungula GEDF, if the altitude CG=the altitude GF, since the two ungulas insist on or above the same base EDF.



Cor. 2. Hence the solidity of the parabolic ungula CDEF is equal to $\frac{8}{15}$ of its circumscribing prism, and the complement of a parabolic ungula is equal to $\frac{4}{5}$ of its circumscribing prism. For the parabolic prism (Prop. XVII.) is equal to $\frac{2}{3}$ of its circumscribing rectangular prism; and since the rectangular prism circumscribing the parabolic prism, may be divided into two equal triangular prisms by a diagonal ie, identical with the plane which divides the ungula from omplement, it follows that the prism circumscribing the ula DECF is equal to that circumscribing the complement CI, equal half that circumscribing the parabolic prism. Let qual the ungula, and C equal the complement, and let P l the triangular prism circumscribing the ungula, and 2P equal the quadrangular prism circumscribing the parabo- prism. Then will $U+C=\frac{4}{3}P$ and $U=\frac{2}{3}C$, hence $C=1\frac{1}{2}U$. stituting the value of C in the first equation, we have

$$2\frac{1}{2}U=\frac{4}{3}P \text{ or } 15U=8P.$$

ence $U=\frac{8}{15}P$,

e ungula is equal to $\frac{8}{15}$ of its circumscribing prism.

nd if we substitute the value of U in terms of C in the equation, we shall have

$$1\frac{2}{3}C=\frac{4}{3}P \text{ or } 5C=4P.$$

ence $C=\frac{4}{5}P$, or the complement of the ungula is equal of its circumscribing triangular prism.

herefore, the portion CLBFE is equal to $\frac{7}{15}$ of the prism EBL.

nd the exterior portion CLEI is equal to $\frac{1}{5}$ of the prism ELI.

PROPOSITION XIX. THEOREM.

rabolic pyramid or cone is equal to two-thirds of its circumscribing rectangular pyramid.

et a pyramid be erected on a base whose figure is a para- , and if this base is circumscribed by a rectangle, it will the rectangle; and because every section of each of the s erected on those figures as bases, and whose sections n a common point are similar figures and similar to the , the solids erected on those bases must be in the relation eir bases; hence the parabolic pyramid is $=\frac{2}{3}$ its circum- ing quadrangular pyramid.

BOOK III.

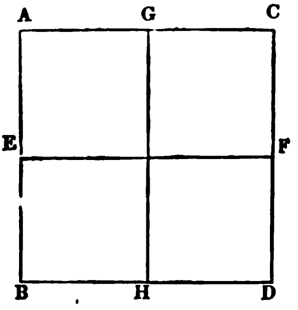
ON REVOLOIDS AND SOLIDS FORMED BY THE REVOLUTIONS OF THE CONIC SECTIONS.

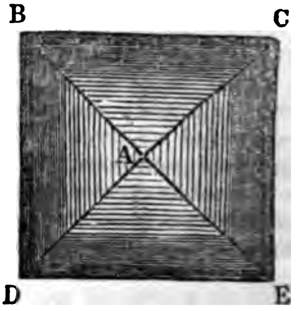
DEFINITIONS.

1. A REVOLOID is a solid generated by the continued semi-revolution of a polygon on axes parallel to the sides of the polygon respectively, and passing through its centre, which is fixed, and it includes all the space that is not cut off by either side of the plane, in their several semi-revolutions.

2. Every revoloid has as many axes of rotation, as the polygon from which it is conceived to be generated, has independent sides; but the axes of any two parallel sides coincide with each other and are identical.

Thus, if the quadrilaterial ABDC be made to revolve on the two axes, EF and GH, parallel to its sides respectively, the solid generated by the revolutions of those sides will be a revoloid, as represented by BCED.



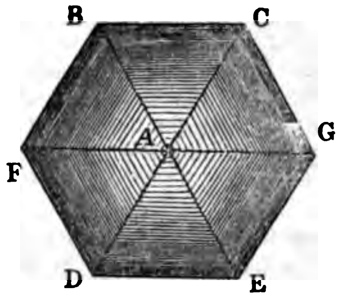


3. A revoloid is designated by the number of sides contained in the figure from whence it is conceived to be generated.

Thus a *triangular revoloid* is one formed by the continued semi-revolution of the sides of a triangle about their respective axes.

A *quadrangular revoloid*, by the revolutions of the sides of a square about their corresponding axes.

lso a *pentagonal*, a *hexagonal*, or other *polygonal revoloid*, rmed by the continued semi-revolution of the sides of a *tagon*, a *hexagon* or other polygon, about their respective s.



B represents a hexagonal oloid.

. The vertices of a revo- are its two extremities, ere its several curve sur- es meet at a point, as at A.

. The line joining these ex- mities, is its transverse or tical axis; and a section ough this axis is a vertical tion.

6. Any diameter perpendicular to the transverse axis, is a njugate axis, and any section perpendicular to the trans- rse axis is a conjugate section.

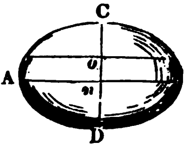
7. The figure from which a revoloid is supposed to be gene- ted, is called its prime; which is always represented by a njugate section through the centre of the revoloid.

8. A revoloid formed by the revolutions of the several sides its prime on axes that are fixed or immoveable during the volutions of each side respectively, may be called right re- loids; and a vertical section through the centre of its op- site sides is a circle.

9. If, during the revolutions of the several sides of the prime a revoloid, their axes are made to move in the line of the nsverse or an opposite conjugate axis, then the revoloid ll assume a different character according to the curve des- bed by its sides. It may be *elliptical*, *parabolic*, or of any *ular curve*.

10. An *elliptical revoloid* is one whose transverse and con- ate diameters are unequal, and whose vertical section ough the centre of its sides is elliptical.

11. A spheroid or ellipsoid is a solid generated by the revo- ion of a semi-ellipse about one of its axes, which remains ed.



If the ellipse revolve round the transverse major axis AB, the figure is called a pro- e or oblong spheroid ; if the ellipse revolve nd the shorter axis CD, the figure is called oblate spheroid.

12. A segment of a spheroid or of an elliptical revoloid is

a part cut off by a plane, parallel to the major or minor axis.

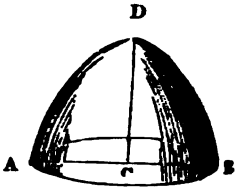
13. A frustum of a spheroid and also of an elliptical revoloid, is a part intercepted between two parallel planes, and is a portion included between two opposite segments.

14. A *parabolic revoloid* is one whose vertical section through the centre of its sides, consists of two parabolas on opposite sides of the same base.

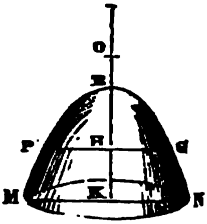
15. A parobolic revoloid is *vertical* when the vertices of the several parobolic sections are identical with the vertices of the revoloid. This may also be called a *parabolic pyramid.*

16. It is a *conjugate parabolic revoloid,* when the vertices of the parabolic sections are all in a plane perpendicular to the vertical axis of the revoloid.

17. A *Parabolic conoid* is a solid formed by the revolution of a semi-parabola about its axis; it is also called a paraboloid.



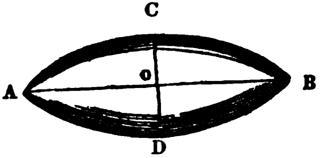
18. A *hyperbolic conoid,* or a hyperboloid, is a solid formed by the revolution of a semi-hyperpola about its greater abscissa, or transverse axis produced. Thus, the hyperbolic conoid MPBGN is formed by the revolution of the semi-hyperbola MPB, about its greater abscissa BK, or the transverse axis BH produced.



19. A *hyperbolic revoloid* is one whose vertical sections through its sides consists of two hyperbolas whose vertices are in the plane of the conjugate axes.

20. A *hyperbolic pyramid* or *pyramoid,* is one whose base is a polygon, and whose vertical sections through the vertice of the pyramoid, is an hyperbola. This may also be called a vertical hyperbolic semi-revoloid.

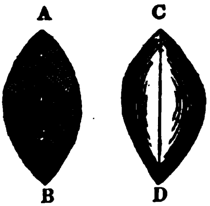
21. A *circular spindle* is a solid generated by the revolution a segment of a circle about its chord, which remains fixed.



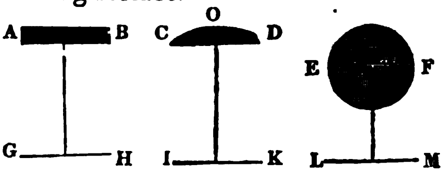
22. An *elliptical spindle* is a lid generated by the revolution the segment of an ellipse about s chord.

23. A *parabolic*, or *hyperbolic spindle*, is a solid formed by e revolution of a segment of a parabola or hyperbola, about s ordinate. Thus, if the segment PBGP of Def. 18 be suposed to revolve about the ordinate PG, which remains fixed, will describe a spindle.

24. A *revoloidal spindle* is a revoloid ircumscribing a spindle, a vertical section f which through the centre of its opposite ides, is equal and similar to that of the pindle through the same plane. It takes articular names according to the designaion of the inscribed spindle.

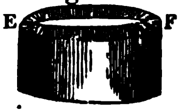


25. A ring is a solid formed by the revolution of a plane surace about an axis exterior to itself, which axis is always in he same plane of the revolving surface.



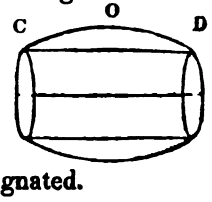
Thus, if the plane urfaces AB, CD, or EF re made to revolve bout their several axs GH, IK, LM, they vill severally describe olids, which are called rings.

26. A ring is designated according to the figure of a conjuate section, or of the plane surface from which it is generated.



Thus from a rectilineal figure AB, is formed prismatic ring.

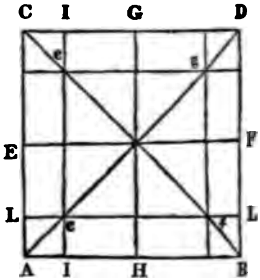
From the circle EF is formed a cylindrical ring.



From the segment CD is formed an unilical ring, which, if the line OP is equal to e radius of the circular area COD, it may called a spherical ring, the curve surface rming a portion of the surface of a sphere. ings formed from other figures or segents of other curves, may be similarly designated.

PROPOSITION I. THEOREM.

A right revoloid is composed of cylindrical ungulas, equal in number to the sides of the revoloid; and these ungulas are such as are formed by plane sections, from one side, meeting in the axis of the cylinder, the intersection of which planes forms a diameter to the cylinder.



For, since the several curve surfaces of a revoloid are conceived to be formed by the revolution of the sides of its prime about axes parallel to those sides respectively; (Def. 1,) the surfaces described by the revolutions of those sides, are cylindrical surfaces, (Def 1. B. III. *El. S. Geom.*) And since the angles formed by the sides of the prime are similar, each to each, in whatever similar part of their revolutions they may be, it follows that they would form similar angles with any plane perpendicular to these sides. Thus, because, when the side AB, of the revoloidal prime ABDC, in its revolution about its axis EF, comes into the position LL; and AC revolving about its axis GH, comes into the position II, &c.; the parts *ce* and *ef* of those sides form the same angle with each other as before; they must also form the same angles with any planes AD or CB perpendicular to the plane ABDC or *cefg*. Hence, the figure *cefg* is similar to the prime ABDC; and because this is true in whatever similar positions, the several sides of the prime may be in the course of their revolutions, it follows, therefore, that sections through the angles meeting in the transverse axis of the revoloid, are plane angles; and as these plane sections cut the cylindric surfaces diagonally through the axis of the revoloid, the segments so cut are cylindric ungulas, (Def. 3. B. II.)

Hence, each side of a right revoloid is a cylindric ungula, such as is formed by plane sections cutting the cylinder diagonally, and meeting in the axis of the cylinder and forming a diameter thereto.

Cor. 1. Hence any vertical section through the angles formed by the meeting of the curve surfaces of the revoloid, or any vertical section of the revoloid not at right angles to the sides, is an ellipse.

Cor. 2. Hence, also, the elliptical revoloid may be conceived to be made up of similar ungulas cut from an elliptical cylinder; see Prop. XIII. B. II., and a parabolic revoloid may consist of ungulas from a parabolic cylinder or prism.

Cor. 3. Since a revoloid is composed of cylindrical ungulas equal in number to the number of the sides, those ungulas are also such as are cut from the cylinder of altitudes or lengths on the cylindric surface equal to the lengths of the sides of the revoloid indicated by the lengths of the sides of its prime.

Cor. 4. All sections of a revoloid perpendicular to the vertical or transverse axis, are similar figures, since the sides of its prime retain their parallel position in whatever part of their revolution they are supposed to be taken.

PROPOSITION II. THEOREM.

The solidity of every right revoloid, bears the same relation to that of its greatest inscribed sphere, as the area of its prime does to that of its greatest inscribed circle.

For since the surface of the revoloid is composed of cylindric surfaces, (Prop. I.) and since vertical sections of a right revoloid through the centres of its opposite sides are circles, (Def. 8.) it follows that a sphere may be inscribed in the revoloid so as to touch its cylindric surfaces through the whole circumference of those circular sections, viz., through the centres of the vertical sides, so that vertical sections through the centres of the sides in the revoloid, will be identical with those of the inscribed sphere through the same planes.

Now, if planes be passed through the two solids perpendicular to the vertical or transverse axis, those planes will cut the solids in the relation of their solidities through such sections; and as all such sections (Prop. I. Cor. 4.) are similar figures and similar to its prime, and as all the corresponding sections of the sphere are similar figures, viz., circles; and because the inscribed sphere touches the surface of the revoloid through their whole vertical sections; each of the conjugate sections of the revoloid are polygons, circumscribing the corresponding circular sections of the sphere. And because an indeffinite number of parallel plane sections may be regarded as a function of the solidities of the bodies through which such sections pass, the sum of a series of the revoloidal sections, equidistant and parallel to each other, is to the sum of a similar series of sections of the sphere, as the solidity of the revoloid to the solidity of the sphere; hence the solidity of the revoloid is to the solidity of the sphere as the area of its prime to the area of its inscribed circle.

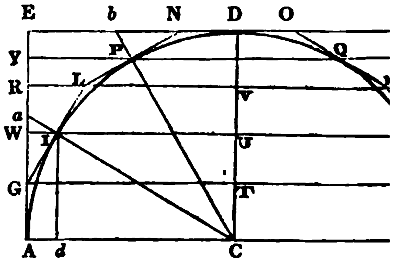
Cor. Hence, also, by a parity of reasoning in refere the perimeters of all the similar polygons formed by th jugate sections of the revoloid which circumscribe and all their corresponding circumferences of the sections inscribed sphere ; the whole surface of the revoloid is of the inscribed sphere as the perimeter of the polygon revoloidal section to the circumference of its inscribed c

Scholium. The proposition and corollary are also ma true in reference to any revoloid, and its inscribed solid volution, if the surfaces of the two solids correspond t vertical sections through the centres of the revoloidal s

PROPOSITION III. THEOREM.

The area of each facial side of a right revoloid is equal of the corresponding side of its circumscribing pris

Let ABDA be a vertical hemisphere of a right quad lar revoloid, and CD its axis; and let AEFB be a pri cumscribing the hemisphere; then will each facial side hemisphere of the revoloid be equal to its correspondin face of the prism.



For first, let a regular semi-polygon AGLNO,&c. be described about the figure representing the revoloid, and from the angles draw the lines GH, LM, &c., and if this is conceived to be done on every facial side of the revoloid included within such l the boundaries of surfaces, it would be a polyedron, whos ral faces AGHB, GLMH, LNOM, &c., correspond wi include the several faces of the revoloid. Now, since pothesis the polygon AGLN, &c., circumscribing the is regular the distance across the face of the polyedron, plane of a vertical section, is equal in each, viz., TV the of the face, GLMH is equal VD the width of the face I and they are each equal to GL or LN; although we ca represent those spaces in the diagram in consequence curvature in a plane perpendicular to this sheet. But it dent that the sides GL, IN of the polygon will truly re the breadth of those faces. And since those several fa trapezium, they are severally equal to half the sum of their parallel sides multiplied by their altitudes or width. Thus the face MH is equal to (½GH+½LM)×LG or VT, and since VT is supposed equal to LG; and since IK is equal ½GH+½LM, IK×LG equal the face GLMH; the face LNOM=PQ×LN, and the face AGHB=AB×AG. Now the areas of the corresponding parallel spaces or sections in the prism being parallelograms, are severally equal to their lengths multiplied by their breadths. Thus the space GRSH corresponding to the face GLMH, is equal GH×GR, and the parallelogram REFS corresponding to the face LNOM is equal RS×RE.

Now let IK be produced each way to W, X and PQ to Y, Z; and through the point I draw the radial lines C*a* C*b*, draw also I*d* perpendicular to AC. Then in the right-angled triangles LRG, *a*IG having an accute angle at G common, they are similar, (Prop. XXII, B. IV. *El. Geom.*), and hence their sides are proportional; and because the right-angled triangle CA*a* has an accute angle at a common with the right-angled triangle GI*a*, this triangle is also similar to the former, as also the triangle C*d*I or UIC. Hence GR : GL : : UI : CI, and because IK=2UI, and AB or GH=2CI, we have GR : GL : : IK : GH. That is, RG which is a factor of the parallelogram GRSH is to GL, which is the factor of the trapezium GLMH as IK, another factor of GLMH to GH another factor of GRSH; whence, by multiplying extremes and means, we have GR×GH=GL×IK, viz., the surface of the face GLMH of the polyedron is equal to the corresponding vertical surface GRSH of the prism; and since the same may be shown in reference to any other of the faces with its corresponding portion of the surface of the prism, it follows that the whole sum of the polyedral faces on one side, is equal to the whole corresponding surface of the prism. Let the number of the polyedral faces be indefinitely increased, and the truth of the proposition is still manifest, but when the faces are indefinitely increased, they become assimilated to that of the body about which they are described; therefore, the facial surface of each side of a right quadrangular revoloid is equal to that of the corresponding side of its circumscribing prism.

Cor. 1. Let us suppose the revoloid, instead of being quadrangular, to consist of any number of facial sides; then, by hypothesis, its circumscribing prism will consist of an equal number of vertical sides and in the same ratio each to each; and hence the proposition is true for a revoloid of any number of sides. Let, then, the number of sides be indefinitely increased, the reveloid then becomes a sphere, and the circumscribing prism becomes a cylinder. Hence the surface of a sphere is equal to the convex surface of its circumscribing cylinder.

Cor. 2. Hence the surface of a revoloid is equal to the perimeter of its centeral conjugate section multiplied by the vertical axis, for the vertical surfaces of the circumscribing prism are equal to this product; and since in a right quadrangular revoloid its circumscribing prism has six equal sides, four of which are equal to the surface of the revoloid, it follows that the whole surface of the revoloid is to the whole surface of the prism as 2 to 3.

If $\downarrow$ represent the perimeter of the central conjugate section, D the vertical axis or diameter, then $\downarrow D$ will represent the surface, and if H be the altitude of any zone by sections parallel to the conjugate axis, then will $H\downarrow=$the curve surface of the zone or segment of the revoloid or sphere.

Cor. 3. Hence also the surface of a sphere is equal to its circumference multiplied by its diameter or altitude, for the curve surface of its circumscribing cylinder is equal to this product. And since the surface of a great circle of the sphere is measured by the product of its circumference into half the radius, or by $\frac{1}{4}$ the diameter, (Prop. XVI. B. V. *El. Geom.*) Therefore the surface of a sphere is four times the area of its great circle: this is equal to $4\pi R^2$, (Prop. XIII, Sch. 3. *El. S. Geom.*) and because the two bases of the circumscribing cylinder are each equal to one of those circles, it follows that the whole surface of the cylinder is equal to six of those circles, and hence that the whole surface of the sphere is to the whole surface of the cylinder as 2 to 3, as before found in the elements of solid geometry.

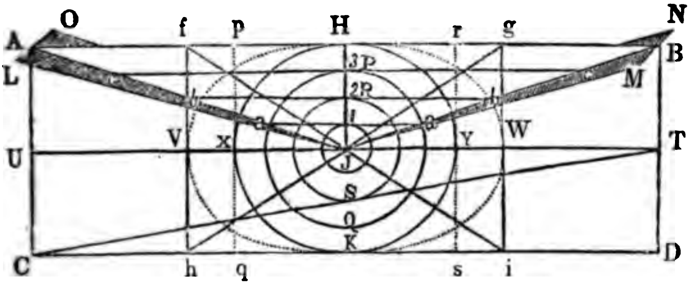
Cor. 4. Since we have shown that the surface GLMH of the polyedron is equal to the surface GRSH contained within the same parallel planes, it follows that the surface of any zone or segment LNOM either of a revoloid or sphere, is equal to the perimeter or circumference of a central conjugate section multiplied by the altitude of such zone or segment.

Cor. 5. The surface of two zones taken in the same revoloid or sphere, or in equal revoloids or spheres, are to each other as the altitudes; and the surface of any zone, is to the surface of the sphere, as the altitude or diameter of the zone is to that of the sphere. Hence the surfaces of every parallel portion of equal altitude are equal.

PROPOSITION IV. THEOREM.

If a cylindrical ungula be cut by two planes from the same side of the cylinder, the intersection of which planes forms a diameter to the cylinder, and if the altitude of the ungula, or the extreme length of the ungula taken in the direction parallel to the axis of the cylinder, is equal to the cylinder's circumference, then the sections or ungulas so cut, will be equal to a sphere described in the cylinder, or to a sphere whose diameter is equal to that of the cylinder.

Let ABCD be a cylinder, and let HK be a sphere of equal iameter described in the cylinder, and let AJB be an ungula cut rom the cylinder by the two planes OLJ, MNJ, from the points A and B, whose distance AB parallel to the axis UT is equal o the circumference of the cylinder, and let the cutting planes meet in J forming a diameter to the cylinder; then will the ection AJB be equal to the sphere HK.



For, let planes be passed through the ungula and sphere erpendicular to the diameter formed by the intersection of the lanes LOJ and NMJ ; and these plane sections, formed by hese planes in each solid, will be proportional to the magni-udes of the solid through such sections. Now, any section f the ungula, by a plane perpendicular to the diameter which asses through the pole J of the sphere, is a triangle; and a ection through the sphere made by the same plane, is a circle; nd the area of a triangle formed by any section, is equal to its ase multiplied by half its altitude on such base. Thus the rea of the triangle JAB=AB×½HJ; the area of a parallel secion J*cc*, is in like manner = the base *cc*×½ the altitude J3; and o for the area of any other parallel section J*bb*, J*aa*, &c.; and ecause these triangles are equiangular, (Prop. XXII. B. IV. El. *Geom.*) their bases are proportional to their altitudes; thus, HJ : AB : : J3 : *cc* : : J2 : *bb* : : J1 : *aa*, &c. The areas of the several circles formed by the same planes passing through the sphere, are equal to their circumferences multiplied by ½ their several radii; thus the area of the great circle of the sphere HXKY is equal to the circumference HXKY × ½ the radius HJ, the area of the circle PQ corresponding to the section J*cc*, is equal to the circumference PQ × ½ radius JP, and the areas of the circles RS, &c., = their several circumferences multiplied by ½ their several radii. And because the circumference of circles are to each other as their radii, (Prop. XV. B. V. *El. Geom.*) as radius JK : circle HK : : radius J3 : circle PQ : : radius J2 : circumference RS, &c.; and, since this is the ratio of the lines AB, *cc*, *bb*, &c., as shown above, the several circumferences are in the same ratio of those lines as bases of their several triangles; but the line AB by hypothesis, is equal to the circumference HK; hence the several circumferences PQ, RS, 1J, &c., are respectively equal to the several bases *cc*, *bb*, *aa*, &c.; hence, also, the areas of the several circles being sections of the sphere, are respectively equal to their several corresponding triangles, being sections of the ungula made by the same planes; and as this is true whatever may be the number of the parallel sections, or in whatever position they are taken, it follows that the solidity of the ungula is equal to that of the sphere.

Cor. 1. Since the section AJB may be regarded as composed of the two unglas AJH, BJH, regarding JH as their common base, and because ungulas of the same base are proportional to their altitudes, it follows that if we cut the ungula H*g*J, whose altitude H*g*=½ HB, and the ungula HJ*f*, whose altitude H*f*=½ HA, then the section *f*J*g* including those ungulas together with an equal opposite section, *h*J*i* are equal to the section AJB, consequently equal to the sphere HK. Or if we take the section CTD, whose base CD=AB, this section will also be equal to the sphere HK.

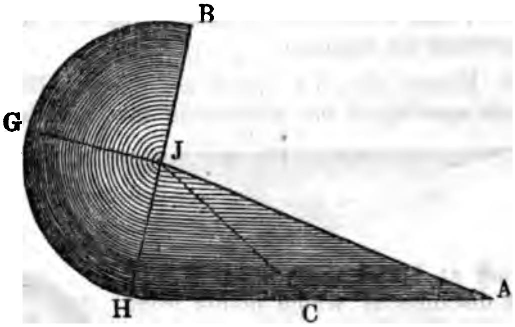
Cor. 2. Since, it may be shown, that all sections of a spheroid or ellipsoid, by planes parallel to its axis of revolution, are similar ellipses; and since ellipses are to their inscribed circles, as the diameter of the circle to the major axis of the ellipse, (Proposition IX of the Ellipse, B. I,) the solidity of an ellipsoid HWKV is to the solidity of the sphere HYKX, as the axis of revolution VW of the ellipsoid is to the axis of revolution XY of the sphere, and hence if an ungula, whose base is equal to half the base of the cylinder be taken, and whose altitude is to that of the ungula CTD as the axis of revolution VW of the ellipsoid to the axis of revolution XY of the sphere, that ungula will be equal to the ellipsoid HWKV made by the revolution of the ellipse on its axis VW.

3. As the ellipsoid HWKV is to the sphere HYKX
axis VW of the ellipsoid to the axis XY of the sphere,
the cylinder *fhig* circumscribing the ellipsoid is to the
r, *pqrs* circumscribing the sphere, as the same axis VW
same axis XY, or as the length of those cylinders res-
ively; the ellipsoid HWKV bears the same ratio to its
scribing cylinder *fhig*, as the sphere HYKX to its cir-
ibing cylinder *pqrs*. If S=the sphere, and E=the el-
and if P=the cylinder circumscribing the ellipsoid, and
cylinder circumscribing the sphere,

XY : S :: VW : E
XY : Q :: VW : P,

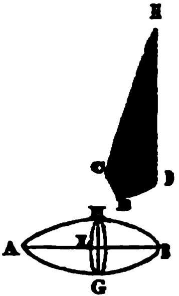
e, (Prop. XIX. B. I. *El. Geom.*) S : Q :: E : P.

lium. The segment AJB being equal to the sphere HK,
nexed figure, may represent the manner in which they

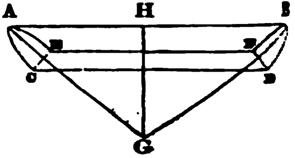


nvertible into each other, as there exists no mathema-
eason why the segment AJB may not be changed, as
represented in the figure.

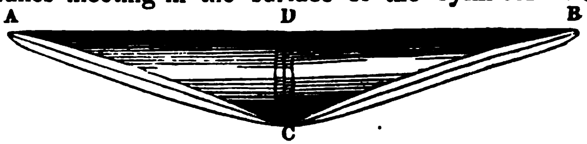
Cor. 4. It is also evident that an ungula BCDH, whose base BCD is the segment of a circle, similar to a segment AHB of a vertical section through this circular spindle AHBG; if the altitude DH of the ungula is equal to the circumference of the conjugate section HG of the spindle, the solidity of the ungula will be equal to that of the spindle, so also will its curve surface.

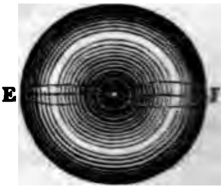


And, if a cylinder segment ABCDH be so cut that the length AB shall be equal to the circumference of a circle of which HG is the radius, and AG and BG lie in the planes ACE, BFD, then may the segment be converted into a ring whose outside diameter is equal 2HG; and every section of the ring will be equal to a section through the segment.

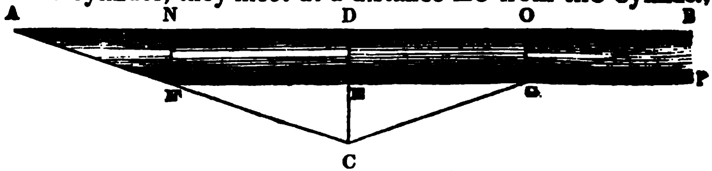


Cor, 5. Hence, also, if a cylindric segment ABC be cut by two planes meeting in the surface of the cylinder at C, and terminating at A and B on the opposite side, the distance of which points from each other is equal to the circumference of a circle whose radius is CD; the segment so cut may be changed in the form represented by EF, which is the form of a cylindrical ring, but without an opening through the centre.

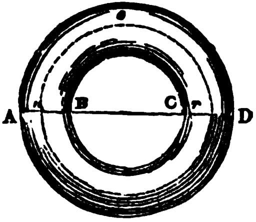




And if instead of the cutting planes meeting in the surface of the cylinder, they meet at a distance EC from the cylinder, and at such an angle that the distance FG shall be equal to the circumference of a circle whose radius is CD, and the section AFGB will form a cylindric ring, whose inner diameter is equal to twice CE.



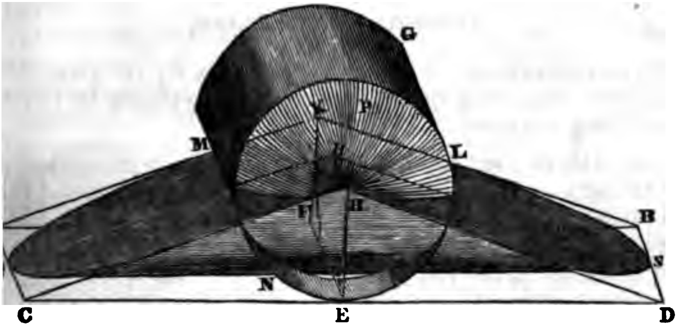
Cor. 6. And because the section ANF, (see diagram above,) cut from its position and placed in the position BGP, comletes the cylinder NFPB, which is ual in length to half the sum of sides G and AB. The solidity of a cylinric ring AB, is equal to that of a cynder whose base is equal to a radial ection of the ring, and whose altitude equal to half the sum of its inner and ter circles. And hence, cylindric rings hose sections are equal, are proportional to their inner or ter circumferences.



PROPOSITION V. THEOREM.

he solidity of a cylinder circumscribing a sphere, is equal to the solidity of a prism circumscribing the cylindrical ungula or ungulas whose solidity is equal to the sphere.

Let MNGL be a cylinder circumscribing the sphere PK; nd let CRDBKA be a prism circumscribing the two similar ngulas QRKV, SRKV described on the base, KRV=half



base of the cylinder from which they are conceived to be en; and if QS, the sum of their altitudes, is equal in length the circumference of the cylinder, then (Prop. IV.) will the gulas equal the sphere of equal diameter to that of the cy-ler, and the cylinder equal MNLG will be equal to the sm CRDBKA.

For the altitude AC or KR of the prism, on the base CRD, qual to the altitude of the cylinder; GL being=the axis, both the cylinder and the sphere; and the length of the side CD the prism is equal to the sum of the altitudes VQ, VS, or length QS of the ungulas taken on the surface of the cy-ler; but the length QS is equal to the circumference of the sphere or cylinder by hypothesis. Now, the area of the base LN of the cylinder, is equal to its circumference multiplied by half the radius RE; and the area of the base CDR of the prism, is equal to the line CD, or the circumference of the cylinder multiplied by half the line RE; hence, the base of the prism is equal to that of the cylinder. And the solidity of the cylinder is equal to its base ENL multiplied by its altitude GL; the solidity of the prism is also equal to its base CDR, or the base of the cylinder multiplied by its altitude RK; hence, the solidity of the cylinder is equal to that of the prism.

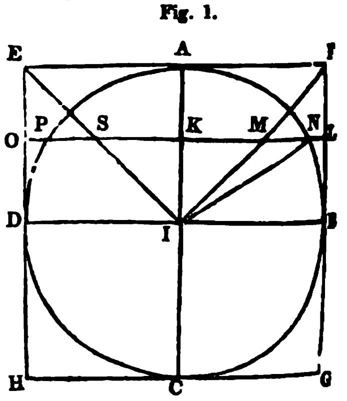
Cor. As the sum of two or more ungulas of equal base, are equal to one greater of the same base, if the sum of their altitudes is equal to the altitude of the greater, (Pr. IX. Cor. 2. B. II.) and because a prism circumscribing an ungula is proportional to the altitude of the ungula, two or more prisms circumscribing ungulas, the sum of which is equal to a sphere, are equal to the cylinder circumscribing that sphere; and also, the several prisms circumscribing ungulas equal to a revoloid, are equal to the prism circumscribing the revoloid composed of those ungulas.

PROPOSITION VI. THEOREM.

Every right revoloid is equal to two-thirds its circumscribing prism; and every sphere is equal to two-thirds its circumscribing cylinder.

Let ABCD, (fig. 1.) be a revoloid, or a sphere circumscribed by the prism or cylinder EFGH; then will the revoloid ABCD be equal to two-thirds the prism EFGH; and the sphere ABCD will be equal to two thirds of the cylinder EFGH.

Fig. 1.



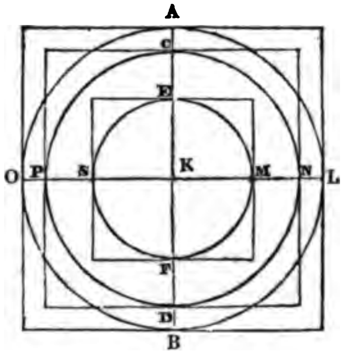
For let the plane EG be a vertical section through the centre of the revoloid and prism bisecting their opposite sides; in which position, (Def. 8.) the section of the revoloid is a circle, and the section of the prism through the same plane is evidently a rectangle. These sections are, also, evidently those of a sphere and its circumscribing cylinder, made by a plane through the common axis, AC, of the sphere and cylin-

ler, (Prop. 1, *Sph. Geom.*) and (Def. 5, B. 1,) through the centre,
, join EI, FI ; also let AIC, as an axis, be parallel to EH or
G; and DIB and KL parallel to EF, or HG, the base of the
ection of the prism or cylinder, the latter line, KL, meeting
I in M, and the circular section of the revoloid or sphere in
; and the plane EIF will represent the vertical section of a
yramid of equal base to that of the prism, and an altitude, IA :
r it will represent a vertical section of a cone of equal base
that of the cylinder and of an altitude, IA.

Now, if the line KL produced, if nececsary, be conceived
revolve on the axis AC, it will cut conjugate sections of those
lids in the relation of their magnitudes, viz: KS the section
f a prism or cylinder, KN the section of a revoloid or sphere,
d KM the section of a pyramid or cone.

Now, AF being equal to AI or IB, and KL parallel to AF,
hen by similar triangles IK = KM, (Prop. XVII, B. IV, *El.
eom.*,) and since, in the right angled triangle, IKN, $IN^2=IK^2$
$+KN^2$, (Prop. XXIV, B. IV, *El. Geom.*) and, because, KL is
qual to the radius IB or IN, and KM=IK, therefore, $KL^2=$
M^2+KN^2; or the longest line forming the section of the
rism or cylinder, is equal to the sum of the squares of the
ther two, forming sections of the revoloid or sphere, the py-
amid or cone.

Fig. 2.



Let now the conjugate sec-
ions of those solids formed by
he revolution of the lines KL,
e represented. (Fig. 2.)

Thus let the square OALB, re-
resent the conjugate section of
prism, described about the
uadrangular revoloid, and let
he circle OALB be the section
f a cylinder circumscribing
he sphere ; the square PCND
ill represent the section of
he the revoloid, and the circle
CND will represent the sec-
ion of the sphere ; the square SEMF will represent the sec-
ion of the pyramid, and the circle SEMF a section of the
one. Now, since we have shown that KL^2, or the square
L described on the line KL is equal to $KN^2 + KM^2$, or the
uare CN + the square EM, being squares described on the
nes KN and KM ; it results that the square OALB, described
n the line OL, which is double the line KL, is equal to the
m of the squares PCND, SEMF, described on the lines
N, SM, being double the lines KN, KM. (Prop. XII, B. I,

El. Geom.) That is, the section of the prism is equal to both the sections of the revoloid and pyramid. And because circles described on the diameters SM, PN, OL, are proportional to the squares described on those diameters, respectively, (Prop. XIV, B. V, *El. Geom.*) it follows that the circle OALB, is equal to the two circles PCND, SEMF; that is, the section of the cylinder is equal to the two sections of the sphere and cone. And because, (Prop. I, Cor. 4,) the sections of a revoloid, of its circumscribing prism, its inscribed pyramid of any number of sides, are similar figures; they are, (Prop. XIV, Cor. 3, B. V, *El. Geom.*) proportional as the squares of the diameters of their inscribed circles, hence the section of a prism of any number of sides circumscribing a revoloid, is equal to the sum of the corresponding sections of the revoloid and pyramid, and as this is always true in any parallel position of the revolving line KL, (Fig. 1.) it follows, that the prism, EB, circumscribing the hemisphere DAB of the revoloid, being composed of all the former sections is equal to the hemisphere DAB of the revoloid and pyramid EIF, composed of all the latter sections; and that the cylinder EB, is, in like manner, equal to the hemisphere DAB and cone IEF. But the pyramid IEF is a third part of the prism DEFB; (Prop. XXVI, Cor. 1, B. II, *El. S. Geom.*) consequently, the hemisphere DAB of the revoloid, is equal to the remaining two-thirds. And the cone IEF, (Prop. VIII, Cor. 1, B. III, *El. S. Geom.*) is equal to one-third of the cylinder, DEFB: hence, the hemisphere DAB, is equal to the remaining two-thirds of the cylinder.

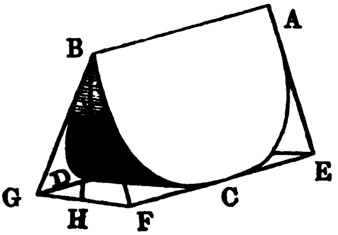
Cor. 1. A pyramid, revoloid, and prism, are to each other as the numbers 1, 2, and 3, when the bases of the pyramid and prism are each equal to the prime of the revoloid, and when their altitudes are all equal. Also, a cone, sphere, and cylinder, are, in like manner, proportional as the numbers 1, 2, and 3, if the base of the cone and cylinder are each equal to the great circle of the sphere, and if the altitude of the cone and cylinder are each equal to the diameter of the sphere.

Cor. 2. All spheres, and all similar revoloids are to each other as the cubes of their diameters, all being like parts of their circumscribing cylinders or prisms.

Cor 3. From the foregoing demonstration, it appears, that the revoloidal frustum DBNP, is equal to the difference between the prism DBLO, and the pyramid SIM, all of the same common height IK. And that the revoloidal segment, PAN, is equal to the difference between the prism, EFLO, and the

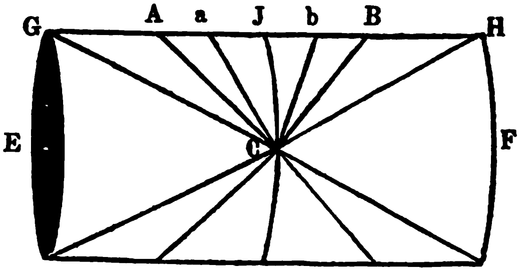
oidal frustum EFMS, all of the same common alti-
AK.
d the same is true of any parallel segment or frustum of
here, cylinder and cone.

r. 4. The sphere may be regarded as a revoloid whose
has an infinite number of sides, which (Prop. XII, Cor.
V, *El. Geom.*) is identical with a circle.



r. 5. From the above de-
tration, and from propo-
fourth, it may be shown
any ungula, being a por-
f the revoloid, is equal to
hirds of its circumscribing
or wedge. Thus, if the
a ABDC be such as forms
tion of a revoloid, it will
ual to two-thirds of its circumscribing prism, BGFEA,
this ungula is the same part of the revoloid as its cir-
cribing prism, is of the prism, circumscribing the revo-

nce, if from the cylinder EF any ungulas GCH, ACB,
GC*a*, &c., are taken, whose cutting planes meet in the
e, C, of the cylinder or anywhere in its axis, those ungu-
e each equal to two-thirds of their circumscribing prism



edge; these being ungulas such as may compose a right
oid, and (Prop. IX, Cor. 2, B. II.) they are proportional to
altitudes or lengths taken on the surface of the cylinder;
the ungula GH, which may be considered as composed
e two ungulas GCJ, HCJ; their common base being the
on through the line CJ, is to the ungula ACB, as the line
to the line AB, &c. And a prism circumscribing an un-

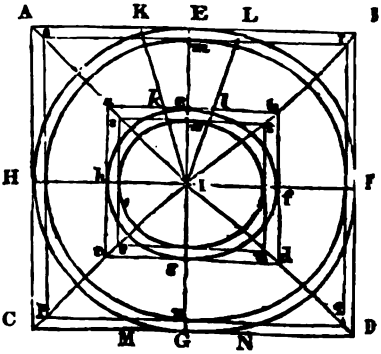
gula is evidently proportional to the length of those lines, forming one of the dimensions of one of its sides.

Cor. 6. Since all the vertical sections of a right revoloid except those bisecting its sides at right angles, are ellipses, (Prop. I, Cor. 1.) it may be inferred that an elliptical revoloid is also equal to two-thirds of its circumscribing prism; and hence that an ellipsoid is also equal to two-thirds its circumscribing cylinder. For the ellipsoid evidently bears the same relation to the elliptical revoloid as the sphere does to the right revoloid, since the elliptical revoloid may, by multiplying the number of the sides of its prime, be shaded off into the ellipsoid without changing its relation to its circumscribing prism, which in such case becomes a cylinder.

Scholium. Lest this latter corollary should not appear sufficiently satisfactory in view of the important principle enunciated; and as it may seem to require a more rigorous demonstration, it will be made the subject of the following proposition.

PROPOSITION VII. THEOREM.

An elliptical revoloid is equal to two thirds its circumscribing prism; and an ellipsoid is equal to two-thirds of its circumscribing cylinder.



Let us imagine the annexed figure ABCD, to be a conjugate section of a right quadrangular revoloid by a plane through its centre, and let *abcd*, be regarded as another conjugate section of the revoloid between the central section and the extremity of the axis, and let them be so projected that the axis I, perpendicular to the plane of projection may be supposed to pass through their centres; and let the circles EFGH, and *efgh* be similar sections of a sphere of the same diameter and similarly posited.

Now let it be conceived that vertical sections or ungulas be taken from each side of the revoloid and sphere; such that KIL, *klI*, shall be sections of the ungula from the side

AIB, then will the sectoral portions of those triangles be respectively, sections through the spherical ungulas; now let the plane sections of the revoloid and sphere be reduced so as to close the spaces formed by the triangles so removed, but retaining their former figures. Thus let ABCD be reduced to *opqr*; *abdc* to *stuv*, &c., then, because (Prop. XXIII., B. I., *El. Geom.*) if any number of squares are proportional, the sides of those squares are proportional, it follows that if P be = the length of the side AB of the revoloidal section, and P' = the length of the sides *or*, and if p = the length of the side *ab* and p' = the length of the side *st*; then will P^2 = the square ABDC and P'^2 = the same square reduced, or *opqr*; p^2 will = the square *abdc* and p'^2 = the same square reduced, or *stuv*.

And we have shown that $P^2 : P'^2 :: p^2 : p'^2$

Hence also (Prop. XX, B. I, *El. Geom.*) $P : P' :: p : p'$

That is, as the side AB is to the same side reduced, so is the side *ab*, to the same side reduced, and hence the sides, or diameters of those sections are reduced by the removal of the ungulas, in the ratio of those sides, or their diameters respectively, and because the circumferences of circles are as their diameters, and their areas as their squares, the diameters of the circular sections of the sphere, are each reduced in the ratio of these diameters; and the same will hold true with regard to any parallel sections of the revoloid or sphere.

Now the several radii EI, *e*I, &c. of the several sections may be regarded as ordinates to the vertical axis, I, of the revoloid and sphere; and the radii *m*I, *w*I, &c., may be regarded as the corresponding ordinates of the vertical sections of the solids so reduced. Now, since these latter ordinates are severally proportional to their corresponding ordinates of a vertical circular section of the revoloid and sphere, it follows that the same sections made by the same plane, through the solids so reduced are ellipses, for (Prop. XII, *Cor. Ellipse*) this is a property of an ellipse, when compared with a circle; hence the revoloid becomes an elliptical revoloid (Def. 10) and the sphere becomes an ellipsoid.

Now if a prism is supposed to circumscribe the revoloid before being reduced, and if a cylinder is supposed to circumscribe the sphere, they must in order to accommodate themselves to the elliptical revoloid and ellipsoid, be reduced in every conjugate section, equal in amount to the reduction of the central conjugate sections of the revoloid and sphere; and hence the prism will have been reduced in the same ratio as that of the revoloid; and the cylinder will in like manner, have been reduced in the same ratio as that of the sphere, so that if (Prop. VI) a right revoloid is equal to $\frac{2}{3}$ of its circumscribing prism, and a sphere is equal to $\frac{2}{3}$ of its circumscribing cylinder, then also will an elliptical revoloid be equal to $\frac{2}{3}$ of its circumscribing prism, and an ellipsoid will be equal to $\frac{2}{3}$ of its circumscribing cylinder.

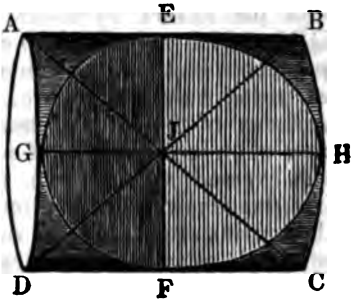
Cor. Since an elliptical revoloid is formed of ungulas, cut from an elliptical cylinder (Prop. I, Cor. 2) whose bases are severally the semi-base of the cylinder, it follows that such ungulas of an elliptical cylinder, are equal to $\frac{2}{3}$ of their respective circumscribing prisms.

Scholium. Since the solidity of a cylinder may be expressed by $\pi R^2 \times H$ (Prop. II., *Sch.* B. III., *El. S. Geom.*,) that is since its solidity is as the square of its radius or diameter multiplied by its height, it follows that ellipsoids are proportional to each other as the square of their revolving axes multiplied by their fixed axes, and hence the same is true also of the elliptical revoloid, If $R^2\pi H =$ the solidity of a cylinder,
$\frac{2}{3} R^2\pi H =$ the solidity of a spheroid
the solidity of a prism will be $4R^2H$,
and the revoloid will be $\frac{8}{3} R^2H$,
or if $D = 2R$ then the revoloid is $= \frac{2}{3} D^2H$.

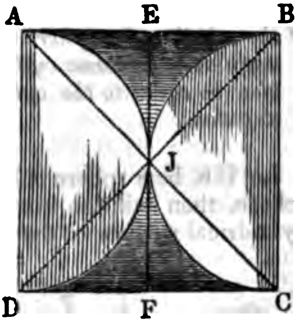
Cor. 2. As in Cor. 3, prop. VI, in relation to the segments of a right revoloid, or a sphere, so in relation to the segments of an elliptical revoloid, or spheroid, they are respectively equal to the difference between the corresponding sections of their circumscribing prisms or cylinders, and inscribed pyramids or cones. (See diagrams Prop. VI).

Cor. 3. Since a spheroid is equivalent to a sphere drawn out as in the case of a prolate, or contracted, as in the case of an oblate spheroid, and in such manner, as that every line or section through the spheroid, in the direction of the expansion or contraction, is drawn out or contracted, in the ratio of the increase or decrease of the axis in such direction, it follows that any segment of a spheroid, by a plane parallel to its axis of revolution, is to a corresponding segment of its inscribed sphere, if a prolate spheroid, or that of its circumscribing sphere if the spheroid is oblate, as the diameter of that sphere to the axis of the spheroid, or as in the conjugate axis of the segment's base to the transverse of the base, or the axis parallel to the axis of the spheroid.

And any segment by a plane perpendicular to the axis is also proportional to a corresponding segment of the sphere from which it may be conceived to be produced as the axis of the sphere to the axis of the spheroid, or as the height of a similar spherical segment to the height or altitude of the segment of the spheroid.

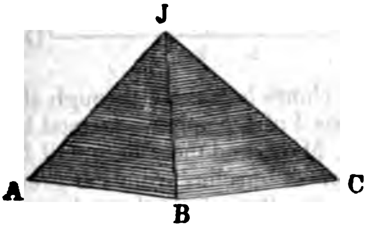


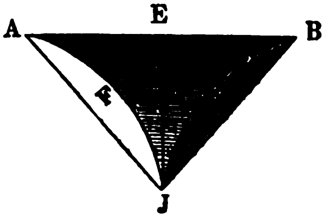
Scholium. Let ABCD be the complement of a cylinder from which is taken the ungula GEFH=a quadrant of a revoloid, and also a similar opposite ungula, cut by planes meeting in the diameter EF; and there may be taken two cones whose bases are the two bases of the complement, and whose vertices are in the centre J, and the parts remaining will be equal to the remaining cylindrical surface $\times \frac{1}{3}$ the radius of the base or distance JF. (Prop. IV., B. II.)



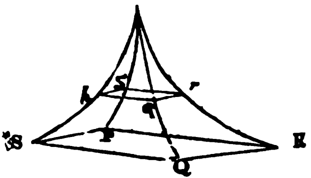
Let the complement be divided in the line EF, and let the segments be inverted so that the bases shall comprehend the line EF, and if the planes ABJ, DCJ cut off the segments AEBJ, DFCJ then there shall be left the pyramid AJ,BJ,CJ,DJ,whose base is equal to a central section of the cylinder along its axis; viz., ABCD, and its vertical height is equal to the radius of the cylinder; and as each side of the cylinder is supposed to be cut alike, we shall have two of those pyramids, which together are equal to one-third of the prism circumscribing the revoloid.

It follows therefore that the two ungulas together with the two pyramids are equal to a full quadrangular revoloid.





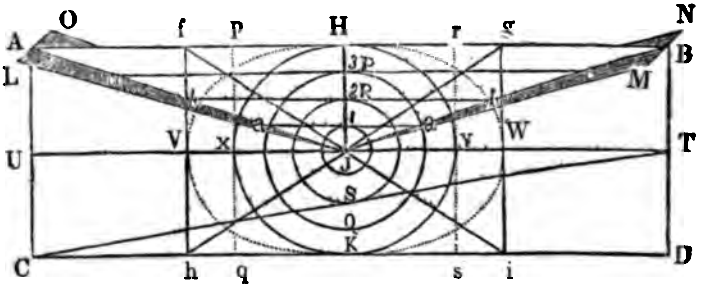
Hence there remains four portions ABJFH to be determined, which when placed together, so that their several vertices J, shall coincide, their cylindric surfaces turned inward, their plane surfaces will be outward, forming a pyramid equal to one of the former pyramids, minus a pyramidal portion PSQRJ, which is required to complete the pyramid. It will be perceived that every section *pqrs* of this latter solid, parallel to the base is a square, and = the square of *pq*, the versed sine of the arc J*p*, therefore this solid is equal to the squares of an infinite series of equidistant versed sines drawn into their distance; or is to its circumscribed prism, erected on the same base PSQR as an infinite series of the squares of equidistant versed sines to a similar series of the squares of radii, as will be more fully discussed in another place.



PROPOSITION VIII. THEOREM.

If the solidity of a sphere is equal to one or several cylindrical ungulas of the same cylinder, the surface of the sphere will also be equal to the cylindrical surface of such ungula or ungulas.

Let HK be a sphere AJB a cylindric ungula equal to the sphere, then will the surface of the sphere be equal to the cylindrical surface of the ungula.



For let an indefinite number of planes be passed through the two solids perpendicular to the axis J of the sphere, formed by the intersection of the planes LOJ, MNJ, and the sphere will be divided into an indefinite number of circles, from the great circle of the sphere down to the smallest about the axis, and the ungula will be divided in like manner into an indefinite number of similar triangles, with bases AB, *cc*, *bb*, &c., which was shown (Prop. IV) to be equal to the circumference of the circles through the corresponding sections of the sphere; and because this is the case throughout, it follows that the surface of the sphere, which may be represented by the sum of the several circumferences of the circles, is equal to the cylindric surface of the ungula, which may likewise be represented by the sum of the several bases of the triangles.

And because cylindric ungulas of the same base are proportional as their altitudes, (Prop. IX, Cor. 2, B. II) and because their cylindrical surfaces are proportional to their altitudes, it follows that if several ungulas, cut from a cylinder of a given diameter are equal to a sphere of the same diameter, the surface of the sphere will be equal to the sum of their cylindrical surfaces.

PROPOSITION IX. THEOREM.

The solidity of a sphere as well as a revoloid, is equal to the product of its surface by one-third of its radius.

For since the revoloid is made up of sections of the cylinder, whose several solidities are equal to their curve surface multiplied by one-third of the radius of the cylinder, whence they are conceived to be taken (Prop. III, B. II) which radius is equal to the vertical height of the several elimentary pyramids of which these sections are formed, and is also the radius of the revoloid, it follows that the solidity of the whole revoloid composed of all the sections, is equal to the whole curve surface of all the sections multiplied by one-third the radius.

Thus, if the revoloid consist of six facial sides a,b,c,d,e,f, the solidity of each of which is equal to its surface × $\frac{1}{3}$ radius or $\frac{r}{3}$, or surface $a \times \frac{1}{3}r =$ solidity a, surface $b \times \frac{1}{3}r =$ solidity b, &c., their surface $\overline{a+b+c+d+e+f} \times \frac{1}{3}r$ equal to the solidity of the whole revoloid, and as the number of sides of a revoloid may be increased indefinitely without altering the relation of its elimentary pyramids, it follows that the same relation exists between the solidity of the sphere, and its surface, as in the revoloid, viz., the solidity of a sphere, as also of a revoloid, is equal to the product of their respective surfaces by one-third of their respective radii.

Scholium. 1. Conceive also a polyedron, all of whose faces touch the sphere; this polyedron may be considerd as formed of pyramids, each having for its vertex the centre of the sphere, and for its base one of the polyedrons faces. Now, it is evident that all these pyramids will have the radius of the sphere for their common altitude; so that each pyramid will be equal to one face of the polyedron multiplied by one-third of the radius; hence the whole polyedron will be equal to its whole surface multiplied by a third of the radius of the inscribed sphere (Prop. XV. Cor. B. III. *El. S. Geom.*) It is therefore manifest that the solidities of polyedrons, as well as revoloids circumscribed about a sphere, are to each other as the surfaces of those polyedrons or revoloids respectively.

Now, also, as with a revoloid, the number of polyedron's faces may be inscribed till the polydron becomes identical with the sphere, and then its solidity is equal to the product of its surface with one-third of its radius; hence the sphere may be conceived to be made up of an indefinite number of indefinitely small pyramidals, whose bases when associated, form the surface of the sphere, and this surface has the same relation to the whole solid, or the sphere, as the base of each individual pyramidal has to the solidity of each.

Cor. 1. Hence the solidity of any sector of a sphere or of a revoloid is equal to its spherical or cylindrical surface multiplied by $\frac{1}{3}$ of the radius, for a sector consists of an association of regular pyramidals, the sum of whose bases form the curve surface of the sector.

Scholium. 2. Since the axis of the cylinder circumscribing a sphere is equal to its diameter, its solidity is equal to its whole surface, including the two ends multiplied by a third of the radius. For it may be conceived to be made up of two cones, whose bases are two ends of the cylinder, and vertical height = half the axis or length of the cylinder = radius; and the elementary pyramids of the curve surface, whose vertices terminate in the centre of the cylinder with those of the cones, and hence, its solidity bears the same ratio to its surface, that the solidity of a sphere, a right revoloid, or polyedron, circumscribing a sphere do to their respective surfaces.

Cor. 2. Hence we have three orders of surfaces, which, taken as bases of pyramids, and multiplied by one-third of the distance of such base to the vertice of the pyramid, will determine the solidity of such pyramid; but, as observed in Schol. to Prop. III, B. III, *El. S. Geom.* in reference to cylindrical surfaces, the vertice of the pyramid with a spherical base, must be in the centre of the spherical curvature.

Cor. 3. Hence, as the solidity of a sphere is equal to two-thirds that of its circumscribing cylinder, and as the surface of the sphere is equal to the curve surface of the cylinder; and as the solidities of each of these bodies are equal to the products of their respective surfaces, by one-third of their common radius, the surface of the two ends of the cylinder is equal to half the curve surface, for if $a =$ the surface of the sphere, and $x =$ the two bases of the cylinder, then will $a + x =$ the whole surface of the cylinder, including the ends. Then $a \times \frac{1}{3}r$ or radius $=$ the solidity of the sphere, and $\frac{1}{3} ra + \frac{1}{3} rx =$ the solidily of the cylinder.

But $\frac{1}{3} ra = (\frac{1}{3} ra + \frac{1}{3} rx) \times \frac{2}{3} = \frac{2}{9} ra + \frac{2}{9} rx$.

Transposing and dividing $\frac{1}{3} a = \frac{2}{3} rx$.

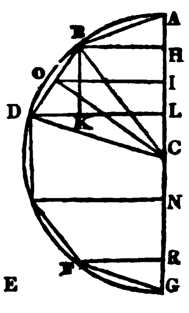
Hence $a = 2x$, therefore the area of the two ends is equal to half the area of the convex surface of the cylinder.

The same may be inferred from the ratio of the inscribed cones, to the remaining portion of the cylinders.

Scholium. 3. Since the surface of a sphere whose radius is R, is expressed by $4\pi R^2$ (Prop. III, Cor. 2,) it follows that the surfaces of spheres are to each other as the squares of their radii; and since their solidities are as their surfaces multiplied by their radii, it follows that the solidities of spheres are to each other as the cubes of their radii or diameters, and the same is true also of revoloids. If the diameter is called D, we shall have $R = \frac{1}{2} D$, and $R^3 = \frac{1}{8}D^3$; hence the solidity of the sphere may likewise be expressed by $\frac{4}{3} \pi \times \frac{1}{8} D^3 = \frac{1}{6}\pi D^3$.

PROPOSITION X. THEOREM.

Every segment of a sphere is measured by the half sum of its bases multiplied by its altitude, plus the solidity of a sphere whose diameter is this same altitude.



Let BH, DL, be the radii of the two bases of a segment, HL its altitude, the segment being generated by the revolution of the circular zone DLHB, about the axis AG passing through the centre of curvature C; from C draw CO perpendicular to the chord DB, draw also the radii CD, CB. The solid described by the section BCD is measured by $\frac{2}{3} \pi$, CB^2, LH (Prop. XXIII, Sch. 2, B. III., *El. S. Geom.*); but the solid described by the isosceles triangle DCB, has for its measure $\frac{2}{3} \pi$. CO . LH, (Prop. XVII, Cor. B. III, *El. S. Geom.*,); hence the solid described by the segment BDO $= \frac{2}{3} \pi$. LH .

$(CB^2—CO^2)$. Now, in the right angled triangle CBO, we have $CB^2 - CO^2 = BO^2 = \frac{1}{4} BD^2$; hence the solid described by the segment BDO, will have for its mearsure $\frac{2}{3} \pi . LH . \frac{1}{4} BD^2$, or $\frac{1}{6} \pi BD^2 LH$.

Again, the solid described by the trapezeium BDLH is $= \frac{1}{3} \pi LH . (BH^2—DL^2+BH . DL,)$ (Prop. X, B. III, *El. S. Geom.*) Hence the segment of the sphere, which is the sum of those two solids, must be equal to $\frac{1}{6} \pi . HL, (2BH+2DL+2 BH . DL+BK^2.)$ But since BK is parallel to HL, we have $DK = DL — BH$, hence $DK^2 = DL^2 — 2DL . BH + BH^2$ (Prop. IX, B. IV., *El. Geom.*); and consequently $BD^2 = BK^2 + DK^2 = HL^2 + DL^2 — 2DL . BH + AH^2$. Substitute this value for BD^2 in the expression for the segment, omitting the parts which cancel each other, we shall obtain for the solidity of the segment $\frac{1}{6} \pi HL . (3BH^2 + 3DL^2+HL^2)$, an expression which may be decomposed into two parts; the one $\frac{1}{6} \pi . HL, (3BH^2+3DH^2)$, or $HL . \frac{(\pi BH^2+\pi DL^2)}{2}$ being the half sum of the bases × by the altitude; while the other $\frac{1}{6} \pi . HL^3$ represents the sphere of which HL is the diameter. The same may be proved of any other segment DE, EF, &c.; hence the proposition is manifest.

Cor. 1. If either of the bases is nothing, the segment in question becomes a spherical segment, with a single base; hence any spherical segment with a single base is equivalent to half the cylinder, having the same base and altitude + the sphere of which this altitude is the diameter.

Cor. 2. If IC is perpendicular to AC the solid described by the revolution of the segment about the axis AC, is a ring, when if DC or BC be the radius of curvature of the arc DB of the segment, it is (Def. 26) a spherical ring, and in such case the ring is equivalent to a sphere, whose diameter is the altitude of the segment from which the ring is taken.

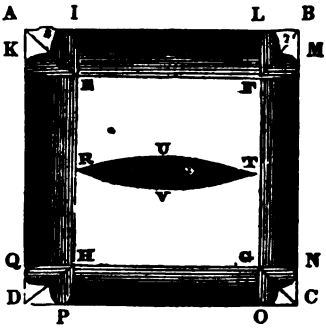
Cor. 3. We may hence infer that every segment of a right revoloid included between two parallel planes perpendicular to its transverse axis, is also measured by the half sum of its bases multiplied by its altitude, plus the solidity of a similar revoloid, whose diameter is this same altitude; and that the solidity of the belt ABCD, (see diagram to next proposition,) taken from the middle segment or zone of a revoloid, is equivalent to a right revoloid of similar type, whose diameter is this same altitude.

4. It may be inferred also that every segment of an
al revoloid, or of an ellipsoid, cut by planes perpendicu-
their vertical or transverse axis, is equal to half the
its bases multiplied by its altitude + the solidity of a
revoloid or spheroid, whose vertical axis is this same
e, since the spheroid is the sphere drawn out as in the
, and constructed as in the oblate spheroid, and since
ne is true of the elliptical revoloid, compared with the
evoloid ; and hence that the solidity of a belt taken from
ldle zone or segment of an elliptical revoloid is equivalent
nilar revoloid, whose vertical axis is the altitude of the
it or breadth of the belt, and the same is also true of the
ken from the middle segment of a spheroid, viz., that it
valent to a similar spheroid, whose axis of revolution is
tude of the segment or breadth of the ring.

PROPOSITION XI. THEOREM.

gular portions of a revoloidal belt taken from the mid-
one of a quadrangular revoloid, are together equal to a
loidal spindle, whose verticle section through its opposite
is a double segment of a circle, equal to the segment con-
ting a conjugate section of the belt.

ABCD be a belt taken from the middle zone of a quad-
ar revoloid, and let its conjugate section IES or KES
segment of a circle ; and the angular portions KAIE,
, NCOG, and PDEQH will be equal to the revoloidal
RUTV whose vertical section through the centre of its
e sides is a similar double segment of the same or an
ircle.



let the prismatic cylin-
ngulas IESLF, FMNG,
, HQ KES, be removed,
t these several angular
be brought in juxta-posi-
r the contact of their cor-
ding faces, thus let FLM
ght in the position IES, so
T shall coincide with IES,
O and HQP be brought in
with FML and EIK, so
GN, HQ, shall coin-
ith FM, EK. For since the faces of these angular
s are all similar and equal to each other, they would co-
when placed in contact each to each, and because the

corresponding faces are parallel to each other in the belt, they would, when placed in contact, form the same angular space as in the belt, and because they form all the angular space about the axis EI when in contact they will form a body, whose conjugate section is a square, and whose vertical section through the centre of its sides is the double segment of a circle, which (Def. 24) is a revoloidal spindle.

Cor. 1. Hence, we may infer that the angular portions of any revoloidal belt taken from the middle frustum or zone of a revoloid of any number of sides, is equal to a revoloidal spindle of the same number of sides, and whose vertical section through its opposite sides is equal to a double segment of the circle composing a section of the belt. And since the angular space is the same in a circular ring as in a circle or about a point, (Prop. XXI. B. V. *El. Geom.*) it follows that the angular space of a belt or ring taken from the middle zone of a sphere, is equal to a circular spindle formed by the revolution of a section of the ring, being a circular segment about its chord as an axis.

Cor. 2. Therefore, a belt from the middle zone of a revoloid of any number of sides, may be resolved into prismatic ungulas equal in number to the number of the sides of the revoloid, the bases of which are severally equal to segments of the circle forming a conjugate section of the belt, and whose altitudes are each equal to the length of the sides of the polygon forming the inner portion of the belt; and one perfect revoloidal spindle whose vertical section through the centre of its opposite sides, is the double segment of the circle, each equal to the segment formed by a section of the belt.

Hence, the solidity of a revoloidal belt is equal to that of a prismatic cylindrical ungula, whose base is the section of the belt and whose altitude is equal to the perimeter of the inner surface of the belt, plus a revoloidal spindle whose vertical section through its opposite sides is the double segment of the section of the belt.

Cor. 3. The solidity of a ring taken from the middle zone of a sphere, is equivalent to that of a prismatic cylindrical ungula, whose base is the section of the ring, and whose altitude is equal to the inner circumference of the ring, plus a circular spindle formed by the revolution of the segment, forming a section of the ring about its chord as an axis.

Scholium 1. Let Z = the middle zone of a revoloid, P =

he prism formed by taking a belt from this zone; then will
Z — P = the belt.

Let S = the segment formed by a section of the belt, and
p = the perimeter of the inner surface of the belt, and S =
he solidity of the spindle formed by the angular portions of
he belt; then will $s \times p$ or sp = the prismatic cylindrical ungulas, and Z — P — sp = s = the revoloidal spindle.

Scholium 2. Let Z = the middle frustum of a sphere, P =
he cylinder after taking away the ring whose section is the
egment of the circle of which the spherical surface of the
egment is formed, and Z — P = the ring.

Let S = the segment formed by a section of the ring and
π = the inner circumference of the ring, and $s\pi$ will = the
rismatic cylindrical ungula which is conceived to form a portion of the ring. Let s = the spindle formed from the angular portions, and Z — P — $s\pi$ = s = the circular spindle.

Scholium 3. Let the length or perimeter of a revolidal belt or
pherical ring, taken from the middle zone of a revoloid or
phere, be increased while the conjugate section remains the
ame, and the portion which may be resolved into the prismatic cylindrical ungula is increased in the same ratio: but the
ngular portion which we resolve into the spindle, remains
onstant.

Therefore, rings formed from portions, whose sections are a
imilar segment of a circle, are not proportional to the circumferences of the rings, unless the chord of the segment is perpendicular to the axis of rotation of the segment generating
he rings. (See Cor. 6th. Prop. IV.)

Cor. 4. If a belt or ring be taken from a middle frustum or
one of an elliptical revoloid or an ellipsoid, the belt or ring
may, in like manner, be resolved into a prismatic ungula of
n elliptical cylinder whose base is the section of the belt or
ing, and whose altitude is its inner circumference; and an
lliptical spindle, such as would be formed by the revolution of
he elliptical segment, formed by a section of the belt or ring
round its chord as an axis, which spindle is equal to all the
ungula space contained in the belt or ring.

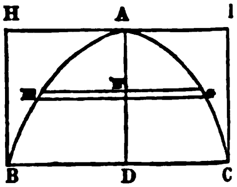
Scholium. 4. The same remarks as were used in reference
o segments Prop. X., Cor. 4, are applicable to the belt
r ring from a middle zone or frustum of an elliptical revoloid
r an ellipsoid, viz: that a belt from a middle zone of an elliptical revoloid is equivalent to a similar revoloid whose axis

is equal to the altitude of that of the zone, and that a ring from the middle zone of an ellipsoid is equivalent to a similar ellipsoid whose axis is equal to the altitude of the segment; and this is true whether it is an oblate or prolate revoloid or ellipsoid.

Scholium. The middle frustum of a circular or an elliptical spindle, may be resolved into two portions; a cylinder whose base is one of the bases of the frustum, and its altitude equal to the length of the frustum; and a ring remaining after this cylinder is withdrawn. The ring may be computed by finding what portion it is of a spherical ring, and the cylinder may be computed according to its dimensions.

PROPOSITION XII. THEOREM.

A vertical parabolic revoloid is equal to half its circumscribing prism; and a vertical paraboloid is equal to half its circumscribing cylinder.



Let ABC be a vertical section of a segment of a quadrangular parabolic revoloid, or of a paraboloid, and let BCHI be a similar section of the prism circumscribing the revoloid, or cylinder circumscribing the paraboloid through the same plane HBADIC, and if we suppose the axis AD to be divided into an indefinite number of equal parts through which, and through the solid, if planes EFG are passed perpendicular to such axis, the sections of the revoloid made by such planes would be squares, all of which make up the revoloid, and the sections of the paraboloid would be circles, all of which constitute the paraboloid, which, as those squares and circular sections are indefinitely near together, may be represented as a function of those squares or circles.

Now, because the square described on FG is equal to $\frac{1}{4}$ of the square on EG, $4FG^2 =$ the area of the square described on EG. But by property of parabola, (Prop. VII., parabola,) $p \times AF = FG^2$ where p denotes the parameter of the parabola; consequently $4p \times AF$ will also express the same square section EG, and therefore $4p \times$ the sum of all the AF's will be the sum of all the square sections, or the same function of the whole content of the revoloid; and because circles are as the squares described on their diameters, the sum of all the AF's $\times \pi$ will be a similar function of the whole content of the paraboloid.

But all the AF's form an arithmetical progression, beginng at 0 or nothing, and having the greater term, and the m of all the terms, each expressed by the whole axis AD. And since the sum of all the terms of such progression is ual to $\frac{1}{2}$ AD, multiplied by AD, or $\frac{1}{2}\text{AD}^2$, half the product the greatest term, and the number of terms ; therefore AD^2 is equal to the sum of the AF's, and consequently $\times\frac{1}{2}\text{AD}^2$, or $2\times p\times\text{AD}^2$ is the sum of the revoloid. But by e properties of the parabola $p : \text{DC} :: \text{DC} : \text{AD}$,

$p = \frac{\text{DC}^2}{\text{AD}}$; consequently $2\times p\times\text{AD}^2$, becomes $2\times\text{AD}$ DC^2 for the solid content of the revoloid. But $4\times\text{AD}$ DC^2 is equal to the prism HIBC, consequently the parabo- revoloid is equal to half of its circumscribing prism, and om a parity of reasoning, with regard to the paraboloid and linder, the paraboloid is equal to half its circumscribing linder.

Cor. Hence each of the ungulas of which the parabolic reloid is composed is equal to half its circumscribing prism, d these ungulas are such as are cut from a parabolic cylinr or prism, by planes meeting in the vertical plane, passing rough the vertices of its two parabolic bases.

PROPOSITION XIII. THEOREM.

he solidity of a frustum of a polar hemisphere of a parabolic revoloid is equal to a prism of equal altitude, and whose base is half the sum of the two bases of the frustum ; and the solidity of a frustum of a polar hemisphere of a paraboloid is equal to that of a cylinder of equal altitude, and whose base is equal to half the sum of the two circular bases of the frustum.

For in the frustum BEGC last proposition.

$2p\times\text{AD}^2 =$ the solid ABC

and $2p\times\text{AF}^2 =$ the solid AEG.

Therefore the difference $2p\times(\text{AD}^2 - \text{AF}^2) =$ the frustum EGC.

But $\text{AD}^2 - \text{AF}^2 = \text{DF}\times(\text{AD}+\text{AF},)$

erefore, $2p\times\text{DF}\times(\text{AD}+\text{AF}) =$ the frustum BEGC.

But, by the parabola $p\times\text{AD} = \text{DC}^2$, and $p\times\text{AF} = \text{FG}^2$,

erefore $2\times\text{DF}\times(\text{DC}^2+\text{FG}^2) =$ the frustum BEGC,

at is frustum BEGC = half the sum DC, FG, of the frusm multiplied by the altitude DF.

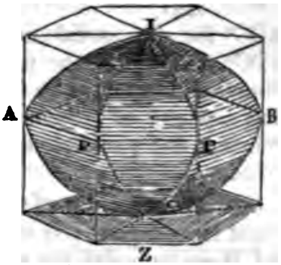
And for the same reasons as adduced in the the last prosition, this demonstration is equally applicable to the frustum a revoloid or a paraboloid.

Cor. Hence any conjugate section of an ungula, being part of a polar hemisphere of a parabolic revoloid, is equal to a prism of equal altitude, and whose base is equal to half the sum of the two bases.

PROPOSITION XIV. THEOREM.

A conjugate parabolic revoloid is equal to $\frac{8}{15}$ of its circumscribing prism ; and a conjugate parobolic spindle is equal to $\frac{8}{15}$ of its circumscribing cylinder.

Let AIBC be a parabolic revoloid whose vertical axis is an ordinate to the several parabolic sections, and whose vertices are all in the plane made by the conjugate section ARPB, and the revoloid will be equal to $\frac{8}{15}$ of its circumscribing prism.



For this revoloid is composed of parabolic ungulas, such as are cut from the vertical side of a parabolical prism, which (Prop. XVIII. Cor. 2, B II.) are equal to $\frac{8}{15}$ their circumscribing prisms ; hence a number of associated ungulas are equal to $\frac{8}{15}$ their associated prisms, and since the inscribed paraboloid bears the same proportion to its circumscribing cylinder, as the revoloid to its circumscribing prism ; hence a conjugate paraboloid or a parabolic spindle is $\frac{8}{15}$ of its circumscribing prism.

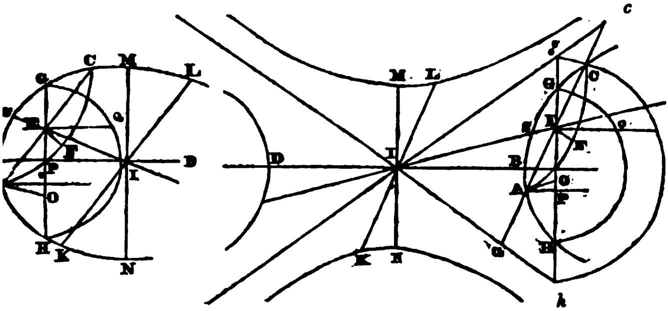
Scholium. The frustum of a parabolic spindle may be resolved into three portions—first, a cylinder whose base is one of the bases of the frustum, and whose altitude is the length of the frustum—second, the angular portion of the ring remaining after taking away the cylinder, which is equivalent to a parabolic spindle formed by the revolution of the section of the ring on the chord or double ordinate—third, a parabolic prism, whose base is a section of the ring, and whose altitude is equal to the inner circumference of the ring.

The first is equal to its base multiplied by its altitude. The second is equal to $\frac{8}{15}$ of its circumscribing prism ; the third is equal to $\frac{2}{3}$ of its circumscribing rectangular prism.

PROPOSITION XV. THEOREM.

any solid, formed by the rotation of a conic section about its axis, that is a spheroid, paraboloid, or hyperboloid, be cut by a plane in any position ; the section will be some conic section, and all the parallel sections will be like and similar figures.

Let ABC be the generating section, or a section of the given id through its axis BD, and perpendicular to the proposed tion AFC, their common intersection being AC ; and GH any other line meeting the generating section in G and H, d cutting AC in E ; and erect EF perpendicular to the plane C, and meeting the proposed plane in F.



Then, if AC and GH be conceived to be moved continually rallel to themselves, will the rectangle AE × EC be to the tangle GE × EH, always in a constant ratio ; but if GH be rpendicular to BD, the points G, F, H will be in the circum- ence of a circle whose diameter is GH, so that GE × EH ll be = EF^2 ; therefore AE × EC will be to EF^2, always a constant ratio ; consequently AFC is a conic section, and ery section parallel to AFC will be of the same kind with it, d similar to it.

Cor. 1. The above constant ratio, in which AE × EC is EF^2, is that of KI^2 to IN^2, the squares of the diameters of generating section respectively parallel to AC, GH ; that the ratio of the square of the diameter parallel to the sec- n, to the square of the revolving axis of the generating ne.

This will appear by conceiving AC and GH to be moved into positions KL, MN, intersecting in I, the centre of the ge- ating section.

Cor. 2. And hence it appears, that the axes AC and 2EF of the section, supposing E now to be the middle of AC, will be to each other, as the diameter KL is to the diameter MN of the generating section.

Cor. 3. If the section of the solid be made so as to return into itself, it will evidently be an ellipse. Which always happens in the spheroid, except when it is perpendicular to the axis; which position is also to be excepted in the other solids, the section being always then a circle: in the paraboloid the section is always an ellipse, excepting when it is parallel to the axis; and in the hyperboloid the section is always an ellipse, when its axis makes with the axis of the solid, an angle greater than that made by the said axis of the solid and the asymptote of the generating hyperbola; the section being an hyperbola in all other cases, but when those angles are equal, then it is a parabola.

Cor. 4. But if the section be parallel to the fixed axis BD, it will be of the same kind with, and similar to, the generating plane ABC; that is, the section parallel to the axis, in a spheroid, is an ellipse similar to the generating ellipse; in the paraboloid, the section is a parabola similar to the generating parabola; and in an hyperboloid, it is an hyperbola similar to the generating hyperbola of the solid.

Cor. 5. In the spheroid, the section through the axis is the greatest of the parallel sections; but in the hyperboloid, it is the smallest; and in the paraboloid, all the sections parallel to the axis, are equal to one another.—For, the axis is the greatest parallel chord line in the ellipse, but the least in the opposite hyperbolas, and all the diameters are equal in a parabola.

Cor. 6. If the extremities of the diameters KL, MN, be joined by the line KN, and AO be drawn parallel to KN, and meeting GEH in O, E being the midde of AC, or AE the semi-axis, and GH parallel to MN. Then EO will be equal to EF, the other semi-axis of the section.

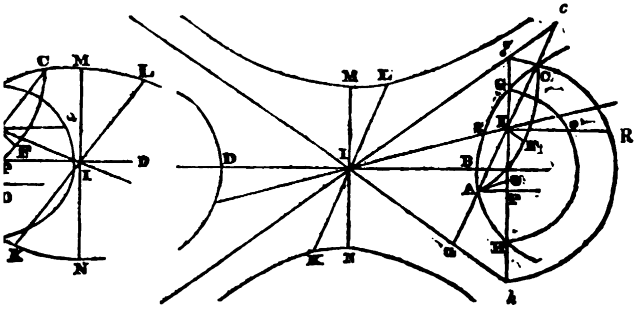
For, by similar triangles, KI : IN : : AE : EO.

Or, upon GH as a diameter, describe a circle meeting EQ, perpendicular to GH, in Q; and it is evident that EQ will be equal to the semi-diameter EF.

Cor. 7. Draw AP parallel to the axis BD of the solid, and meeting the perpendicular GH in P. Then it will be evident that, in the spheroid, the semi-axis EF = EO will be greater than EP; but in the hyperboloid, the semi-axis EF = EO, of the elliptic section, will be less than EP; and in the paraboloid, EF = EO is always equal to EP.

ɔlium. The analogy of the sections of an hyperboloid to ɔf the cone, are very remarkable, all the three conic ıs being formed by cutting an hyperboloid in the same n as the cone is cut.

ıs, let an hyperbola and its asymptote be revolved toge- ɔout the transverse axis, the former describing an hyper- , and the latter a cone circumscribing it ; then let them posed to be both cut by a plane in any position and the ctions will be like, similar, and concentric figures ; that ıe plane cut both sides of each, the sections will be con- :, similar ellipses ; if the cutting plane be parallel to the ıtote, or to the side of the cone, the sections will be para- and in all other positions, the sections will be similar ncentric hyperbolas.



t the sections are like figures, appears from the forego- rollaries. That they are concentric, will be evident we consider that Cc is = Aa, producing AC both ways :t the asymptotes in a and c. And that they are similar, ve their transverse and conjugate axes proportional to ther, will appear thus : Produce GH both ways to meet ymptotes in g and h ; and on the diameters GH, gh, be the semi-circles GQH, gRh, meeting EQR, drawn ıdicular to GH, in Q and R ; EQ and ER being then tly the semi-conjugate axes, and EC, Ec, the semi-trans- axes of the sections. Now if GH and AC be conceived noved parallel to themselves, AE × EC or CE^2, will GE × EH or EQ^2, in a constant ratio, or CE to EQ : a constant ratio ; and since cE is as Eg, and aE as Eh Ec or cE^2, will be to gE × Eh or ER^2, in a constant ɔr cE to ER will be a constant ratio; but at an infinite dis- from the vertex, C and c coincide, or EC = Ec, as also : Eg, consequently EQ is then = ER, and CE to EQ will be = cE to ER ; but as these ratios are constant, if they be equal to each other in one place, they must be always so ; and consequently CE : Ec : : QE : ER.

And this analogy of the sections will readily be recognised, when we consider that a cone is a species of the hyperboloid; or a triangle a species of the hyperbola, whose axes are infinitely small.

PROPOSITION XVI. THEOREM.

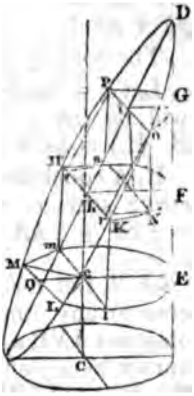
If SI *be the semi-diameter belonging to the double ordinate* AEC *of the generating plane,* AEC *being the diameter of the section* AFC, *conceived to be moved continually parallel to itself ; and* x *denote any part of the diameter* SI, *intercepted by* E *the middle of* AC, *and any given fixed point taken in* SI ; *then will the section* AFC *be always as* $a + bx + cxx$; a, b, c, *being constant quantities ;* b *in some cases affirmative, and in others negative ;* c *being affirmative in the hyperbola, and negative in the ellipse, and nothing in the parabola ; and* a *may always be supposed to denote the distance of the given fixed point from the vertex* s.

In any conic section, AC^2 is as $a + bx + cxx$; but all the parallel sections are like and similar figures, and similar plane figures, are as the squares of their like dimensions ; therefore the section AFC is as AC^2, that is, as $a + bx + cxx$.

Cor. If the given fixed point, where x begins, coincide with the vertex s, then will a be equal to nothing, and the section will be as $bx \pm cxx$, or as $x \pm dxx$, in the hyperbola and ellipse, and as bx, or as x, in the parabola.

The subject of hyperbolic revoloids and hyperboloids will be considered in another place ; we will here add a few general scholia and formulæ in relation to cylindric and conical ungulas.

SCHOLIA AND FORMULÆ IN RELATION TO CYLINDRIC AND CONICAL UNGULAS.



Scholium. It has been shown (Prop. VI., Cor. V,) that an ungula pertaining to a right revoloid is equal to $\frac{2}{3}$ its circumscribing prism, and Prop. III, that its curve surface is equal to that of the adjacent side of its circumscribing prism; and because the ungula FHKD is equal to an ungula of a right revoloid, if the intersection HK of the plane FHK, DHK passes through the axis of the cylinder; therefore the convex surface of the ungula FHKG is equal to $Fh \times FD$; let GP*o*D be an ungula cut from the former by the plane GP*o*, parallel to the base of the former; and the convex surface of the ungula GP*o*D is = the surface FHKD—the convex surface of the segment HKFGP*o*, and this may be divided into two portions; viz., the curve surface on the segment P*o*N*n* FG, and the convex surface of the segment *o*N*n* PHK, the latter of which is equal $(r\text{K} \times \text{FD} + n\text{K} \times \text{FD}) = 2r\text{K} \times \text{FD}$, and the former is equal the arc PG*o* $\times$GF or *o*N.

It will be perceived therefore that, although we have the quadrature of the whole convex surface of a revoloidal ungula, and also of any portion NK*o* in absolute terms, yet any portion *o*GFN, or *o*GPD is known only in terms of the arc of the circle, and consequently depends on the circle's quadrature, but we are enabled by the principles referred to in this scholium, to determine both the surface and solidity of any portion PBF, with the same degree of accuracy as we have the surface and solidity of the whole cylinder, from which the ungula is derived, and since a revoloidal spindle is conceived to be made up of partial ungulas, the revoloidal spindle is susceptable of the same degree of accuracy in its determination.

Let FD = h the altitude, and hG = Hh = r the radius of the cylinder from which the ungula is derived; the whole convex surface of the ungula is $= 2rh$ - - - - (1)

Let k = the altitude of the ungula GP*o*D, and let oi = half the chord oP = c, then will the surface ONK $= (r-c) \times h$ $= rh-ch$ - - - - - - - - - (2)

Let the arc PG*o* $= p$, and the surface *o*NGFP*n*, will be $=$ $p \times (h-k) = ph-pk$ - - - - - - - (3)

Hence the surface of the ungula PG*o*D $= 2rh - rh + ch - ph + pk = rh + ch - ph + pk$ - - - - - - (4)

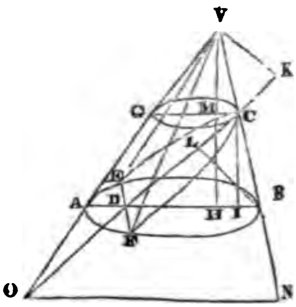
Let an ungula ELMD whose base is greater than half that of the cylinder be considered: if EF = FG, then will the convex surface of this, equal that of the ungula FKHD + the arc (HFK or $\frac{1}{2}\pi r$) × EF + $\frac{1}{2}\pi r$ (KL or N*o*) — $rh + ch - ph + pk$ - - - - - - - - - - (5)

The solidity of the ungula FKHD is $= 2r^2h$ - - (6)

Let a = the base PG*o* of the ungula GP*o*D, and the solidity will be $(rh + ch - ph + pk)\frac{1}{3}r - a \times \frac{1}{3}(h - k)$ (Prop. V, B. II.) $= \frac{1}{3}r^2h + \frac{1}{3}rch - \frac{1}{3}rph + \frac{1}{3}rpk - \frac{1}{3}ah + \frac{1}{3}ak$ (7)

If A equal the base of the ungula ELMD, and if S = its curve surface, its solidity will be $S \times \frac{1}{3}r + A \times (\frac{1}{3}h - \frac{1}{3}k) = \frac{1}{3}Sr + \frac{1}{3}hA - \frac{1}{3}kA$ - - - - - - - (8)

Sch. 2. Let AEBF be the base of a cone or any other pyramid, right or oblique; AVB a section through the vertex by a plane perpendicular to the base; EVF, ECF two other sections perpendicular to AVB, the former through the vertex, and the latter through the side at C, between V and B. On AB let fall the perpendiculars VH, CI; and on DC the perpendiculars VK, BL, draw CG parallel to AB, meeting AV and VH in G and M.



Then it is evident that EFBV is a pyramid, whose base is EFB, and whose altitude is VH; let the base be called A and the altitude a, and the solidity will be $\frac{1}{3}Aa$; and it is evident that EFCV is a pyramid whose base is EFC and altitude VK; let the base be called B and the altitude b, and its content will be $\frac{1}{3}Bb$, it is evident also that the ungula EFBC $= \frac{1}{3}Aa - \frac{1}{3}Bb$.

But, by similar triangles, ABV, GVC, it is AB−CG : CI or HV−VM : : AB : HV, or $a = \frac{AB \times CI}{AB - GC}$; also AB—GC : CI : : AB : HV : : GC : VM $= \frac{GC \times CI}{AB - GC}$; and DC : BD : : (by the similar triangles ICD, DBL) CI : BL : : (because of the similar triangles BCI and CVM, VKC and CBL) VM $= \frac{GC \times CI}{AB - GC}$: VK $(b) = \frac{GC \times CI \times DB}{DC \times (AB - GC)}$

Therefore the ungula EFBC will be

$$= \frac{\frac{1}{3}CI}{AB-GC} \times (A \times AB - B \times \frac{GC \times DB}{DC}); \quad - \quad - \quad (1)$$

which is a general formula for the ungula of any pyramid.

If the base be circular, or the pyramid a cone, and the angle CDB be less than the angle VAD; or which is the same, if CD and VA, produced, intersect in N; the section ECF will be a segment of an ellipse, whose transverse axis is CN, and conjugate $\sqrt{(NO \times GC)}$, NO being drawn parallel to AB, and meeting VB produced in O. And then the formula will become $\frac{\frac{1}{3}CI}{AB-GC} \times$ (AB × circular segment EBF $- \frac{GC \times DB}{DC} \times$ elliptic segment ECF) $= \frac{\frac{1}{3}CI}{AB-GC} \times AB \times$ circular segment EBF $- \frac{GC \times DB \times \sqrt{(NO \times GC)}}{DC \times CN} \times$ circular segment, whose diameter is CN, and height CD) =, since sim. seg. are as the squares of their diameters, $\frac{\frac{1}{3}CI}{AB-GC} \times$ (AF × circular segment EBF $- \frac{GC \times DB \times CN \times \sqrt{NO \times GC}}{DC \times AB^2} \times$ segment of the circle AEBF whose height is $\frac{AB \times DC}{CN}$) = the content of the elliptic ungula EFCB - · - - (2)

But, by similar triangles, GC—AD : DC : : GC : CN = $\frac{GC \times CD}{GC-AD}$, and GC—AD : DB : : GC : NO $= \frac{GC \times DB}{GC-AD}$, which values of NO and NC being substituted in the above expression of the elliptic ungula, will give

$\frac{\frac{1}{3}CI}{AB-GC} \times$ [AB× circular segment EBF $- \frac{GC^3}{AB^2} \times \frac{DB}{(GC-AD)}^{\frac{3}{2}}$ × segment of the circle AEBF whose height is $\frac{AB \times (GC-AD)}{GC}$] $= \frac{\frac{1}{3}h}{D-d} \times$ [D × circular segment EBF $- \frac{d^3}{D^2} \times (\frac{DB}{BD-D+d})^{\frac{3}{2}} \times$ segment of the circle AB whose height is $\frac{D \times (DB - D + d)}{d}$]. - - - - - (3)

= the elliptic ungula EFCB; putting *h* for the height of the ungula, D and *d* for the diameters of the base and end, or top, of the frustum, respectively.

If this value be taken from $\frac{1}{3}hn \times \frac{D^3-d^3}{D-d}$ = the whole conic frustum, the remainder will express the complemental elliptical ungula EFCGA $\frac{\frac{1}{3}h}{D-d} \times (D^3-d^3) \times n$ − D circular segment, whose height is $\frac{BD}{D} + \frac{d^2}{D^2} \times \left(\frac{BD}{BD-D+d}\right)^{\frac{3}{2}} \times$ segment whose height is $\frac{BD-D+d}{d}$, or $\frac{\frac{1}{3}h}{D-d} \times -nd^3 + D \times$ segment whose height is $\frac{AD}{D} + \frac{d^2}{D^2} \times \left(\frac{BD}{BD-D+d}\right)^{\frac{3}{2}} \times$ segment whose height is $\frac{BD-D+d}{d}$ . . . . (4)

If the points D and A coincide, the section EFC becomes a whole ellipse, and the formula above, become $\frac{1}{3}Dhn \times \frac{D^2-d\sqrt{Dd}}{D-d}$ = the elliptical ungula ACB - - - - - - (5)

And the complemental ungula ACG = $dhn \times \frac{D\sqrt{Dd-d^2}}{D-d}$ - - - - - - - - - - - - - - - - (6)

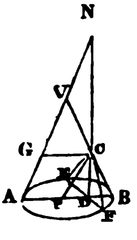
If the angle CDB be equal to the angle VAB, the section will be a parabola, whose axis is CD, and base EF $=2\sqrt{(AD \times DB)} = 2\sqrt{(D-d) \times d}$, and its area, by prop. VI, B. I, $= \frac{2}{3}DC \times EF = \frac{4}{3}DC\sqrt{(D-d)\,d}$; and therefore the expresssion becomes $\frac{\frac{1}{3}h}{D-d} \times (D \times$ segment EBF $-\frac{Dd-d^2}{DC} \times \frac{4}{3}DC\sqrt{Dd-d^2}$ = the parabolic ungula EFBC - - - (7)

If this be taken from $\frac{1}{3}hn \times \frac{D^3-d^3}{D-d}$, the remainder will express the complemental ungula EFCGA, viz.

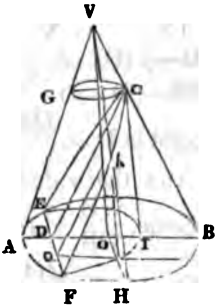
$\frac{1}{3}h \times (\frac{4}{3}d\sqrt{Dd-d^2} - \frac{nd^3}{D-d} + \frac{1}{D-d} \times$ segment whose height is $\frac{d}{D})$

or $\frac{\frac{1}{3}h}{D-d} \times \left((D-d)\ \frac{4}{3}\sqrt{d\,(D-d)} - nd^3 + D \times \text{segment whose height is } \frac{D}{d}\right)$ - - - - - - - - - - - - - - (8)

If the angle CDB exceed the angle VAB, the ection will be a an hyperbola, whose transverse axis is CN, but the transverse axis $= \frac{CG \times CD}{PD}$ CP being drawn parallel to VA, or $\frac{d \times CD}{D - d - DB}$ nd the conjugate also $= GC \sqrt{\frac{DR}{PD}} = d\sqrt{\frac{DB}{D - d - DB}}$ and the area of the hyperbolic section substituted in the general formula (1) will give the solidity of the hyperbolic ungula.



Scholium 3. If from the perimeter of the section EFC here be projected, the surface EIF by perpendiculars hP,CI, &c., to the base of the cone, the surface EBFIE on the base, will be to that, perpendicular above, viz., ECFB on the surface of the cone as OB to AV; as the radius of the base to the slant height of the cone, (Prop. XII, B. II.,) moreover the area of the section EFC is o the area EIF as DC : DI; hence the convex surface FCEB $= S = \frac{BV}{BO} \times$ EBFIE - - - - - - - - - - - - - - - - - - (1)



And the area FIE $= \frac{DI}{DC} \times$ FCEF

Let ECEF=B and the expression for the surface of the base FIE of the ungula FIEC becomes $\frac{DI}{DC} \times B$ - - - - (2)

Let the base EFB of the ungula EFBC be called A, and the area EBFIE will be $A - \frac{DI}{DC} \times B$ - - - - - - - (3)

And consequently $S = \frac{CB}{IB} \times A - \frac{DI}{DC} \times B$ - - - (4)

If from ($\frac{VB}{OB} \times$ base AFBE of the cone) = the convex surface of the whole cone, there be taken that of the ungula found above; the remainder $\frac{VB}{OB} \times (FAEF + \frac{DI}{DC} \times FCEF)$ - (5) will express the convex surface of the remaining part EFCVA of the cone. And if from the value last found, there be taken $\frac{VB}{OB} \times$ circle CG, $= \frac{VC}{OI} \times$ circle GC the convex surface of the

cone GVC, the remainder $\frac{BV}{OB} \times (TAEF \times \frac{DI}{DC} \times FCEF -$ circle GC$)$ - - - - - - - - - - - - - - - (6)
will express that of the complement EFAGC.

If the section FCE be an ellipse, the surface of the ungula will be $\frac{VB}{OB} \times$ (circular segment FBE $- \frac{DI}{DC} \times$ elliptical segment FCE) ; and hence $\frac{VB}{OB} \times EBF - \frac{GC^2 \times DI}{AB^2 \times (GC - AD)} \sqrt{\frac{DB}{GC - AD}} \times$ (segment circle AB, whose height is $AB \times \frac{GC - AD}{GC}$,)$= \sqrt{\frac{(4h^2 + (D - d)^2)}{D - d}} \times \Big(EBF - \frac{d^2}{D^2} \times \frac{DB - \frac{1}{2}(D - d)}{DB - (D - d)} \times \sqrt{\frac{DB}{DB - (D - d)}} \times$ seg. cir. AB, whose height is $D \times \frac{DB - D - d}{d}$ - - - - - - - - - - - (7)

And the value of the surface EFCVA, (Formula 5,) will become, $\frac{VB}{OB} \times$ (circ. seg. FAE $\times \frac{DI}{DC} \times$ ellip. seg. FCE) (8)

Or it is $= \frac{\surd(4h^2 + (D - d)^2)}{D - d} \times (FAE + \frac{d^2}{D^2} \times \frac{\frac{1}{2}D + d - AD}{d - AB} \sqrt{\frac{D - AB}{d - AB}} \times$ seg. circle AB, whose height is $D \times \frac{d - AB}{d}$
- - - - - - - - - (9)

And the value of the surface of the complemental ungula, (Formula 6,) will become $\frac{VB}{OB}$ (cir. seg. FAE $+ \frac{DI}{DC} \times$ ellip. seg. FCE $-$ circle CG$) = \frac{\surd(4h^2 + (D - d)^2)}{D - d} \times (-nd^2 + FAE + \frac{d^2}{D^2} \times \frac{\frac{1}{2}(D + d) - AD}{d - AD} \sqrt{\frac{D - AD}{d - - AD}} \times$ seg. circle AB, whose height is $D \times \frac{d - AD}{d}$) - - - - - - - - - - - (10)

When D coincides with A the expression will become, $\frac{VB}{BO} \times AB^2 \times n - n \times AB \times \surd(AI \times GC) = \frac{n \surd(4h^2 + (D - d)^2)}{D - d} \times (D^2 - \frac{D + d}{2} \surd Dd)$ for the convex surface of the ungula ABC. - - - - - - - - - - - - - - - (11)

And $\frac{VB}{OB} \times AI \surd(AB \times GC) \times n = 2n \times VB \times AI \sqrt{\frac{GC}{AB}}$

$= \frac{n\sqrt{(4h^2 + (D-d)^2)}}{D-d} \times \frac{D+d}{2} \sqrt{Dd}$ for the oblique cone ACV. - - - - - - - - - - - - - - - - - - (12)

Also $\frac{VB \times n}{OB} \times (AI\sqrt{(AB \times GC)} - GC^2)$

$= \frac{n\sqrt{(4h^2 + (D-d)^2)}}{D-d} \times (\frac{D+d}{2} \sqrt{(Dd - d^2)})$ for the complemental elliptic ungula ACG. - - - - - - (13)

If DC be parallel to AV, or the section a parabola; since its area B is $\frac{4}{3}$ DC × DF = $\frac{4}{3}$ DC $\sqrt{(AD \times DB,)}$ the general formula for the ungula will become

$\frac{VB}{OB} \times$ seg. (FBE $- \frac{4}{3}$ DI $\sqrt{(AD \times DB)}) = \frac{\sqrt{(4h^2+(D-d)^2)}}{D-d}$ × [seg. FBE to height DB $- \frac{2}{3}(D-d) \sqrt{(d(D-d))}$] for the convex surface of the parabolic ungula FEBC. - - - (14)

And the expression in Formula 5 will become,

$\frac{VB}{OB} \times$ (seg. FAE $+ \frac{4}{3}$ DI $\sqrt{(AD \times DB)}) = \frac{\sqrt{(4h^2+(D-d)^2)}}{D-d}$ × [(seg. FAE to height AE $+ \frac{2}{3}(D-d) \sqrt{(d(D-d))}$] for that of the part AEFCV.

Also, that in Formula 6, will be $\frac{VB}{OB} \times$ seg. FAE $+ \frac{4}{3}$ DI $\sqrt{(AD \times DB)} - AD^2 \times n) = \frac{\sqrt{(4h^2+(D-d)^2)}}{D-d} \times$ segment FAE to height AD $+ \frac{2}{3}(D-d) \sqrt{(d(D-d))} - nd,)$ for that of the complimental parabolic ungula FAEDG. (15)

If the angle CDB be greater than the angle VAB, or the section be an hyperbola, its area being found, and substituted for B in the general formulæ, will give the surfaces ofthe hyperbolic ungulas.

If the hyperbolic section be perpendicular to the base, DI will vanish, and the expressions will become,

$\frac{\sqrt{(4h^2+(D-d)^2)}}{D-d} \times$ seg. of cir. AB, whose height is $\frac{D-d}{2}$, for the curve surface of the perpendicular ungula CIB. (16)

$\frac{\sqrt{(4h^2+(D-d)^2)}}{D-d} \times$ seg. of the circle AB, height $\frac{D+d}{2}$, for that of the remaining part AICV. - - - - - - - - (17)

$\frac{\sqrt{(4h^2+(D-d)\ )}}{D-d} \times$ (seg. of the circle AB, whose height is $\frac{D+d}{2} - nd^2$,) for that of the complemental perpendicular ungula AIGC. - - - - - - - - - - (18)

BOOK IV.

ON THE REVOLOIDAL CURVE, THE RECTIFICATION OF THE ELLIPSE, AND OTHER CURVES, AND ON THE QUADRITURE OF THE CIRCLE, &C.

DEFINITIONS.

1. The *revoloidal curve* is the curve forming the contour of one of the facial sides of a revoloid; since this designation may apply to any revoloid, therefore, if the revoloidal curve is mentioned without reference to the species, the curve of a *right revoloid* is understood.

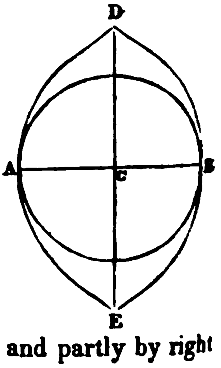
2. If the revoloidal surface is extended on a plane, its contour is called a *plane revoloidal curve;* and the surface is called a *plane revoloidal surface.*

3. The *vertices* of a plane revoloidal surface are the two angular extremities, as D and E.

4. The *vertical* or *transverse* axis of the plane revoloidal surface, is the right line drawn through the vertices, as DE.

5. Its *conjugate axis* is a line drawn at right angles to its transverse, which it bisects, terminating in the curve, as AB.

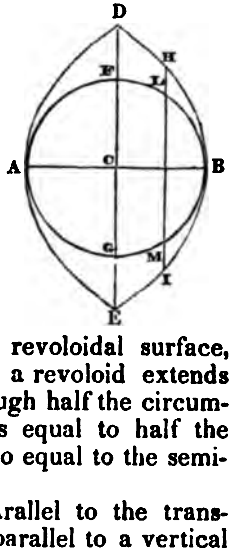
6. A *quadrant* of a revoloidal surface is a portion cut off by the two axes, as ACD or BCD.



7. Any area bounded partly by curves and partly by right lines, is sometimes called a *mixtilineal area* or space.

PROPOSITION I. THEOREM.

The vertical length of one of the plane surfaces of a quadrangular revoloid, is equal to the semi-circumference of its inscribed circle, and the length of any double ordinate to its conjugate diameter, is equal to the arc of that circle cut off by such ordinate toward the extremity of the conjugate diameter.



Let ADBE be a plane surface from a quadrangular revoloid, and AFBG its inscribed circle, and the vertical length DE of the revoloidal surface will be equal to the semi-circumference FBG, and the length of the double ordinate HI will be equal to the arc LBM cut off by such ordinate.

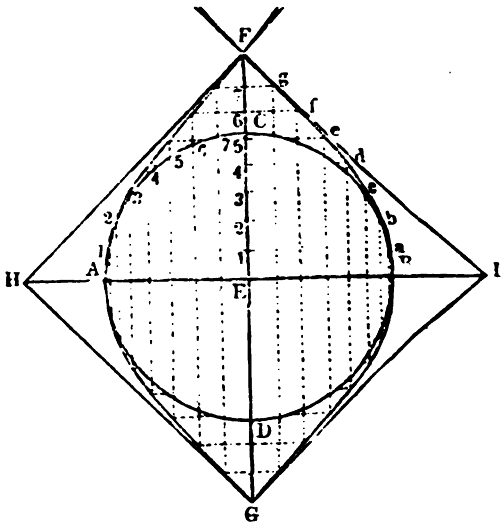
For since (Def. 8, B. III,) the vertical section of a right revoloid through the centre of its opposite sides, is a circle, and since in a quadrangular revoloid this circle is such as may be described on a diameter equal to the conjugate axis of the revoloidal surface, and because each of the facial surfaces of a revoloid extends from one vertice to the other, passing through half the circumference, it follows that its length, DE, is equal to half the length of that circumference which is also equal to the semicircumference FBG.

Again, since any ordinate HI drawn parallel to the transverse axis DE, is the representative of a parallel to a vertical section through the line DE, it follows that the section formed by a plane passing through the ungula of which this face is the surface, is a segment of a circle whose chord terminates in the curve forming the edge of the ungula, and the section of this ungula is similar to the section of its contiguous ungula by a plane perpendicular to this section; which section, through the contiguous ungula, may be represented by a segment LBM; for a quadrangular revoloid has its sides at right angles to each other in a plane perpendicular to the transverse axis. Hence, if HI is equal to the arc containing a segment equal to the segment LBM, it is therefore equal to the length of the arc LBM.

PROPOSITION II. PROBLEM.

To make a plane projection of the facial surface of a right quadrangular revoloid.

With a radius EA, equal the radius of the circle forming a vertical section of the revoloid, describe a circle ACBD, and draw the diameter AB, and from E perpendicular to AB draw the lines EF, EG, each equal to one-fourth of the circumference of the circle, ACBD, or equal to one-fourth of the circumference of the revoloid. Divide these lines into any number of equal parts, as 1, 2, 3, 4, &c., on the line EF. In like manner divide each quadrant of the circumference into the same number of equal parts, 1. 2, 3, 4, &c.; through the divisions on the circumference draw lines from 1, 2, 3, &c., parallel to DC, and through the divisions on the line EF draw lines both ways parallel to AB, as 7*g*, 6*f*, 5*e*, &c., and where these lines meet the former corresponding lines through the divisions corresponding to the same numbers, will be points in the curve forming the boundary of the surface, through which, if a curved line *a*, *b*, *c*, *d*, *e*, *f*, *g*, &c. is drawn, this line will represent the revoloidal curve, and the space enclosed will represent the plane surface of a right quadrangular revoloid.



Scholium 1. The nature of this curve is such, that, as it passes off at the vertices, it reproduces itself again, passing into nother curve of the same character but of opposite curvaure, for where it passes the vertices F and G, the signs beome changed from positive to negative and from negative to ositive, so that the curve is reproduced indefinitely as the xes EF and EG are continued.

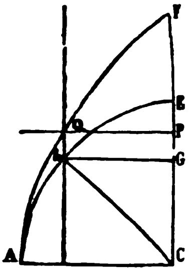
Scholium 2. The revoloidal curve passes into and becomes lentical with the circle while passing the extremities of the iameter, but its fluxion carries it out of the circle as it leaves ese points, and it becomes incorporated with and identical ith a right line as it passes off at the extremities of the axis, ut its fluction carries it out of the right line as it becomes xtended.

For let the diameter AB be produced each way to H and I, o as to be equal to FG, and from the extremities of these nes draw HF, IF, IG, GH, forming a square circumscribing e revoloidal surface ; let HF be extended to M, and IF be xtended to L, then these lines so produced will cross each ther at right angles in F, and form an angle with the axis FG f 45°. Let the revoloidal curves extend to P and P ; now the vertice F be brought to its natural position, on the reoloid, these curve lines evidently cross each other at right ngles also, and at the point of contact form an angle of 45° ith the axis, which is the same as that formed by the right nes ; hence these curve lines agree with the right lines, HM nd IL, at that point both in position and inclination, and thereore are identical.

And also, as the revoloidal curve passes into and occupies he space of the circle at the extremities of the diameter, A nd B having, in its original position, formed a part of the ircle at that point, and the position of that point not having een changed in reference to the axis or diameter AB, it follows, that it is still equal to and identical with the circle at hose points, but its fluction, or the law of its propagation, auses it to leave the circle after passing those points.

Cor. Each of the ordinates through the quadrant, AEF, paallel to the axis, EF, is equal to the portions A1, A2, A3, &c., f the arc of the quadrant, intercepted by those lines respecively toward the point A.

Scholium 3. Hence, this curve is generated by the locus of he intersection of two right lines, AC and EF equal the raius and semi-circumference of a circle moving uniformly from ny point E or A, perpendicular to each other, through heir respective lengths.



Let A be the origin, and since CF equal the quadrant AE, hence $CF=\frac{1}{4}\pi$, $AC=r$.

Let arc θ' equal any arc A*h*, measuring the angle AC*h*, then since A*h*=CP it is also $\theta'=y$; hence, the equation to the curve is $y = \text{arc } \theta'$

Let F be the origin, and we have $y=QP$ $=hG=\sin.\ \theta'=\sin.\ hCE$.

Then the equation, considering F as the origin, is $y=\sin.\ \theta'$

PROPOSITION III. THEOREM.

The contour of a plane revoloidal surface from a right revoloid is equivalent to the perimeter of an ellipse, formed by a vertical section through the angles of the revoloid.

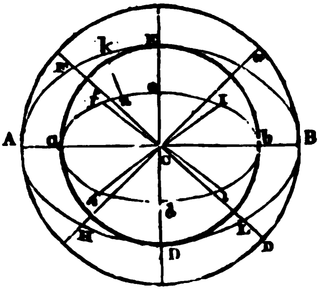
For the section of a revoloid through the angles is an ellipse by definition, and this ellipse terminates the facial surface of the revoloid when in its proper position. Now if the cylindric surface of the revoloid is extended on a plane, its parts are not altered in relation to each other; its vertical length on the plane is equal to its length on the cylindric surface of the revoloid; and its conjugate suffers no change, being a right line parallel to the axis of the cylindric surface while on the revoloid and a right line still when extended; for the surface may be extended in like manner as we would unroll a piece of cloth, or a piece of paper, made to agree with its surface, which suffers no contortion of any of its parts in the change, but the whole surface is the same in reference to its edges after the change as before, and as each facial surface of the revoloid extends through half the circumference of the circle of the revoloid, viz: from one vertice to the other, each side of the surface is terminated by one-half of the ellipse formed by a section through the angles, and as the angles of the revoloid cause generally the ellipses to cross each other at the vertices, forming a vertical angle also on the facial surface, the other side is bounded by one-half of a similar ellipse, so that the whole perimeter of the facial surface of a revoloid is equal to the perimeter of an ellipse by a plane passing through the angles of the revoloid.

Cor. Hence, the perimeter of a right quadrangular revoloidal surface is equal to that of an ellipse whose conjugate or *minor* axis is equal to that of the transverse axis of the revoloid, and whose *major* axis is in the ratio to its minor axis as the $\sqrt{2} : 1$.

For the plane passing through the angles of the revoloid forming the ellipse, cuts the plane forming a circular section through the centre of the sides at an angle of 45°, therefore, the transverse axis of the ellipse is as the diagonal of a square of which the vertical axis of the revoloid or its diameter forms a side.

PROPOSITION IV. THEOREM.

If there be described two ellipses concentric with each other, on axes of which those of the outer one exceed those of the inner one by N, *then the two ellipses will not be equidistant throughout, but will be nearer to each other at points in the curve between the vertices of the axis, than at the vertices.*



Let AB, ED, and *ab*, *ed*, be the two axes of two concentric ellipses, and let the axis $ab=Q$, and $ed=R$, and let the axis $AB=Q+N$ and $ED=R+N$, then the distance between the two ellipses, between the vertices E and A will be less than at those vertices.

For, draw the two equal conjugate diameters, HG, FL, and also the two *hl*. *fl*,

Then, HG^2+FL^2, or $2HG^2=AB^2+ED^2$ (Prop. XV, of the Ellipse,) and $2hl^2=ab^2+ed^2$

Now, because the sum of the squares of the axes AB^2+ED^2 are not greater than HG^2+FL^2, those squares cannot be proportional (Prop. XVIII, B. I, *El. Geom.*) hence also. (Prop. XXIII. B. I., *El. Geom.*) the axes themselves cannot be proportional.

Now it is evident, that when the axes AB and ED are nearly equal, then also they will very nearly form the extremes of a proportion of which the two diameters HG, FL are the means; which is the more nearly true the nearer the two axes are to an equality, or the nearer the ellipse approaches to a circle, and hence they are more disproportional, the greater the eccentricity of the ellipse.

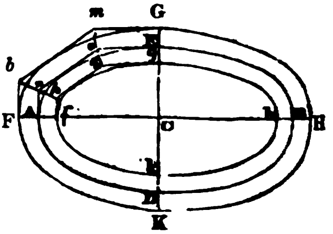
Now it is evident, that the inner ellipse, *a*, *e*, *b*, *d*, is more eccentric than the outer one, AEBD, since the two axes of the inner one are less than those of the outer one by the same constant quantity N, (by hypothesis,) hence the conjugate diameters *hl*, *fl*, are more nearly equal to HG and FL than *ab* to AB, or than *ed* to ED. Hence the elliptical curves are nearer each other at any point F or *k* between the vertices of the major and minor axes, than at those vertices.

Cor. 1. Hence, if about an ellipse there be described a curve concentric thereto and equidistant throughout, such curve will not itself be an ellipse, but if it is described very near to the elliptical curve and equidistant, it may be regarded as an ellipse without much error in so considering it, and this will be more nearly correct the more nearly the ellipse approaches to the circle.

Cor. 2. If about any curve except the circle, another curve be described, in such manner as to be equi-distant in all its parts from the former, these two curves cannot be similar curves, neither in properties nor in figure, all which may be shown by the same reasoning as in the proposition, viz: a curve described equidistant from a parabola, either within or without, cannot be a parabola, and a curve described equidistant from a hyperbola, cannot itself be a hyperbola; all of which is evident from the properties of those curves, but when they are drawn exceedingly near to those curves they may be regarded as curves of similar character as those near which they are so drawn.

PROPOSITION V. THEOREM.

If there be described two curves, one within and the other without an ellipse, of such kind that they shall both be equi-distant from the ellipse, or the ellipse shall be midway between the two, then will the space included between the two curves so described be equal to the circumference of the ellipse multiplied by the distance between the two curves.



Let FGHK, *fghk*, be two concentric curves described so as to be equi-distant from the elliptical circumference AEBD, one without and the other within, so that AEBD shall be midway between the two, then will the curvilinear space F*f*, G*g*, H*h*, K*k*, F*f*, be equal to the circumference AEBD multiplied by their common distance F*f*, or G*g*.

For, describe about each of those curves polygons F*bm*G, &c., A*no*E, &c., *fpqg*, &c., such that their corresponding sides will be parallel, and draw *bp*, *mq*, &c., and the surface included etween the outer and inner polygon will be divided into the rapeziums F*fpb*, *bpqm*, &c., each of which is equal to the half um of its parallel sides multiplied by their distance F*f*; but ny side, A*n* of the polygon described about the middle curve equal half the sum of its corresponding parallel sides of the apeziums; hence the trapezium F*fpb* is equal to A*n* × F*f*, nd the trapezium *bpqm* is equal to *no* × F*f*; and since this is ue for each of the trapeziums, it follows, that the sum of all equal to the sum of all the sides of the polygon described out the ellipse AEBD, multiplied by the common distance *f*. And this would be manifestly true, whatever be the imber of the sides of the polygon described about the ellipse, it when the number of the sides of the polygon is indefitely increased, the polygon becomes a curve similar to that out which it is described. (Prop. XII, Cor. 4, B. V, *El. eom.*) Hence as in the proposition.

Cor. It is evident, also, that if there be described curves ithin and without a parabolic or any other curve, so as to equi-distant from it, then the space included between the iter and inner one will be equal to the curve situated midway itween them, multiplied by the distance between the outer id inner curves, and the same may be affirmed of the revoidal curves so drawn.

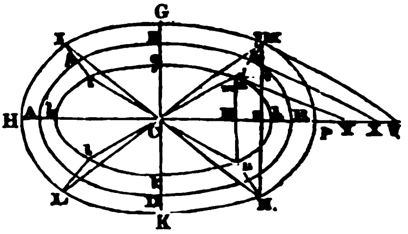
Scholium 1. The polygon described about two eccentric irves drawn so as to be equidistant throughout, are not simir polygons, since the figures about which they are described e not similar. (Prop. IV., Cor. 1.)

Scholium 2. The last two propositions suggest a method of ctifying the elliptical circumference, and also of finding the ngths of other curves whose quadratures are correctly or proximately known.

For, if about an ellipse AEBD, whose major axis AB=P, id whose minor axis ED=Q, another ellipse FGHK be desibed, whose major axis FH=P+N, and whose minor axis H=Q+N, and if another concentric ellipse be also desibed within the former, whose major axis *fh*=P—N and hose minor axis *gk*=Q — N, we shall have an elliptical ring , G*g*, H*h*, K*k*, whose area may be found by subtracting the ner ellipse from the outer one, then if this area is divided by e distance of the inner and outer circumference the quotient ill be the elliptical circumference AEBD.

PROPOSITION VI. PROBLEM.

Let it be required to find the distance Mm, *between two concentric ellipses, the difference of whose major and minor axes are each equal to a quantiy* N.



Let HGPK, *bghk* be two concentric ellipses, the difference of whose axes, HP —*bh*, is N, also GK—*gk* =N; draw the equal conjugate diameters ML, NI, *ml*, *ni*; draw the right coordinates MN, *mn*, draw *mt* parallel to HP, and join M*m*, which will be the distance between the ellipses at the points M, *m*; we shall have $sC^2=\frac{1}{2}PC^2$ and $sM^2=\frac{1}{2}CG^2$, also $EC^2=\frac{1}{2}hC^2$ and $st^2=\frac{1}{2}Cg^2$. (Prop. XXI. Cor. 1 of Ellipse.) and $sC-EC=sE$ or mt, and $sM-st$ or $mE=Mt$. Hence $\sqrt{Mt^2+mt^2}$ $=Mm=$the distance of the two curves at the points M, *m*.

Let the axis HP=25, and GK=19, and if N=2, then

$$bh=23 \quad \text{and} \quad gk=17$$

$$\text{and} \quad sC=8.83175 \quad EC=8.13172,$$

$$\text{hence } tm=.70003$$

$$sM=6.71751 \text{ and } mE=6.00999,$$

$$\text{hence } Mt=.70752$$

$$\text{and } Mm=.995302.$$

Scholium. 1. But because the line M*m* is not perpendicular to the curve AFRD at the point of contact, but very nearly perpendicular to MW, it is therefore greater than the true distance of the curves, which we will suppose is *uv*, to find which, we have

$$YC=\frac{Ch^2}{EC}; \text{ and } WC=\frac{PC^2}{sC}$$

$$YE=YC-CE. \text{ and } Ws=WC-Cs$$

$$mY=\sqrt{mE^2+EY^2},\ MW=\sqrt{Ms^2+Ws^2}.$$

Therefore, we have the sides of the right-angled triangles *m*EY, M*s*W given to find the angles W and Y, the difference of which, when found, is equal to the angle made by the lines MW and *m*Y with each other produced. Then we have a right-angled triangle M*m*, MW produced, and *m*Y produced right-angled at M, to find the sides MW produced, and *m*Y produced, and also the base *uv* of an isosceles triangle having the same vertical angle, which line *uv* will be the shortest distance required.

Scholium 2. Let the equal conjugate diameters IN, LM, *in*, *lm*, be drawn, and the difference of the semi-diameters CI−C*i* will be very nearly equal to the distance I*i* of the curves at he points I, *i*. But IN=$\sqrt{\frac14 PH^2+\frac14 GK^2}$, and *in*=$\sqrt{\frac14 bh^2+\frac14 gk^2}$, ence, $\sqrt{\frac14 PH^2+\frac14 GK^2}-\sqrt{\frac14 bh^2+\frac14 gk^2}$=the distance I*i* very early.

Let the axis AR=24, and FD=18; then, if we make PH =25, and GK=19, IN will=22.2036.

lso, we may have *bh*=23, *gk*=17; hence, *in* or *ml*=20.2237.

Therefore, $\frac{22.2036-20.2237}{2}$=.9899=the difference CI and *i*, which is nearly equal the true distance of the curves hrough the point *h*, which will be more accurate as the axes pproach equality, and will be approximately true till the ecentricity of the curves becomes very great.

PROPOSITION VII. PROBLEM.

o find the length of the elliptical circumference, approximately.

It has been observed, (Prop. V. Scholium,) that the cirumference of an ellipse inscribed between two other concenric ellipses, is equal to the area or space included between the wo divided by their distance from each other; but since the istance of the curves is not constant in every part, we must ake their average distance

If G*g*. P*h*, (see diagram to Prop. VI,) be the distance of the wo extreme ellipses through the lines of their axes, and *uv* the istance through the point *e*, then the average distance will be ery nearly equal $\frac12$G*g*+$\frac12$*uv*; hence, if the area of the ring within e exterior and interior circumferences is divided by $\frac12$G*g*+$\frac12$*uv*, e quotient will be the length of the whole circumference FRD; or if a quadrant of the ring is divided in like maner, the quotent will be the length of a quadrant F*e*R of the ircumference.

If we assume the axes AR=24, FD=18, as in the last proosition. and the axes of the other ellipses as there assumed, e shall have for the area of the greater ellipses, $\frac12$PH $\times\frac12$GH$\times\pi=12.5\times9.5\times\pi=373.06381$; and the area of the maller ellipse $\frac12 gk\times\frac12 bh\times\pi=11.5\times8.5\times\pi=307.09042$, therere the area of the ring is equal to 65.97339.

Let this be divided by $\frac{995302+1}{2}$, the average distance as ound in the last proposition. and we have 66.128 for the cirmference of the ellipse AFRD.

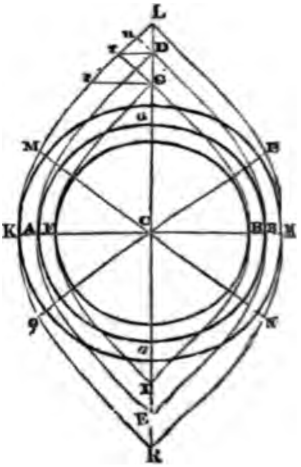
Let the area be divided by the distance as found in (Sch.

2. Prop. V,) and we have $65.97339 \div \frac{9899+1}{2} = 66.3115 =$ the elliptical circumferance, which is true to four places of figures, and very nearly to six; hence, this mode of computing the elliptical circumference is sufficiently accurate for any ordinary calculation.

Scholium. The length of the parabolic or hyperbolic arc may be approximately determined in the same manner as is here suggested for the ellipse.

PROPOSITION VIII. PROBLEM.

Let it be required to find the length of a revoloidal curve.



Let ADBE be a revoloidal curve from a right quadrangular revoloid whose length is required; let two other concentric curves KLMR, FGHI, be described on each side of the first and equidistant therefrom, and if these two last described curves are at a small distance only from the former, they will be very nearly revoloidal curves likewise.

Let *aa*, the axis of the revoloid from which the surface ADBE is supposed to be taken, equal P; and since, by hypothesis, ADBE is the surface of a right quadrangular revoloid, the conjugate is also equal to P, and the vertical length DE of the revoloidal surface will be equal $\frac{1}{2}\pi \times P$; since DE is equal to half the circumference A*a*B*a* of the revoloid. Now, let N be the distance AK, or AF, that the concentric curves are proposed to be drawn; and since the angles DL*n* and LD*n* are each=45° in a right quadrangular revoloid, make the semi-circumference of the revoloid from which the surface KLMR is taken, equal $\frac{1}{2}\pi P+2\sqrt{Dn^2+Ln^2}=\frac{1}{2}\pi P+2\sqrt{2N^2}=\frac{1}{2}\pi'$, and make the circumference of the revoloid from which the surface FGHI is supposed to be taken, $=\frac{1}{2}\pi P-2\sqrt{Dn^2+Ln^2}=\frac{1}{2}\pi P-2\sqrt{2N}=\frac{1}{2}\pi'$, then may P'=the diameter of the greater revoloid, and P'' equal the diameter of the lesser, and $P+2N$ equal the conjugate axis KM of the larger surface, and $P-2N$ equal the conjugate axis of the smaller.

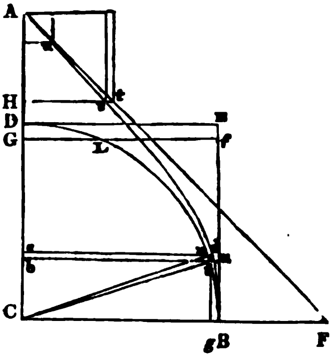
And we have (Prop. III. B. III.) the area of the surface LMR=(P+2N)×P′, and the area FGHI=(P—2N)×P″, nd (PP′+2P′N)—PP″+2P″N equal the space KF, LG, H, RI between the inner and outer curve. Let this area be ivided by the distance KF or N, and the quotient will be the ngth of the revoloidal curve ADBE very nearly.

Scholium 1. Since the major and minor axes of the outer nd inner curves vary in very nearly the same ratio, it follows at the outer and inner curves must be very nearly similar to e central one where the distance KF is small, even if the urface possesses a considerable degree of eccentricity, proacted in the direction KM; but the greater the eccentricity hen protracted in the direction LR, the greater is the simila-ty of the concentric figures, and the greater is the accuracy ith which we can rectify the curve.

2. Since the perimeter of a revoloidal curve is also the pemeter of an ellipse, such as is formed by a vertical section rough the axis and the angles of a revoloid, this mode of recfying the revoloidal curve furnishes also a mode of rectifying e elliptical circumference; and reciprocally.

PROPOSITION IX. PROBLEM.

To find the vertical length of the revoloidal surface, and consequently the circumference of the circle.



Let ABC be the quadrant of ne of the facial surfaces of a uadrangular revoloid, and BCD quadrant of the inscribed cirle; AC the semi-transverse, nd CB the semi-conjugate axis; xtend CB to F, making it equal AC, draw AF, and the triangle ACF will be equal to oneurth of the square circumscribg the whole revoloidal surce. From the extremities of e radii CD and CB, draw the nes DE and BE, perpendicular to those radii respectively, en will BCD be a square circumscribing the quadrant equal one-fourth of the square circumscribing the circle.

On the line BE take any distance B*a*, and from the point *a*, raw the line *ab* parallel to the semi-conjugate, BC; then set f from C on the transverse CA, the distance C*c* equal the arc

B*i* cut off by the line *ab*; and draw *cd* also parallel to BC, cutting the curve AB, and from *d* draw *dg* parallel to AC, then will the rectangle CB*ab* equal the portion of the revoloidal surface C*cd*B. (Prop. III.Cor. 1, B.III,) and $cd=ib=$cosine of the arc B*i*, and $Cb=ig=$sine of the arc. The portion of the revoloidal surface CB*dc* may consist of the rectangle C*cdg*, and the segment B*dg*; the former of which is equal to the product of the line C*c* or *dg*, into the cosine *ib* or *cd*; and the latter may be divided into the two portions B*ig*, a segment of the circle, and the trilinear space B*id*, the space included between the circle and the revoloidal curve; the former of which is equal to the difference between half the product of the arc B*i*, or its equivalent C*c*, into the radius CB; and half the rectangle of the sine *ig*, into the cosine *ib*; and the latter may be approximately estimated by considering, that the distance between the circle and the curve at any point, and in direction parallel to the base line *id*, is equal to the difference between the arc of the circle included between such point, and the radius CB, and its sine. Thus, the distance $id=$arc B*i* or *dg*, its equivalent, —*gi*, its sine. Hence, it will be perceived that its value converges rapidly as we approach the semi-transverse AC. Though the ratio of this space is constantly changing with regard to its linear dimensions, as the vertical length of a conjugate section is varied; yet, when the distance B*a* is taken very small, its area may be regarded as equal to one-half the product of the base *id*, into its vertical height perpendicular to that base, viz., into *g*B.

Let $x=$the arc $\mathrm{B}i=\mathrm{C}c$,
and $s=$sine of the arc B*i*,
and $c=$cosine.

Then will $cx=$the rectangle C*cdg*,
and $\frac{1}{2}rx-\frac{1}{2}cs=$the segment B*ig*,
and $\frac{rx}{2}-\frac{rs}{2}-\frac{cx}{2}+\frac{cs}{2}=$the space B*id*,
and . . . . $rs=$the rectangle BC*ba*.

Hence, $cx+\frac{1}{2}rx-\frac{1}{2}cs+\frac{1}{2}rx-\frac{1}{2}rs-\frac{1}{2}cx+\frac{1}{2}cs=rs$.

By transposing and condensing $\frac{1}{2}cx+rx=\frac{3}{2}rs$

Therefore, ... $x=\frac{\frac{3}{2}rx}{r+\frac{1}{2}c}=\frac{3is}{2r+c}$ . . (1)

Or if we make the curvilinear space $id\mathrm{B}=ai\mathrm{B}$, making the mixtilineal space C*cdb* equal to the arithmetical mean between the mixtilineal space *di*BC*c* and *dia*BC*c*, we shall have

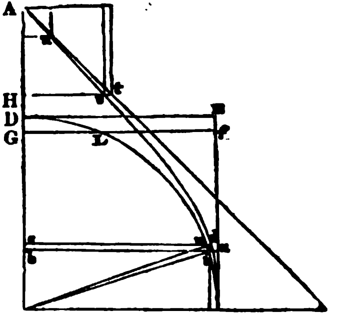
$xc+\frac{1}{2}rx-\frac{1}{2}sc=$the space *di*BC*c*
and $xc+rs-cs=$the space *dia*BC*c*

half the sum of which make=the rectangle
$\mathrm{C}ba\mathrm{B}=rs$.

$$\frac{2cx+\frac{1}{2}rx+rs-\frac{2}{3}cs}{2}=rs$$

$$2cx+\tfrac{1}{2}rx=\tfrac{2}{3}cs+rs.$$

, $$x=\frac{\frac{2}{3}cs+rs}{2c+\frac{1}{2}r} \quad \text{- - - - - -} \quad (2.)$$



, draw *f*G parallel to
, cutting off any arc DL,
ff from A the distance,
al the length of the arc
from the point H, draw
H*t*, perpendicular to the
to cut the curve in *v*,
rilinear space AH*v* will
to the rectilineal DE*f*G,
I, and Cors. B. III.)
if we compute the area
rilinear space AH*v* in
its sides, those sides may be rectified by its known
re. The length of the side AH equal the arc DL, and
AH*v* is less than half the rectangle of AH into H*v*,
ter than $\frac{1}{2}v\text{H}^2$; but will be very nearly an arithme-
n between the two.

$$\text{AH}=x=\text{arc DL},$$

$$\text{H}v=s=\text{sine of } x,$$

$$\text{DG}=v=\text{versed sine of } x.$$

$$\frac{\frac{1}{2}x^2+\frac{1}{2}xs}{2}=\text{area AH}v,$$

$$vr=\text{area DG}f\text{E};$$

s been shown that the area AH*v*=area DG*f*E.

, $$vr=\frac{\frac{1}{2}x^2+\frac{1}{2}sx}{2}$$

$$4vr=x^2+xs.$$

ting the square $x^2+sx+\frac{s^2}{4}=4vr+\frac{s^2}{4}$,

, $$x=\sqrt{4vr+\frac{s^2}{4}}-\frac{s}{2} \quad \text{- -} \quad (3.)$$

naller the arc is taken, the greater the accuracy; for
arc is taken very small, the sine and arc are very
entical; when no appreciable error would occur in
lering them, and the area AH*v* would still be found
half the square of AH, and half the rectangle of AH
; and if the difference of *x* and *s* becomes $=o$, then
ssion for the surface becomes barely $\frac{x^2}{2}$

and $x=\sqrt{2rv}$ - - - - - - - - (4.)

Let the two arcs, DL, *i*B, be taken such that the diagonal A*t* is nearly equal the arc *i*B, and if we make AH*t* equal the rectangle DG*f*E, and *n*BC*c* equal the rectangle *a*BC*c*, the result will be a close determination of the length of the arc of the circumference; for the area AH*t*, in such case, is very nearly as much in excess, above the area AH*v*, as the area *n*BC*c* is in defect of the area *d*BC*c*, and if these arcs are taken very small, any error may be rendered evanescent.

For this purpose, let $x=$ the arc *i*B

and $x'=$ the arc DL.

Then let $cx-\frac{1}{2}cs+\frac{1}{2}rx=rs$

which reduced gives $x=\frac{\frac{1}{2}cs+rs}{c+\frac{1}{2}r}$ - - - - - - - - (5.)

And $\frac{1}{2}x'^{2}=rv$

Then $x'=\sqrt{2vr}$ - - - - - - - - (6.)

Whence, if we take an arithmetical mean between the results of these two equations, we shall arrive at a very approximate determination of the circumference, if the arc is taken very small.

For an example, let us take an arc$=\frac{1}{24576}$ part of the circumference. We have, by trigonometry, the sine of that arc $=.00025566346=s$, and its cosine$=.99999996732=c$. Hence, by formula 1, we have

$$x=\frac{\frac{2}{3}rs}{r+\frac{1}{2}c}$$

Therefore. $00025566346\times\frac{3}{2}=\frac{3}{2}rs=.00038349819$

and $1+9999996732\div2=r+\frac{1}{2}c=1.4999998366$

And $\frac{^{3}rs}{r+\frac{1}{2}c}=.00038349819\div1.4999998366=.000255663462 78$

hence $x=.00025566346.278$, which multiplied by 24576, gives the whole circumference$=628318526133$ when the diameter is$=2$, or $3.1415926306=\pi$, when the diameter is 1, which result is true to eight places of figures, but the 9th should be 5 instead of 3.

If instead of this arc, we take that of $\frac{1}{40000}$ of the circumference, the sine of which is$=.000157079632033$, and the cosine $=.999999987462994$

$$\frac{3}{2}rs=.000235619448050$$

$$r+\frac{1}{2}c=1.499999993831497$$

$$\frac{\frac{3}{2}rs}{r+\frac{1}{2}c}=000157079632676$$

whence $\pi=.000157079632676\times\frac{40000}{2}=3.1415926535.2,$

which is true to eleven places, but the last figure should be 8 instead of 2.

Hence, it will be perceived, that, by taking a very small arc, we can rectify the circumference to any degree of exactness.

Cor. 1. Since in formula (6) $\frac{1}{2}x^2=vr$; if $r=1$, then $v=\frac{1}{2}x^2$ (1.)

Hence, the versed sine may be taken as half the square of the arc, which is approximately true when the arc is small, and if very small, may be taken as an accurate determination, but as the arc is increased, if we proceed as in formula (3,) where $rv=\frac{1}{4}x^2+\frac{1}{4}sx$, we have the value of v or the versed sine, equal to $\frac{1}{4}$ of the square of the arc$+\frac{1}{4}$ the rectangle of the arc and sine; which is approximately true for any arc of the quadrant, and may be taken for an accurate determination when the arc is small.

Hence this general formula $v=\frac{1}{4}x^2+\frac{1}{4}s3x$ - - - - (2.)

Cor. 2. Hence, also, from formula (1,) $x=\dfrac{\frac{2}{3}rs}{r+\frac{1}{3}c}$ we may deduce expressions for the sines, of small arcs, in terms of the cosine and arc; and also for the cosine in terms of the sine and arc. For from this equation we obtain

$$s=\tfrac{2}{3}x+\tfrac{1}{3}cx \quad - - - - - - - - - \quad (1.)$$

and

$$c=\frac{3s}{x}-2 \quad - - - - - - - - - \quad (2.)$$

Scholium. 1. If s is taken equal to the sine of 30°, we shall avoid in some measure the inconvenience of using imperfect decimal terms; for the sine of 30° is equal to half the radius, and we then have only one decimal term entering into the expression, viz., the cosine which is, 866025, &c.; but we have a more difficult determination of the area of that portion of the revoloidal curve existing without the circle, viz., the space B*id*; which, if determined, would lead to the true determination of the arc of the circumference.

2. If the ratio between the quadrant of the circle and the portion of the revoloid without the quadrant, viz., if the portion ADB is determined, then the ratio of the circumference may also be determined.

Thus, if $\frac{1}{2}rx$=the area of the quadrant CDB, and z=the area of the space included between the quadrant and the curve, viz., ABD, and if the ratio between these terms are known, that is, if $\frac{1}{2}rx : z :: m : n$

then $\frac{1}{2}rx+z=r$ (Prop. III, B. III.)

by uniting extremes and means,

$$\tfrac{1}{2}rnx=zm$$
$$z=r^2-\tfrac{1}{2}rx$$

By substituting this value of z in the former equation $\frac{1}{2}rnx=mr^2-\frac{1}{2}rmx$; by dividing and transposing $\frac{1}{2}nx+\frac{1}{2}mx=mr$;

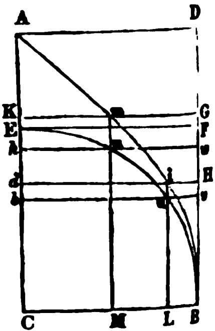
hence $$x=\frac{mr}{\frac{1}{2}n+\frac{1}{2}m}=\tfrac{1}{4}\text{ of the circumference.}$$

3. Or if the ratio between the quadrant of the revoloidal surface and its circumscribing parallelogram is determined, the circumference may be computed also.

For the revoloidal quadrant being equal to the square of the radius, (Prop. III. B. III,) it is in the same ratio to its circumscribing parallelogram, as the radius to one-fourth of the circumference, seeing the parallelogram is equal to the product of the radius by one-fourth of the circumference.

PROPOSITION X. THEOREM.

If a quadrant ABC *of a plane revoloidal surface from a right quadrangular revoloid be described, and a rectangle* ACBD *circumscribe the quadrant, then if the axis* AC *be divided into any number of equal parts, and if ordinates be drawn from the points of division across the rectangle, then the parts of those ordinates included between the axis* AC, *and the revoloidal curve* BA, *will be equal to the sum of a series of sines of arcs of the quadrant* EB, *of a circle in arithmetical progression, equal in number to the number of the ordinates, and the parts of those ordinates intercepted by the curve* AB, *and line* CD*; will be equal to the sum of a similar series of versed sines, and the sum of the whole ordinates will be equal to an equal series of radii of the quadrant* EB.



Draw any ordinate as KG, and from m, where it cuts the curve, draw mM parallel to AC, and let it cut the circumference of the circle described on the ordinate CB, in n draw nh parallel to mK, and the line mK=the line nh, is the sine of the arc nE, and because (Prop. II,) the line AC=the quadrant EB of the circumference, and the line mM=KC is also=the arc nB, the line KA equal the arc nE; take Kd=KA, and from d, draw the ordinate dH parallel to the former ordinate, and from the point i, where it cuts the curve, draw iL, and from the point *a* where the latter line cuts the quadrant, draw *ab*, which will be the sine of the arc *a*E=to the part *id* of the ordinate *d*H; and because *i*L is equal to *a*B (Prop. I,) and *m*M =*n*B, and AC=EB, and because AC —*m*M=*m*M—*i*L, E*n* =*na*; and since the same may be shown in reference to any other ordinate, drawn from a point on the axis AC, whose distance from *d* toward C is equal to K*d* or AK, it follows that the parts of equidistant ordinates drawn across the rectangle intercepted between the curves AB, and the axis AC, are equal to the sum of a similar series of sines of arcs of the quadrant EB taken in arithmetical progression; which is the first branch of the proposition.

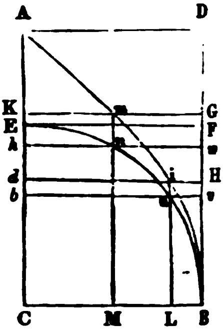
And since *av*=*i*H, is the versed sine of the arc *a*B corresponding to the sine *a*L with its complement *ab*, and *nw*=*m*G versed sine of the arc *n*B, its complement being *m*K; and since they have been shown to be at distances from each other proportional to the arcs E*n*, *na*, &c.; and since the same may be shown in reference to the portion of any ordinate intercepted by the curve AB, and line BD, wherever drawn, it follows that the sum of the portions of the equidistant ordinates intercepted by the curve AB and line CD, is equal to the sum of a similar series of versed sines of arcs taken in arithmetical progression; which is the second branch of the proposition. And since the lines CA and BD are parallel by hypothesis, the ordinates are all equal in length, and equal to the radius CB, hence the whole series of ordinates is a similar series of radii.

Cor. 1. From the preceding demonstration, it appears that any line or ordinate drawn from any point on the axis AC, parallel to the conjugate CB, and terminating in the curve AB, is equal to the sine of the arc on the quadrant BE, cut off by a line drawn from the point of termination of such ordinate in the curve parallel to the axis AC.

Cor. 2. Hence, also, the sum of the series of sines is to the sum of a similar series of radii as the square ECBF, described on the radius, to the rectangle CB, AC, of the radius and arc of the quadrant. Since the square ECBF is equal (Prop. III. B. III.) to the surface ABC.

PROPOSITION XI. THEOREM.

The area of a plane revoloidal surface is to that of its circumscribed parallelogram, as the sum of an indefinite series of sines in the circle, to the sum of an equal series of radii.



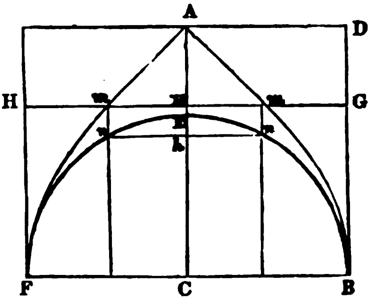
Let ACBD be a parallelogram circumscribing the quadrant of the revoloidal surface ABC, and let EBC be a quadrant of the circle, then will the area of the quadrant of the revoloidal surface ABC be to that of a parallelogram ACBD, as the sum of an indefinite series of sines of the quadrant BCE, to the sum of an equal series of radii.

For, since all ordinates *vb*, *d*H, &c., cut the surfaces of those figures in relation of their magnitudes in the sections through which such ordinates pass; and since (Prop. XI,) if ordinates be drawn through those figures equidistant from each other, the portions of the ordinates intercepted by the curve and axis, are equal to the sum of a series of sines of arcs in arithmetical progression for the whole quadrant equal in number to the number of the ordinates, and if these ordinates are equidistant from each other, the sum of the portions passing through either surface, drawn into their common distance, may be taken for the surface; and since the distance of the ordinates is equal by hypothesis, both for the parallelogram and revoloidal surface, the portions of the ordinates intercepted by each, will be in relation to their surfaces respectively, when their number is indefinitely increased, and their distance becomes indefinitely small. Hence, as the sum of a series of sines of arcs of the whole quadrant taken in arithmetical progression, is to the area of a quadrant ACB of the revoloidal surface, so is the sum of an equal series of radii to the area of the parallelogram ACBD; and what has been shown for one quadrant of the revoloidal surface, is also true for the whole.

Cor. Hence, the space BDA without the revoloidal surface, is to the revoloidal surface, as a sum of an indefinite series of arcs in arithmetical progression to the sum of a similar series of sines.

PROPOSITION XII. THEOREM.

If the quadrant AB *of a revoloidal curve be made to revolve about its axis* AC, *and if a plane hemisphere of a quadrangular revoloid be described about the solid so generated, having the same axis* AC, *then the revoloid will be to its circumscribing prism, as the sum of the squares of a series of sines of the quadrant, to the sum of the squares of an equal series of radii.*



Let an indefinite number of planes be passed through the revoloid perpendicular to the axis, and at equal distances from each other, and the sections made by these planes will all be squares, (Prop. 1, Cor. 4 B. III.) and their sides will all be equal to the ordinates H*m*, *hn*, &c., drawn through the intersection of such planes with the vertical sections; and hence the side of each parallel section, is equal to twice the sine *nh* of the quadrant corresponding to such section, being = *nn*. Now let each of those parallel planes be extended to HG passing through the prism, and it is evident that each of the sections of the prism will be squares, whose sides are severally equal to twice the radius CB, of an inscribed circle. Now the magnitudes of these solids through each section, are evidently in the relation of the magnitudes of such sections; and if the number of these equidistant planes are indefinitely increased, and they are indefinitely near together, their sum will represent the whole of each of the solids in the relation of their whole magnitudes, and since each conjugate section of the revoloid is the square described on double the sine, answering to the ordinate in reference to the quadrant CEB, and each section of the prism is the square described on the line HG, equal twice the radius, it follows that the solidity of the revoloid, is to that of the circumscribing prism, as the sum of the squares of a series of the sines of the quadrant, to the sum of the squares of an equal series of radii.

Cor. Hence, the solidity of the space between the surfaces he revoloid described as above, and that of its circumscribing prism, is to that of the revoloid, as the sum of the squares of a series of versed sines of the quadrant, to the sum of the squares of an equal series of sines, and this space is to that of the prism, as the sum of the squares of the versed sines, to the sum of the squares of an equal series of radii.

Cor. 2. Let the prism and also the revoloids, be divided into four quadrants by planes through the vertical axis, and passing through the centres of the opposite sides; and the solid so described. will be truly represented by the value of the conjugate parallel sections, passing through them, viz: the segment of the revoloid, will be represented by the sum of the squares of the sines; the space between the revoloid and surface of the prism, by the sum of the squares of an equal series of versed sines, and the prism by the sum of the squares of an equal series of radii.

Cor. 3. Since the revoloid has the same ratio to its circumscribing prism, as the solid of revolution about which it is described, has to its circumscribing cylinder; the solid formed by the revolution of the revoloidal quadrant AB, will be to its circumscribing cylinder, as the sum of the squares of a series of sines of the quadrant, to the sum of the squares of an equal series of radii.

PROPOSITION XIII. PROBLEM.

Let it be required to find the circumference of the circle from the ratio of the sum of the series of sines for every minute of the quadrant to the sum of an equal series of radii.

The number of the series of sines to every minute is 5400 = the number of minutes in the quadrant, which is the number of radii to be compared with the series of sines, and if the radius = 1, then 5400 is the sum of the series of radii; and the sum of the series of sines to every minute is by Trigonometry = 3438.2467465.

And (Prop. XI,) the area of the revoloidal surface is to that of its circumscribing parallelogram, as the sum of an indefinite series of sines to the sum of an equal series of radii; but the series of sines to every minute being a definite number, and such that the surfaces between the lines may be rendered appreciable, they do not represent those spaces or traziums in their exact ratios, but represent the longest sides of se trapeziums, making up the revoloidal surface, but in or- r that they may be true indices of those trapeziums, they ould be such as pass through the centres, when each would reduced, by a quantity equal to half the difference between elf and the next greater one, and as the sum of all their ferences, is evidently equal to the radius, half of the sum of eir several differences is equal to half the radius ; therefore, e sum of the natural sines must be reduced by that quantity, z : $3438.2467165 - ,5 = 3437.7467465$, when if we make

$r =$ radius

d $x = \frac{1}{4}$ of the circumference, we shall ve, (Prop. IX., Sch. 3.)

$$r : x :: 3437.7467465 : 5400$$

Hence, $x = 1.570.796337 = \frac{1}{4}$ the circumference when e diameter is 2, which is true to 8 places of figures, viz: to 5707963, but the 9th figure should be 2 instead of 3.

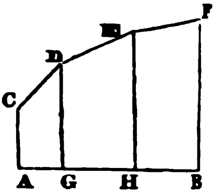
Cor. 1. Because the cosine of 60° is equal to half the ra- us, and because the surface of any conjugate section of the voloid is equal to the radius multiplied by the cosine corres- nding to each section, (Prop. XI, Cor. 1.) the sum of the nes for 60° is equal to half the sum for 90° or the whole adrant; and consequently, is equal to the sum of the series, r the arc from 60° to 90°.

Cor. 2. Hence, by proceeding as in the proposition, using e sum of the sines for an arc of 60° in comparison with a rresponding portion of the circumscribing parallelogram, e ratio of the circumference of the circle to its radius, may e determined by this arc, in the same manner as by the hole quadrant.

Scholium. It will be perceived that the sum of all the natural sines of the quadrant, to any number denoting the series, ay be calculated by reversing the operation, viz: $x : r :: r :$ sum of series of sines minus $\frac{1}{2}r$, when $n =$ the number enoting the series: and this may evidently be effected for the hole quadrant or any portion of it.

PROPOSITION XIV. THEOREM.

If there be any number of equidistant ordinates of different lengths drawn from a right line AB, *and terminated by the vertices of a polygon, then the area comprehended between the greatest and least ordinate, and the right line* AB *and polygonal line* CDEF *is equal to the sum of all the middle ordinates + half the sum of the extreme ordinates drawn into the common distance* AG.

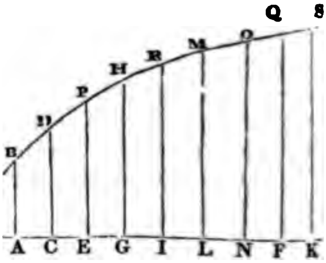


For the quadrilateral ACDG is = $\frac{1}{2}$ (AC+GD) AG, the quadrilateral EDGH = $\frac{1}{2}$ (G*D*+HE) AG or GH, and the quadrilateral HEFB = $\frac{1}{2}$ (HE+BF) HB ; hence by addition we have

$$\left(\tfrac{1}{2}\,AC+GD+EH+\tfrac{1}{2}\,BF\right)AG.$$

PROPOSITION XV. THEOREM.

If a right line AK *be divided into any even number of equal parts* AC, CE, EG, *&c. ; and at the points of division their be erected perpendicular ordinates* AB, CD, EP, *&c., terminated by any curve* BDPS ; *and if* a *be put for the sum of the first and last ordinate* AB, SK, b *for the sum of the even ordinates* CD, GH, LM, FQ ; *and* c *for the sum of all the rest,* EP, IR *and* NO ; *then* (a+4b+2c) × $\frac{1}{3}$ *of the common distance* AC *will be the area,* ABSK *very nearly*



Through the first three points BDP, let a parabola be conceived to be drawn, having its axis parallel to the ordinates; the parabolic area ABPE, (Prop. VI. *Schol* .B. I.) will be (AB + 4CD+EP) × $\frac{1}{6}$ AE = (AB+ 4CD+EP) × $\frac{1}{3}$ AC; and when the points of BDP are at no great distance from each other the parabolic curve will very nearly cioncide with any other regular curve, drawn through the same points.

Let us now take the ordinates EP, GH, IR ; then will (EP +4GH+IR) × $\frac{1}{3}$ EG=the area EPRI; and (IR+ 4LM+NO)

$\frac{1}{3}$ IL=area IRON. Also (NO+4FQ+KS)×$\frac{1}{3}$ NF=area OSK, whence by addition, we have [(AB+KS) + (4DC+4H+4LM+4FQ)+(2EP+2IR+2NO)]×$\frac{1}{3}$CA=(a+4b+2C) $\frac{1}{3}$ AC.

Cor. This theorem may be applied for computing the con-nts of solids, by using parallel sections instead of the ordi-ıtes, as will appear in Prop IV., Corollaries and Scho-ım, B. I.

Schol. It is evident that the greater the number of ordinates ıd the nearer the points DBP, &c., are to each other, the ore nearly will any curve, drawn through them, agree with e parabola, and hence the greater accuracy will be ob-ined; the same remark will also apply to solids.

Cor. 2. Hence if the area of any space AB, KS is known, ıd the ordinates AB, IR,KS, the value of the line AK may be termined.

For if AB + KS = a and IK = b, we have, by considering e curve as a parabola, whose axis is parallel to KS, the area = (a + 4b) × $\frac{1}{6}$ AK, let AK = p, hence we have

$$= \frac{A}{a + 4b} \text{ or } p = \frac{\frac{1}{6}A}{a + 4b} \quad - \quad - \quad - \quad - \quad - \qquad (1)$$

Or, if the number of ordinates is increased, we have by proposition, A = (a + 4b + 2c) $\frac{1}{3}p \div n$, n being the mber of divisions in the line AK, or the number of ordinates s 1, and we have $p = \frac{3nA}{a + 4b + 2c}$ - - - (2)

PROPOSITION XVI. PROBLEM.

find eleven equidistant ordinates to hyperbola between the asymptotes, and by means of those ordinates to find the area.

Taking the equation $a^2 = xy$, and assuming $a = 10$, and the st value of x, or the distance from the centre to the first linates = 10, and if the lower distance is 1 = d, we shall ve for the ordinates.

$$\frac{10}{11} \quad \frac{10}{12} \quad \frac{10}{13} \quad \frac{10}{14} \quad \frac{10}{15} \quad \frac{10}{16} \quad \frac{10}{17} \quad \frac{10}{18} \quad \frac{10}{19} \quad \frac{10}{20}$$

sum of the first and last or $a = \frac{10}{10} + \frac{10}{20} = 1{,}5$

the sum of the even ordinates or b

$$=\frac{10}{11}+\frac{10}{13}+\frac{10}{15}+\frac{10}{17}+\frac{10}{19}=3.4595393$$

The sum of all the recurring ordinates or c

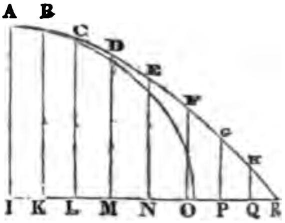
$$=\frac{10}{12}+\frac{10}{14}+\frac{10}{16}+\frac{10}{18}+=2.7281744\tfrac{5}{8}$$

Therefore, by the proposition,

$(a+4b+2c)\frac{1}{3}d=6,9315021$, is the area required.

PROPOSITION XVII. PROBLEM.

To find the value of π by equi-distant ordinates to the revoloidal curve.



If A = the area of a quadrant, AIR of the revoloidal curve, and a = the minor semi-axis, AI = the radius of a circle inscribed in the curve, and $\frac{1}{2}\pi$ = the semi-major axis, IR; and if a = an ordinate EN to the major axis, equi-distant from I to the vertex R, then will the area of the curve, considering it as a parabola, be $= a+4\,b\times\frac{1}{6}\frac{\pi}{2}$; and if A equal the area of the revoloidal quadrant, the semi-transverse $=\frac{\pi}{2}=\frac{A}{\frac{1}{6}(a+4b)}=\frac{6A}{a+4b}$. (1)

Let $a=1$, and $b=\sqrt{\frac{1}{2}}=70712$, &c.

and A will $= r^2 = 1$,

hence we have $\frac{A}{a+4b}=1.5672239=\frac{\pi}{2}$

this is true to two places of figures only, but the third should be 7 instead of 6.

Let now two other ordinates, CL, GP, be taken in addition to this, equidistant therefrom, and from the extremities of the axis, and we have by Prop. XVI, Cor. 2, $\frac{3An}{a+4b-2c}=\frac{1}{2}\pi$

Let the two ordinates b be the sines of 22° 30′ = .382683, and 67° 30′ = .923880, and c = sine of 45° = .707107, and we have $a=1$, $4a=5.226252$, $2c=1.41424$, $A=r^2=1$.

Hence $\frac{3nA}{a+4b+2c}=\frac{12}{7.640466}=1.5705848$, &c. $=\frac{\pi}{2}$

which is true to four places, viz: 1,570, but the fifth should be 7 instead of 5.

Let, now, seven ordinates be taken between the two extremes, and the distance of the ordinates will be reduced $\frac{1}{2}$.

Thus the ordinates will be the sines of 11° 15′, 22° 30′, 33° 45, 45°, 56° 15′, 67° 30′, 78° 45′, 90° = .195090. .382683. 555570, .707107, .831470, .823880, .980785, 1.

$$a = \left\{\begin{array}{l}\text{sin. } 0^\circ = 0\\ \text{sin. } 90^\circ = +1\end{array}\right\},\quad b = \left\{\begin{array}{r}\text{sin. } 11^\circ\ 15' = .195090\\ 33^\circ\ 45' = .555570\\ 56^\circ\ 15' = .831470\\ 78^\circ\ 45' = .980785\\ \hline 2.562915\end{array}\right.$$

$$c = \left\{\begin{array}{r}\text{sin. } 22^\circ\ 30' = .382683\\ 45^\circ\ \ 0' = .707107\\ 67^\circ\ 30' = .923880\\ \hline 20.18670\end{array}\right.$$

Hence, $a=1$, $4b=10.251660$, $2c=4027340$, and $n=8$.

Therefore, in the formula $\frac{3n\mathrm{A}}{a+4b+2c} = \frac{\pi}{2}$ we have $\frac{1}{2}\pi$ = 1.5707833, which is true to five places, but the sixth should be 9 instead of 8.

By comparing each of the results obtained above with the true numbers, we shall have the ratio of its approximation.

Thus, the difference between the first result and the true number is, .0035724

that of the second, .0002115

that of the third .0000180.

Hence, it will be seen that the result approximates in nearly a geometrical progression, to the true value of π as we increase the number of ordinates, or as the distance between the ordinates is decreased. We may, therefore, determine the number of ordinates that must be taken, in order to give an accurate result to any number of decimal places; for it will be perceived that the ratio of the above variations are nearly 16 to 1. Hence, we may safely infer, that it will approximate at the rate of three decimal figures in every two subdivisions. Instead of computing the value of $\frac{1}{2}\pi$ from the ordinates drawn in the whole quadrant, we may take any small arc of the quadrant, and having found its quadrature let it be called A′; and if we proceed as before, by drawing one ordinate equidistant from its extremes, we shall have, according to the formula, $\frac{\frac{1}{6}\mathrm{A}'}{a+4b} = \pi' =$ the assumed arc, which, multiplied by the number of times this arc is contained in the quadrant, will give the same result as though the ordinates are taken for the whole quadrant.

Let the formula $\frac{\frac{1}{6}A'}{a+4b} = \pi'$ be applied to a segment of the revoloidal surface whose arc is 30°, and whose greater ordinate is the radius.

Here, $A' = \text{sin. } 30° \times r = \frac{1}{2}$; hence, $A' = \frac{1}{2}A$,

$$a = r + \text{sin. } 60° = 1.866025$$
$$b = \quad \text{sin. } 75° = .965926$$
$$\frac{6A'}{a+4b} = \frac{3A}{5.729709} = .52358670 = \pi' = \text{arc of } 30°,$$

radius 1; hence the arc of 90°, or $\frac{1}{2}\pi = 1.57076010$; which is true to five places of figures.

Let there be two other ordinates drawn across the segment, then by the formula, $\frac{3nA'}{a+4b+2c} = \pi'$, since $\pi = \frac{1}{2}\pi \div 3$, we shall have $\frac{9nA'}{a+4b+2c} = \frac{1}{2}\pi$, and $\pi = 4$.

$$a = \left\{ \begin{array}{l} \text{sin. } 60° = .866025 \\ \text{sin. } 90° = .100000 \end{array} \right\} = 1.866025$$
$$b = \left\{ \begin{array}{l} \text{sin. } 67° \ 30' = .923880 \\ \text{sin. } 82° \ 30' = .991445 \end{array} \right\} = 1.915325$$
$$c = \text{sin. } 75° = .965926$$

Hence, $a+4b+2c = 11.459157$

and, $9nA' = 18$,

$$\frac{9nA'}{a+4b+2c} = 1.57079616.$$

which is true to seven places, which is as far as the sines are calculated, on which the value is predicated.

Let the first and second results be compared, and we shall have the difference of the first result and the true value,

$$\left. \begin{array}{r} 1.57079632 \\ -1.57076010 \end{array} \right\} = .00003622$$

that of the second,

$$\left. \begin{array}{r} 1.57079632 \\ -1.57079616 \end{array} \right\} = .00000016$$

Divide the first error by the last, and we have the quotient = 226 = the ratio of the approximation, or the proportional accuracy of determination, by varying the number of the ordinates. It will be perceived, therefore, that this portion of the revoloidal curve approximates much faster than the whole quadrant, and is, therefore, more nearly similar to a parabola than the whole curve; it may hence be inferred, that if any small segment of the revoloidal surface is taken adjacent to the conjugate diameter, such segment will be very nearly a portion of a parabola, and that, by so considering it, the value of the length of the arc may be determined with any required degree of accuracy.

For, in taking the whole quadrant, we found the ratio of convergency, by doubling the number of the ordinates, to be 1 to 16; and in the segment embracing the arc of 30°, adjacent to the conjugate diameter, we find the ratio of convergency to be 1 to 226; and if an arc is taken still smaller, the ratio of convergency will become proportionally greater.

Let any segment of the revoloidal surface be taken, and if the value of its arc, or the value of π, be computed by any number of ordinates, and if the number of ordinates is then increased so that the common distance is reduced one-half, and the value of π is again estimated by the increased number of ordinates, and if the variations of the two results from the true value be compared with each other, they will show the ratio of convergency of the process for that arc by increasing the number of its ordinates, or the rate of approximation by any specific increase of the ordinates for such arc or segment.

Hence, we may at all times determine the value of π to any required degree of exactness; for if we wish to determine its value to any given number of decimal places, we have only to assume some given arc and find its rate of convergency, then take such an arc as, according to this rate, will give the required result.

The arc of 90° gave the true result only to 2 places, that of 30° to 5 places, with three ordinates; and we may expect a still greater ratio of convergence for a smaller arc; let us take an arc of 10°; we may, according to this ratio, only have the value to 8 places, and by proceeding to decrease the arc, we should, by taking 1°, have the value to 16 places; but since the curve approaches more and more to a similarity with that of a parabola as we approach the vertices of the conjugate axis, the ratio of convergency increases also rapidly as we approach that point, or as the arc assumed is decreased; so that, by taking an arc of one minute of a degree, the accuracy of determination would extend to many places of decimals; and if the arc should be reduced still further, to seconds and fractions of a second, the result would come out true to several hundred decimal places; all of which is manifest by pursuing the investigation.

Let a distance be taken on the axis equal the arc of 2 minutes of a degree from the conjugate diameter. Then having

the sine $89° 58' = .9999998308$
that of $89° 59 = .9999999577$

the cosine of 2 is .00058177637, which, multiplied by radius, is the value of A′

We have, in the formula $\frac{6A'}{a+4b}$

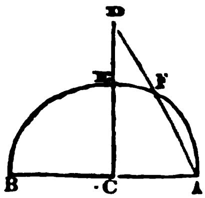
$$6A' = .00349065837$$
$$a + 4b = 5.9999996616$$

and $\frac{6A'}{a+4b} = 0005817764172 = \pi' =$ the arc of 20° 2′, which multiplied by 2700, the number of such arcs contained in a quadrant, we have the value of $\frac{1}{2}\pi = 1.57079632644$, which is true to 10 places, or as far as the sines on which its value is predicated.

For more extended investigations on this subject, see notes.

PROPOSITION XVIII. THEOREM.

If a circle be described, and from its centre a line equal to one-fourth of the circumference be drawn perpendicular to a radius, the triangle formed by connecting the extremity of this line with the extremity of the radius, will be equal to the quadrant of the circle.



Let ABE be a semicircle, described on the radius AC, from the centre C draw the line CD equal to one-fourth of the circumference perpendicular to the radius AC; join DA, then will the triangle ACD be equal to the quadrant AEC.

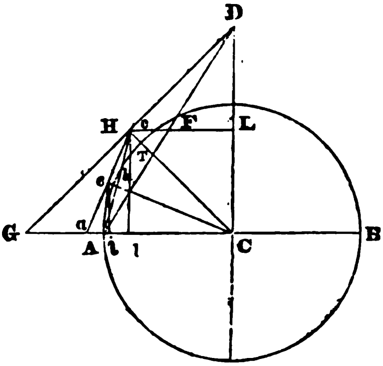
For, according to Prop. XV, Cor. 1, B. V. *El. Geom.* the area of a sector of a circle is equal to half the product of the arc of the sector multiplied by the radius. Now, the quadrant AEC is a sector of the circle, and the triangle ACD is equal to half the product of the arc AE, or the line CD multiplied by the radius AC; hence the triangle ACD is equal to the quadrant AEC.

Cor. Hence we may infer that AF, the segment of a circle cut off by the line DA, is equal to the portion of the triangle DEF cut off by the arc EF; for the triangle ADC is equal to the quadrant AEC, and if the line AD cuts off a segment AF from the quadrant, then it necessarily includes an equal space DEF within the triangle and without the quadrant; otherwise the triangle ADC, could not be equal to the quadrant AEC.

PROPOSITION XIX. THEOREM.

If a circle be described, and from the centre two radial lines be drawn perpendicular to each other, equal in length to the arc of the quadrant of the circle; and if the extremities of these radial lines be connected with another line as hypothenuse, forming with those lines a triangle, and if this third line is bisected by another radial line from the centre, bisecting also the arc of the quadrant; and if a line be drawn from the point of the hypothenuse cut by the last mentioned radical line, to the extremity of the radius, or the point where one of the firstmentioned radial lines cut the circle; then will the triangle formed by these three lastmentioned lines, be equal to the sector included between the two radial lines forming sides of this triangle; and the lastmentioned line will cut off a segment from the sector without the triangle, equal to the portion included in the triangle without the sector, and a perpendicular let fall from the point of bisection of the hypothenuse on the radius, is equal in length to the arc of the circle included between the bisecting line and radius.

From the centre C of the circle AB draw the radial lines CD and CG, each equal in length to the arc of the quadrant AE, forming a right angle at C, draw DG; draw also CH bisecting DG in H; and draw HA; then will the triangle ACH be equal to the sector ACT, and the segment A*b* cut off from the sector by the line HA, will be equal to the portion HT*b* included in the triangle ACH, but without the sector ACT; from the point H of the intersection of the line CH with DG, draw HI perpendicular to AC, and the line HI will be equal to the arc AT included between the radial lines AC and CH.



Draw the line AD forming with the radius CA, and the radial line CD the triangle ACD; and (Prop. XVIII.) the triangle thus formed is equal to the quadrant AEC; draw also HI perpendicular to CD; and because HL is parallel to AC, the two sides DG and DC will be cut proportionally by the line HL, (Prop. XIV, B. IV, *El. Geom.*); so that if the line DG is bisected in the centre at H, the line CD is also bisected in the centre at L, so that LC or HI$=\frac{1}{2}$CD.

Now, the two triangles ADC and AHC, having a common base, viz., AC, are as their altitudes, (Prop. VIII, Cor. B. IV

El. Geom.); but the altitude of the triangle AHC is HI, equal to LC, equal to half the altitude CD of the triangle ACD; therefore, the triangle ACH is equal to half the triangle ACD; but the triangle ACD is equal to the quadrant ACE; therefore, the triangle ACH is equal to half that quadrant, or is equal to the sector ACT, which is the first branch of the proposition.

Now, because the triangle ACH is equal the sector ACT, the segment A*b* cut off from the sector by the line AH, is equal to the portion HT*b* included in the triangle and without the sector, (Prop. XVIII. Cor.) which is the second branch of the proposition.

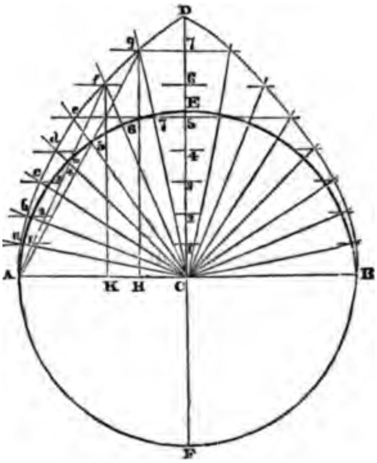
And because the line CD is equal to the arc AE of the quadrant, and because the arc AT is equal to half the arc AE, it is also equal to half the line CD=CL=HI, which is the last branch of the proposition.

Cor. If BA be extended to *a*, so that C*a* shall be equal CH, and if a line H*a* be drawn and bisected by C*e*, and a line be drawn from *e* to A, forming the triangle AC*e*, this triangle so formed will be equal to the sector of the circle intercepted by the lines CA and C*e*; and the line A*e* will cut off a segment of the circle without the triangle equal to the space included in the triangle without the sector, and a perpendicular *ei* let fall from *e* on the radius AC, is equal to the arc intercepted by the lines CA and C*e*. And the same may be inferred from any further divisions or subdivisions of the circle.

PROPOSITION XX. THEOREM.

With the radius CA *let there be described a circle* AEBF, *and from the centre* C *draw the line* CD *perpendicular to the diameter* AB, *or radius* CA, *and equal in length to the arc of the quadrant* AE, *let the line* CD *be divided into any number of equal parts, as* 1, 2, 3, 4, *&c., and let the arc of the quadrant be divided in like manner into a similar number of equal parts* 1, 2, 3, 4. *&c. From the divisions on the line* CD *draw the lines* 1a, 2b, 3c, *&c., parallel to the radius* CA; *and through the divisions* 1, 2, 3, *&c., on the arc draw the radial lines* Ca, Cb, Cc, *&c. Then, if from the points of intersection* a, b, c, *&c., of the radial lines, with their respective parallel lines according to their respective numbers, lines be drawn as* fA, *to the extremity of the radius* CA, *then will this line, together with* CA *and* Cf *form a triangle which is equal to the sector* CA6, *included within the radial line* Cf *and the radius* CA.

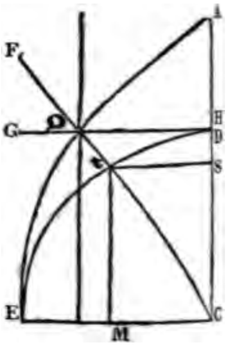
And the segment Am *cut off from the sector by the line* fA *is equal to the portion* fm6 *included in the triangle without the sector; and if lines be drawn from the several points of intersection of the parallel and radial lines perpendicular to the radius as* fK, *the lines so drawn will be respectively equal to the arcs intercepted by the radial lines from the extremities of which they are drawn, and their radius* CA.



For each of the radial lines C*a*, C*b*, &c., cut the arc AE in the same ratio that the corresponding parallel lines 1*a* 2*b*, &c., cut the perpendicular CD; thus the radial line C*g*, passing through the point or division 7 on the arc, cuts off one division from the arc, and intercepts with the radius CA all the rest, and the corresponding line 7*g* parallel to the radius CA cuts off D7 on the perpendicular CD, so that if the whole line CD is equal to the arc AE, the portion C7 of the line CD = H*g* is equal to the arc A7 of the circle. And the radial line C6*f* cuts the arc AE in the same ratio as the line *f*6 parallel to CA, cuts the line CD, viz., the radial line C6*f* cuts the circle through the division marked 6, and the line *f*6 parallel to CA, cuts the line CD in the division marked 6, so that if the whole line CD equal the arc AE, then will the portion C6 of that line = *f*K be equal to the portion A6 of the arc. Now, the area of every sector of the arc AC6, is equal to the arc of the sector multiplied by half the radius CA; but the triangle A*f*C is equal to the line *f*K multiplied by half the base or radius CA; therefore, the triangle A*f*C is equal to the sector AC6, and hence the segment A*m*, cut off from the sector by the line A*f*, is equal to the portion *fm*6 included within the triangle, but without the circle, and, as has been shown, *f*K=C6 is equal to the arc A6. And as the same holds true in each of the points of intersection *a*, *b*, *c*, *d*, &c., it follows that the result corroborates the affirmation expressed in the proposition.

Cor. If a curve line be drawn through the several points *a*,*b*, *c*,*d*, &c., and a triangle be formed by, two lines from any point in the curve drawn to the two extremities of the radius, the triangle so formed, will be equal to the sector included between the radius and the other line terminating in C, and the same relation of areas and lines will exist with regard to the triangles and lines drawn from any point of the curve, as though they were drawn through the points *a*, *b*, *c*, *d*, &c.

Scholium 1. The curve BD may be described about the quadrant BE in a similar manner; and since from any point in this curve, if a triangle is constructed on the radius as a base, this triangle is equal to the sector of the circle included between the radius and one of its sides, the curve may be called the curve of the circle's quadrature. This curve varies from the revoloidal curve, inasmuch as the revoloidal curve is formed by drawing a line through the points of intersection of a series of lines parallel to the radius drawn through their respective divisions on the perpendicular, with an equal series of lines perpendicular to the former, drawn through their respective divisions on the arc (see Prop. II.)



Scholium 2. This curve is generated by the locus of the intersection Q. of the two right lines CF,HG; HG being made to pass uniformly along from A to C, being always perpendicular to AC, while CF revolves about the centre through the arc ED.

Let the origin be at A.

Let AH $= x$, HQ $= y$

AC $=$ DE $= \pi'$, angle ACQ $= \theta$, tS $=$ sine $\theta = s$, tM $=$ *cosine* $\theta = c$

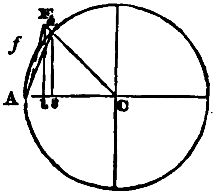
Then $c : s :: \pi' - x : y$

Hence $y = \dfrac{s\pi' - sx}{c}$ which is the equation to the curve.

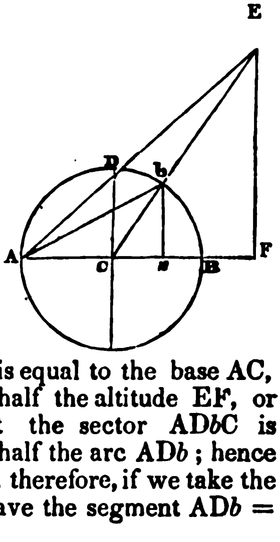
PROPOSITION XXI. THEOREM.

If from the extremity of the radius of a circle, a chord be drawn cutting off any segment less than a semi-circle, and if from the centre of the circle a secant be drawn through the opposite extremity of the segment, and if the secant be produced so that a line drawn from its extremity, perpendicular, to meet the diameter produced, shall be equal to the arc of the segment, then the area of the segment will be equal to that of a triangle formed by the chord of the segment, and the part of this secant line without the circle, and a line joining the opposite extremities of this line with that of the chord.

For it is evident from the converse of Prop. XVIII Cor., that if CE be so drawn that the perpendicular E*t* shall be equal to the arc A*a*, then the triangle CEA will be equivalent to the quadrant A*a*C; take away the triangle AC*a*, then there will remain the triangle A*a*E = the segment A*fa*.



First, let A*a* be a chord cutting off the segment A*fa*; from the centre C through the extremity of the segment at *a*, draw a secant line C*a* produced to E, so that the perpendicular E*t* on the diameter, shall be equal to the arc A*fa* cut off by such secant; then will the area of the segment A*fa* be equal to that of the triangle A*a*E formed by the chord A*a*, with the part *a*E of the secant without the circle, and the line AE joining the opposite extremities.



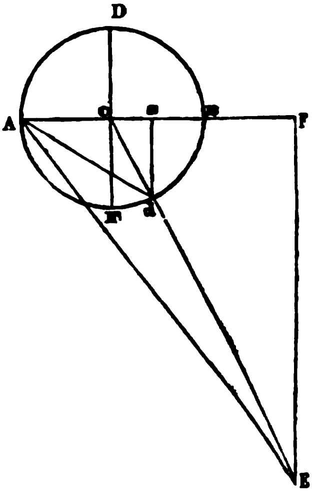
Secondly, let the chord A*b* extend into the second quadrant, cutting off the segment AD*b*, draw C*b*, and extend it to E, so that the perpendicular EF drawn from the point E to the diameter AB, produced, shall be equal to the arc AD*b* cut off by the chord A*b*, or the secant CE, and the area of the segment AD*b* will be equal to that of the triangle A*b*E formed by the chord A*b*, the part of the secant *b*E, and the line AE joining their opposite extremities.

For, in the triangle ACE, the area is equal to the base AC, the radius of the circle multiplied by half the altitude EF, or the arc AD*b* of the segment, but the sector AD*b*C is equal to the radius AC multiplied by half the arc AD*b*; hence the triangle ACE = the sector AD*b*C, therefore, if we take the triangle AC*b* from each, we shall have the segment AD*b* = the triangle A*b*E.

PROPOSITION XXII. THEOREM.

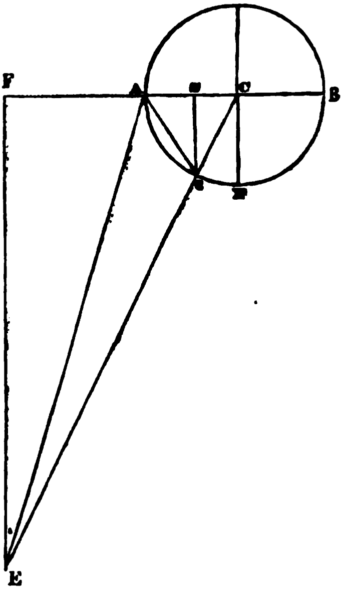
If from the extremity of the radius of a circle, a chord be drawn cutting off a segment greater than a semi-circle, and if through the opposite extremity of the segment, a secant be drawn from the centre of the circle, and if the secant be produced, so that a line be drawn from its extremity on the diameter, produced, if necessary, shall be equal to the arc of the segment, then the area of the segment will be equivalent to the triangle formed by the radius, secant line, and a line joining the opposite extremities of these lines, plus *a tri-*

angle formed by the radius and the chord, with the line joining the opposite extremities of those lines.



For, let the chord extend into the third quadrant, cutting off the segment ADB*d*A, greater than a semi circle, draw C*d*, and produce it to E, so that the perpendicular EF, on the diameter AB produced, shall be equal to the arc ADB*d* of the segment, and join AE, and the area of the triangle ACE, *plus* AC*d* will be equal to the segment ADB*d*A.

For the area of the triangle ACE is=the base AC, or radius, multiplied by half the perpendicular FA, which is equal to the arc ADB*d*, by hypothesis; but the area of the sector ADB*d*CA is equal to the radius CA multiplied by half the arc ADB*d*, hence the triangle ACE = the sector ADB*d*CA. Add to both the triangle AC*d*, and we have the segment ADBCA = the triangle ACE + the triangle AC*d*.



Secondly, let the chord A*e* extend into the fourth quadrant, cutting off the segment ADBFE. Through the point *e* draw the line CE produced, so that the line EF perpendicular to BA produced, shall be equal to the arc ADBF*e*. Then the segment ADBF*e* will be equal to the trian. ACE + the trian. A*e*C.

For the area of the triangle ACE is equal to the base AC, or radius multiplied by half the perpendicular FE, which is, by hypothesis, equal the arc ADBF*e*; but the area of the sector CADBF*e*C is equal to the radius AC multiplied by half the arc ADBF*e*; hence the triangle ACE is equal the sector

CADBF*e*C. Add to each the triangle AC*e*, and we have the segment ADBF*e* = the triangle ACE + the triangle AC*e*.

Scholium. It may be observed, that, as the termination of the segment approaches the point B, or as the segment becomes equal to the semi-circle, its equivalent triangle becomes infinitely extended in the line AE, and at the same time the sine becomes infinitely small, and while it passes the point B, the sine is equal to 0, and AE is infinite. The same may be said as it approaches the point A on the fourth quadrant, and becomes equal to a complete circle.

PROPOSITION XXIII. THEOREM.

The area of a segment of a circle is equal to half the product of the difference between the arc of the segment and its sine multiplied by the radius.

First, let A*fa* (see first diagram to Prop. XXI) be a segment of a circle cut off by the radial line C*a*, and produce it to E, so that the perpendicular E*t* on the radius will be equal to the arc of the segment; join EA and (Prop. XXI.) the area of the triangle AE*a* will be equal to the segment A*fa*; from the point *a* draw the perpendicular *as*, which is the sine of the arc A*a* of the segment, and the area of the triangle AE*a* will be equal to (E*t*—*as*) ½AC (Prop. XXXI, B. IV, *El. Geom.*) Hence, the segment being equal to the triangle, AE*a* is equal to half the product of the difference of the arc, and its sine multiplied by the radius.

Secondly, let the chord A*b* (see second diagram to Prop. XXI) extend into the second quadrant, draw CE, making EF = the arc AD*b*; draw AE, and the triangle AE*b* will be equal to the segment AD*b*; draw *bs*, the sine of the arc AD*b* of the segment; then will the area of their triangle AE*b* be equal to (EF -- *bs*) ½AC; (Proposition XXXI, Book IV, *Elements Geom.*); hence the segment, being equal to the triangle, is equal to half the difference of the arc and its sine multiplied by the radius.

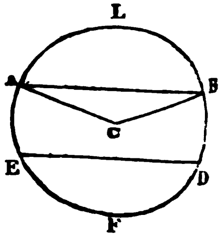
Thirdly, let the chord A*d* (see first diagram to last prop.) extend into the third quadrant, cutting off the segment ADB*d*, greater than a semi-circle.

Draw C*d* and extend it to E, making EF equal to the arc ADB*d*; join EA, and the area of the segment ADB*d* will be equal to the triangle ACE+AC*d*, (Prop. XXII); draw *ds* perpendicular to the radius CB, and this line will be the sine of the arc ADB*d*; and since it is a sine of an arc greater than a semi-circle, its value is to be considered negative, by Trigonometry.

From EF substract — dS, and we have EF + dS, which, if we multiply by AC, will give the area of the triangle AEC +ACd, which (Prop. XII.) is also equal to the segment ADBd. Hence the area of the segment ADBd is equal to the difference of the arc, and its sine multiplied by the radius.

Fourthly, let the chord Ae (see 2d fig. last prop.) extend into the fourth quadrant, cutting off the segment ADBFe; draw EC so that EF shall be equal to the arc ADBFe of the segment join EA, and the area of the segment ADBFe will be equal to that of the triangle ACE + the triangle ACe (Prop. XII); draw es, the sine of the arc ADBFe, which, by Trigonometry, is negative, being below the diameter, and in the fourth quadrant, which sine substract from the line EF, and we have EF + es, which, multiplied by half AC, gives the value of the segment ADBFe. Hence we have, as before, the segment equal to half the product of the difference of the arc of the segment, and its sine multiplied by the radius.

Hence we may infer, that a circular zone or a portion of the circle included between two segments is equal to half the product of the difference of the arc of the zone, and one of the segments included, and the sine of such arc, — the difference between the arc of the included segment, and its sine multiplied by the radius.



For if ABDE be a zone, and ABL be a segment of the circle C, the segment EALBD equal to the zone ABDE, and segment ABL; but these segments are respectively equal to the excess of their several arcs above the sines multiplied by half the radius. Therefore, the zone ABDFE, is equal to the difference of the excess of the arcs EALBD, and ALB, above their respective sines, multiplied by half the radius. The same may be shown in relation to the area ABDFE in reference to its external segment EDF, or AEFDB, regard being had to the positive and negative value of the sines.

Scholium. Let A represent the area of the segment of a circle; let x = the arc of the segment, and s = sine of that arc, and let r = radius, and the segment may be expressed as in the following formula

$$A = \tfrac{1}{2}(rx - rs) \quad . \quad . \quad . \quad (1)$$

If r = 1, then the expression becomes

$$A = \tfrac{1}{2}(x - s) \quad . \quad . \quad . \quad (2)$$

whence the segment is expressed in terms of the arc and sine

$A' =$ the area of a circular zone ABDE

$x' =$ the arc EALBD

$s' =$ the sine of that arc.

n will the area of the zone be expressed by

$$A' = \tfrac{1}{2}(rx' - rs') - \tfrac{1}{2}(rx - rs) \quad \text{- - - -} \quad (3)$$

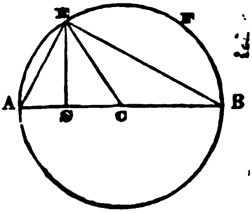
l $$A' = \tfrac{1}{2}(x' - s') - \tfrac{1}{2}(x - s) \quad \text{- - - - -} \quad (4)$$

arc may be expressed in terms of the numbers represent- segment, and the sine of the arc, by the formula, $x =$

$$\text{- - - - - - - - - -} \quad (5)$$

l the sine, in terms of the arc and segment by

$$s = x - 2A \quad \text{- - - -} \quad (6$$

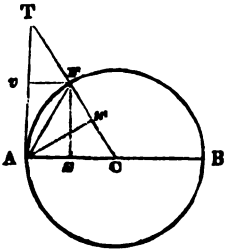


That the area of the segment of e is equal to half the difference of c of the segment, and its sine × by lius may also be shown as follows. E be the segment of a circle, radius is AC, draw ES perpendi- o the radius, and ES will repre- ie sine of the arc AE; let AC be nted by r, and ES by s, and the arc AE by x, then will = the sector ACE, and $\frac{1}{2}rs =$ the triangle AEC, and $\frac{1}{2}rs =$ the difference of the sector and triangle = the nt AE, viz., the seg. is = half the difference of the arc, sine multiplied by the radius.

, in the segment EFB let CB, be represented by r, and c EFB by x, and ES will be the sine of the arc EFB, = the sector ECBF, and $\frac{1}{2}rs =$ the triangle ECB. $\frac{1}{2}rx$ = their difference = the segment EBF.

PROPOSITION XXIV. THEOREM.

ea of the space intercepted by a tangent and secant with- circle, is equal to half the product of the difference of angent and arc, intercepted by the secant, multiplied by adius.



ATF be the space intercepted by gent AT, and the secant CT, with- circle; and the area of that space equal to half the product of the ce of AT and the arc AF, multi- y the radius AC.

the arc of the triangle ATC is equal ×AC, and the area of the sector s equal to half the arc AF×AC.

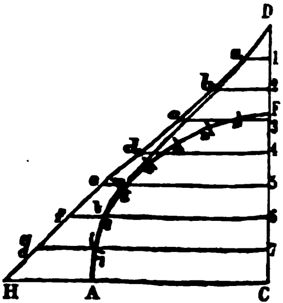
$t =$ the tangent AT, and $x =$ the arc AF, $r =$ AC.

Then we have for the triangle ACT, $\frac{1}{2}rt$, and for the sector AFC, $\frac{1}{2}rx$; if from the triangle we take the sector, we have $\frac{1}{2}rt-\frac{1}{2}rx=$the space AFT, hence as in proposition.

Scholium. Draw F*s*, FA, and F*v*; then F*s* will represent the sine of the angle ACF, and *s*A or T*v*, the versed sine; hence the traingle ATF equals one-half the product of the difference of the tangent and sine multiplied by radius equals one-half the product of the tangent and versed sine; draw A*s'* perpendicular to CF, and this will also represent the sine of the angle ACF on the radius CF; hence, also, the triangle ATF is equal to one-half the difference of the secant CT and radius multiplied by the sine.

PROPOSITION XXV. THEOREM.

If about a plane revoloidal surface from a quadrangular revoloid, a square be described, the space enclosed by the square, but without the revoloidal surface, will be to that contained by the square, as the sum of an infinite series of segments of the quadrant of the circle, whose arcs are in arithmetical progression, to the sum of a similar series of sectors subtended by the same arcs.



Let ACD be a quadrant of a plane revoloidal surface, and let HCD be a triangle forming one-fourth of a square circumscribing the whole revoloidal plane surface. Divide the axis CD into any number of equal parts as 1, 2, 3, 4, &c., and through the points of division draw ordinates 1*a*, 2*b*, 3*c*, &c., parallel to HC, and these ordinates will cut the surface of the quadrant of the revoloid and its circumscribing triangle, in the relations of their magnitudes, through the portions where such ordinates pass; and if the number of those ordinates be indefinitely increased, the sum of those drawn across the triangle HCD, will be to the sum of those drawn through the revoloidal surface ADC, as the area of the triangle to the area of the revoloidal surface. Now, the ordinates *a*1, *b*2, *c*3, &c., are equal to the arcs represented by numbers on the arc AF of the quadrant, corresponding to those on the axis, (Prop. II, and Cors.); and the portions *i*7, *b*6, *m*5, &c., of those ordinates, are severally equal to the sines of those arcs, (Prop. XIII, Cor. 1,); hence the portions of those ordinates, intercepted by the curve AD and line DH, are severally equal to the difference of the arcs and sines represented by these ordinates. And since we have shown (Prop. XXIII.) that the area of the segment of a circle is equal to half the difference of the arc of the segment and its sine multiplied by radius, and since the area of the sector of a circle is equal to half the arc of the sector multiplied by radius, it follows that the sector and segment containing the same arc, are to each other, as the arc is to the difference of the arc and sine.

Let a equal the arc, s equal the sine, r equal the radius.

Then $\frac{1}{2}ra=$the sector,

and $\frac{1}{2}ra-\frac{1}{2}rs=$the segment containing the same arc, which are to each other in the ratio of

$$\tfrac{1}{2}ra : \tfrac{1}{2}ra-\tfrac{1}{2}rs, \text{ or of } a : a-s.$$

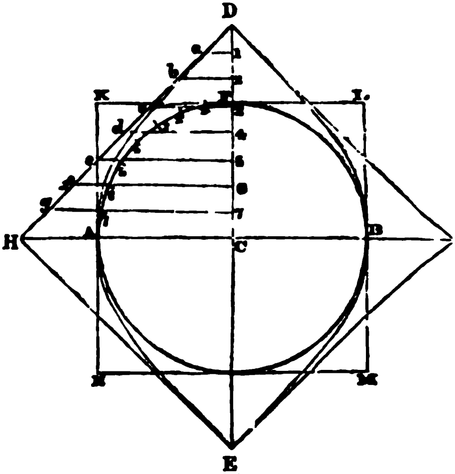
And since this is true for a segment and sector, contained by any arc in that quadrant, that is, the sector of a circle is to the segment containing the same arc as the arc is to the arc minus the sine, or as the ordinates $g7$ to gi, or as $f6 : f\,l$, &c.; and as this relation evidently exists in reference to each of the parallel ordinates, and as the ordinates represent arcs of the circle in arithmetical progression, it follows that those ordinates drawn across the triangle, may also represent a series of sectors of a circle, while the portion of those ordinates intercepted by the curve AD and line HD, may represent a similar series of segments of the circle. And as we have shown above that the surfaces HCD and HAD are to each other in the relation of the ordinates passing through each, it follows that the trilinear space ADH will be to the triangle HCD as the sum of an infinite series of segments of the circle whose arcs are in arithmetical progression, whose first term is equal to the common difference, and whose last term is the quadrant of the circumference to the sum of a similar series of sectors with the same arcs. And since the whole plane revoloidal surface consists of four quadrants, ADC, and its circumscribing square consists of four triangles, HCD, it follows that the whole revoloidal surface will be to the whole circumscribing square in the same ratio.

Cor. The revoloidal quadrant ACD, may be represented by a similar series of isosceles triangles formed by the chords of the series of segments with two radii drawn from the extremities of those chords. For we have shown that the series of sectors may be represented by the area HCD, while the series of segments are represented by HAD; if we take the segments from the sectors, we shall have the triangles, evidently equal HCD $-$ HAD$=$ACD.

PROPOSITION XXVI. THEOREM.

The area of a quadrangular revoloidal surface, from a right quadrangular revoloid, is to that of its circumsribing square, as the sum of an infinite series of sines of the quadrant whose arcs are taken in arithmetical progression, and whose common difference is equal to the first term, to a similar series of arcs of such sines.

Let DHEI be a square circumscribing the plane revoloidal surface DAEB, and let ADC be a quadrant of that surface, the triangle HDC being its corresponding portion of the square; divide the semi-axis CD into any number of equal parts, as 1, 2, 3, 4, &c., and through the points of division draw the ordinates $1a$, $2b$, $3c$, &c., parallel to CH, those lines will each be equal to such arc of the quadrant AF as corresponds to the point of division on the line CD, in reference to similar divisions of the quadrant AF; thus the line $1a$ equal the arc F1 on the quadrant, since it is evidently equal to 1D on the axis CD; and (Prop. II, Cor.) $1D=1F$; also, if we take the line $2b=2D$, this is also, for similar reasons, equal to the arc 2F; and hence each line parallel to CH, terminated by the lines CD, HD, are equal to the arc, corresponding to the divisions on the line CD, through which it passes.



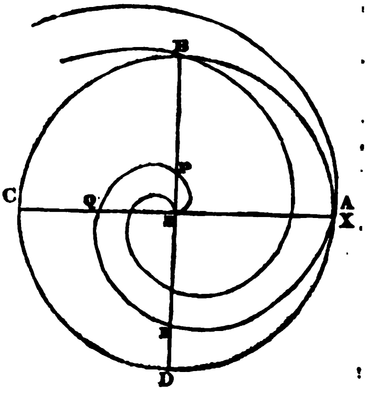
Again, the portion of these lines or ordinates intercepted by the axis CD, and the curve AD, (Prop. X, Cor.) are severally equal to the sines of the arcs, corresponding to the divisions from which they are drawn on the axis. And since, by hypothesis, the arcs are taken in arithmetical progression, they must be equidistant; hence, if the number of such ordinates are indefinitely increased, the sum of the portions of them intercepted by the axis CD, will represent the area or the surface ADC, as the whole ordinates drawn across the triangle HDC represent the area HDC. Therefore the surface ADC is to the surface HDC as the sum of the series of sines, to the sum of a similar series of arcs taken as above. Now, since this is the case with one quadrant, it must also be true in relation to the whole revoloidal surface and its circumscribing square.

Cor. 1. Hence, if a square KLMN be described on the diameter of the inscribed circle, the square so described will be to the square DHEI as the sum of the series of sines of the quadrant, to the sum of a similar series of arcs of the sines; since the square KLMN (Prop. III, B. III,) is equal to the area of the revoloidal surface DAEB.

Cor. 2. Each of the ordinates *a*1, *b*2, *c*3, &c., drawn across the triangle HCD parallel to HC, is equal to the arc of the circle represented by the corresponding numbers in the divisions of the quadrant AF. Thus, *a*1 equal the arc 1F, *b*2 equal the arc 2F, and *g*7 equal the arc 7F or the quadrant.

ON SPIRALS.

There is one class of curves which are called spirals, from their peculiar twisting form. They were invented by the ancient geometricians, and were much used in architectural ornaments. Of these curves, the most important as well as the most simple, is the spiral invented by the celebrated Archimedes.



This spiral is thus generated: Let a straight line SP of an indefinite length move uniformly round a fixed point S, and from a fixed line SX, and let a point P move uniformly also along the line SP, starting from S, at the same time that the line SP commences its motion from SX, then the point will evidently trace out a curve line SPQRA, commencing at S, and gradually extending further from S.

When the line SP has made one revolution, P will have got to a certain point A, and SP still continuing to turn as before, we shall have the curve proceeding on regularly through a series of turnings, and extending further from S.

To examine the form and properties of this curve, we must express this method of generation by means of an equation between polar ordinates.

$$\text{Let } SP = r,\ SA = b,\ ASP = \theta\ ;$$

then since the increase of r and θ is uniform, we have

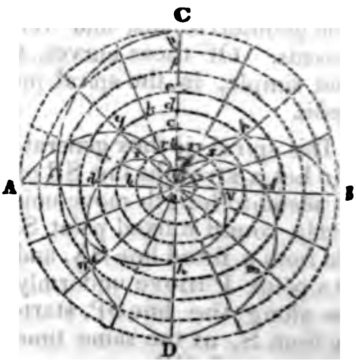
$$SP : SA :: \text{angle } ASP : \text{four right angles} :: \theta : 2\pi$$

$$\therefore r = \frac{b\theta}{2\pi} = a\theta, \text{ if } a = \frac{b}{2\pi}.$$

From this equation it appears that when SP has made two revolutions or $\theta = 4\pi$, we have $r = 2b$, or the curve cuts the axis SX again at a distance, 2SA ; and similarly after 3, 4, n revolutions it meets the axis SX at distances 3 ; 4, n times SA.

Let any number of concentric circles be described, whose radii IA, IC are in arithmetical progression, and if the circumference of the outer circle is divided into any number of equal parts, and radii are drawn from each of the points of division in the circumference of the circle to the centre I, lines drawn from the centre in such manner as to pass through the points of intersections, of the several radii with the curves in consecutive order, will be spirals similar to those of Archimedes.

If the curve line pass from I, through *rp*B, or through *rq*A, the curves I*rq*A, I*rp*B will be spirals ; so also the lines I*sdm*D, I*sdn*D, and I*vhf*C and spirals ; all generated by the same laws, but with different ratios of their angular, compared with their rectilenear motion ; or their circular, with their radial motion. The two spirals I*sdm*D, I*sdn*D, commencing on the line IC, and terminating at D, form the heart-like figure I*sdm*D*nds*I.



These spirals, and especially the one with a heart-like form, are extensively used in mechanical operations ; to communicate a uniform rectilenear, reciprocating, from a rotary motion; it is therefore important that they receive some consideration.

The area of any portion included between the spiral and its circumscribing arc of the circumference terminated by the radius, its origin, is equal to two-thirds the sector, having the same arc of the circumference as its base.

For let I*a*, I*b*, I*c*, &c., be the radii of circles in arithmetical progression, then will the arcs of these circles intercepted by two radii, be also in arithmetical progression, and since the value of θ also increases uniformly in arithmetical progression along with r or π, hence the value of the intercepted parts of the several arcs will be a series of arithmeticals multiplied into another corresponding series of arithmeticals, therefore their products will be a series, of the squares or a series of numbers proportional to a series of squares of a series of arithmeticals.

It has been shown (Prop. IV, Cor. 3, B. I) that the sum of an infinite series of the square of a series of numbers in arithmetical progression increasing from 0, is equal to $\frac{1}{3}$ of the last term multiplied by the number of terms.

But the arc intercepted by the spiral with the radius, as its origin is the last term, and the radius represents the number of terms; hence the area I*rp*BCI is equal to $\frac{1}{3}$ of the product of the arc BC × IC; and the area I*sdm*DBCI is = $\frac{1}{3}$ arc CBD × IC; also the area I*vhf*CADBCI is = $\frac{1}{3}$ circumference ACBD × IC. But the area of the whole sector IBC, ICBD, or ICBDAC in either case is equal to half the product of their respective arcs, multiplied by the radius; hence the space intercepted by the spiral in each case is $\frac{1}{3}$ that of their respective sections.

Cor. 1. Hence the space I*rp*BI, is = $\frac{1}{3}$ the sector ICB; the area I*sdm*DI is = $\frac{1}{3}$ the sector ICBD or $\frac{1}{3}$ the semi-circle; and the area I*vhf*CI is = $\frac{1}{3}$ of the whole circle,

Also the heart I*sdm*D*nds*I = $\frac{1}{3}$ of the whole circle.

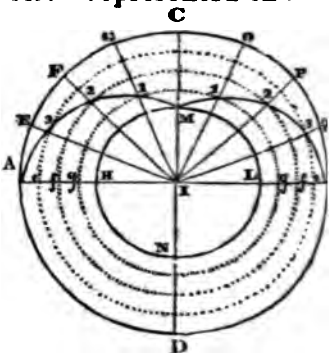
Scholium 1. The spiral of Archimedes is sometimes used for the volutes of the capitals of columns, and in that case the following description by points is useful. (See first diagram.)

Let a circle ABCD be described on the diameter CSA, and draw the diameter BD at right angles to AC; divide the radius SC into four equal parts, and in SB take SP=$\frac{1}{4}$SC, in SA take SQ=^{1}SC, and in SD take SR=$\frac{3}{4}$SC; then from the equation to the curve these points belong to the spiral; by subdividing the radius SC and the angles in each quadrant we may obtain other points as in the figure. In order to complete the raised part in the volute, another spiral commences from SB.

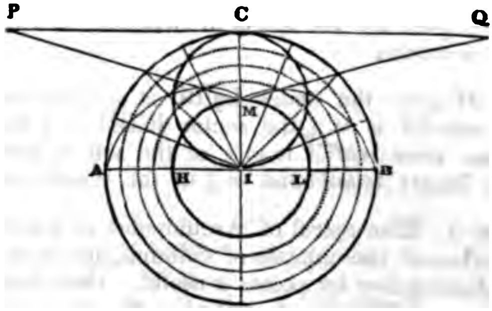
Scholium. These spirals are of the same kind, as th formed by winding a chord around a conical spire, from vertex to the base, in such manner, as to encircle the spire equal distances; the exact length of such curve is of diffic determination.

The same spiral would be represented by the convoluti of a conical screw; also, by a screw represented on a di

If the origin of the spiral is at any point M, not in the centre of the concentric circles, then the area AFCM123A between the spiral and the outer circumference is $=\frac{1}{2}$ of the product of the arc ACB through which the curve would have passed from the centre I, multiplied by the radius $-\frac{1}{3}$ of the arc $LM \times IM - \frac{1}{2}$ (arc $LM +$ arc $BC) \times MC$.



If PQM be a triangle, whose base $PQ =$ the semi-circu ference $ACB =$ the angular space passed through by the tv spirals MA, MB, then either portion PCM, QCM of the t angle may be expressed by $\frac{1}{2}$ PC or $\frac{1}{2} AC \times CM = \frac{1}{2}$ arc CB $IC - \frac{1}{2}$ arc $LM \times IM - \frac{1}{2}$(arc $LM +$ arc $BC) \times MC$, from tl



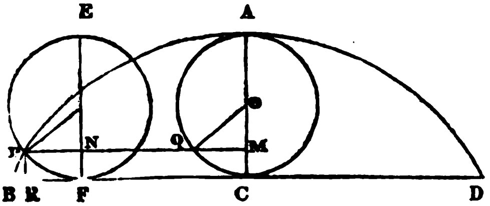
substract the expression for the area included within the spir and the arc AB, and we have $\frac{1}{4} ACB \times IC - \frac{1}{6} LM \times IM$ the difference of the areas.

THE CYCLOID.

If a circle EPF be made to roll in a given plane upon a straight line BCD, the point in the circumference which was in contact with B at the commencement of the motion, will, in a revolution of the circle, describe a curve BPAD, which is called the cycloid.

This is the curve which a nail in the rim of a carriage-wheel describes during the motion of the carriage on a level road. The curve derives its name from two Greek words signifying "circle formed."

The line BD which the circle passes over in one revolution is called the base of the cycloid; if AQC be the position of the generating circle in the middle of its course, A is called the vertex and AC the axis of the curve. The description of the curve shows that the line BD is equal to the circumference of the circle, and that BC is equal to half that circumference. Hence also if EPF be the position of the generating circle, and P the generating point, then every point in the circular arc PF, having coincided with BF, we have the line BF = the arc PF, and FC = the arc EP or CQ;



Draw PNQM parallel to the base BD.

Let A be the origin of the rectangular axes,

AC the axis of x, and O the centre of the circle AQC.

Let AM $= x$, AO $= a$,

MP $= y$, angle AOQ $= \theta$:

then by the similarity of the position of the two circles, we have

$$PN = QM, \text{ ond } PQ = NM;$$

$$\therefore MP = PQ + QM = NM + QM = FC + QM = \text{arc } CQ$$

$$+ QM \text{ that is, } y = a\theta + a \sin. \theta = a(\theta + \sin. \theta) \qquad (1)$$

$$x = a - a \cos. \theta = a \text{ vers. } \theta \qquad (2)$$

The equation between y and x is found by eliminating θ between (1) and (2)

$$\cos.\ \theta = \frac{a-x}{a} \quad \therefore \sin.\ \theta = \frac{\sqrt{2\,a\,x-x^2}}{a}$$

$$\text{and } y = a\,\theta + a \sin.\ \theta$$

$$= a \cos.^{-1}\left(\frac{a-x}{a}\right) + \sqrt{a\,x-x^2}$$

But we can obtain an equation between x and y from (1) alone; that is from the equation, AP = arc CQ + QM.

For arc CQ = a circular arc whose radius is a and versed sine x

$$= a \left\{ \text{a circular arc whose radius is unity and vers. sin.} \frac{x}{a} \right\}$$

$$= a \text{ vers.}^{-1} \frac{x}{a}$$

$$\therefore y = a \text{ vers.}^{-1} \frac{x}{a} + \sqrt{2\,a\,x-x^2}$$

If the origin is at B, BR = x and RP = y, the equations are

$$x = a\,\theta - a \sin.\ \theta$$

$$y = a - a \cos.\ \theta.$$

We shall not discuss these equations at length, as the mechanical description of the curve sufficiently indicates its form.

The cycloid, if not first imagined by Galileo, was first examined by him; and it is remarkable for having occupied the attention of the most eminent mathematicians of the seventeenth century.

Of the many properties of this curve the most curious are, that the whole area is three times that of the generating circle, that the arc CP is double of the chord of CQ, and that the tangent at P is parallel to the same chord. Also that if the figure be inverted, a body will fall from any point P on the curve to the lowest point C in the same time; and if a body falls from one point to another point, not in the same vertical line, its path of quickest descent is not the straight line joining the two points, but the arc of a cycloid, the concavity or hollow side being placed upwards.

BOOK V.

ON THE PRODUCTION AND RESOLUTION OF GEOMETRICAL MAGNITUDES, CONSIDERED AS LINES, SURFACES, AND SOLIDS, EXISTING IN THEIR SPECIFIC RELATIONS OF FORM AND PROPORTIONS.

CHAPTER 1.

DEFINITIONS AND PRINCIPLES.

ART. 1. We have hitherto referred lines, surfaces and solids, in all their varieties of figures and species, to some specific quantities and relations which were cognizable in such magnitudes, and whose properties were rendered evident to our consideration. Magnitude we have compared with magnitude; figure with figure; and we have thereby established their relations, under arbitrary considerations.

We will now consider magnitudes in the relation of their organization, or in the relation of their laws of production; and instead of referring magnitudes to specific magnitudes arbitrarily chosen, we will refer them to others, only in the relation of their laws of generation.

2. Since a point by definition is locality without extension, any number of associated points cannot possess magnitude, hence a magnitude is not a multiple of one, or any number of points.

3. Neither can any number of lines, however associated, constitute a surface, since lines are supposed to possess no breadth or thickness, one of which is essential to a surface; for if one line does not possess breadth, neither can any number of associated lines; and if a line be multiplied by any abstract number, since it is expressed only in relation to its length, it can only be multiplied or increased in that relation.

4. So, also, if a surface be multiplied by any number, in itself considered, the product cannot be a solid; for since the surface possesses no thickness, it does not possess the characteristic of the solid, and hence any number of such surfaces, or multiple of such surfaces in themselves considered, cannot be a solid.

5. The distance between any two points is a line. For a point being locality without extension, if there be two locali-

ties, they must be seperate from each other, and their distan from each other is necessarily extension in space, which agre with our definition of a line, viz., "extension in one dimensio

6. Space is a medium in which all positive objects, and a local relations exist; its existence is only indicated by its u versal property of extension; it is infinitely divisible in ea or all its three dimensions of extensions, and infinitely exte sible.

7. Any definite portion of space, or any extension in spac is magnitude.

Magnitude may possess extension in one, two, or three d mensions, but space can properly exist only in its three dime sions of extension; if it can be divested of extension in o dimension, it can in another, and so on till its extension is e tinguished. Magnitude may be properly applied to extensio in whatever degree it exists; but space cannot properly exi independent of its three dimensions, which are its essenti properties.

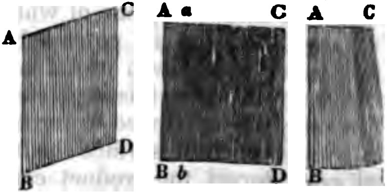
8. If there be two points A, B, the first point A, draw through the distance AB, produces or describes the line AB; that is, the distance from A to B in the portion of space passed through by the point A, or the locality occupied by the point A in its passag is the line AB.

A———B

9. If a line be moved through any space in a direction n agreeing with its length or extension, the locality pass through by the line is a surface.

Thus, if a line AB, be moved from its position AB to CD, will, by that means, generate the surface ABDC, for the lin will have occupied every portion of the extension between th two lines AB and CD, which cannot be said of any limite number of lines placed in juxtaposition across the figure ABD

If, in this motion, the line AB always maintains its parall position, and if any point A in the line, describes a right line AC, the surface will be a rhomboid or parallelogram; and if, in addition to this, the line AC, or the direction of its motion is perpendicular to AB, then will the rhomboid be a rectangle.



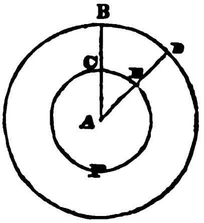
But if the line AB in its motion should not preserve its par allel position, or if the distance BD, passed through by the poi B, is greater than AC, then the figure generated will depend o the nature of the lines which serve as its boundaries, but i general, in such case, one or both of the lines AC, BD will b curves, the nature of which will depend on the specific variation of the course and inclination of the generating line.

If instead of one line AB being drawn through the whole distane AC, a series of parallel lines AB, *ab*, &c., are drawn through their respective distances from each other, the result will be similar; or the same surface will be described by the series drawn through their several small distances, as by the single line drawn through the greater distance.

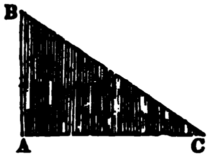
The geometrical operation of drawing a line through a given distance to produce a surface, is equivalent to that of multiplying a line by a line, the product of which we have shown in the elements of geometry to be a surface.

The measure of the surface generated by the motion of a line, is the length of the line multiplied by the distance passed through by the line; which distance may always be regarded as the distance moved by the centre of the line.

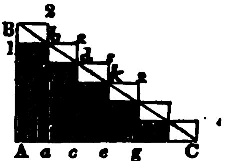
10. If the point A remains fixed while the line revolves around it till it, comes again into the position AB from whence it started, the surface generated is a circle; and the line described by the point B, will be the circumference of the circle. If it moves only from the position AB to AD, the surface will be a sector of a circle, and the line described by the point B, is the arc BC of the circumference, and because the circumference CEFC described by the centre C of the revolving line represents the whole motion of the line AB in its revolution; hence, AB drawn into the circumference CEFC, represents the whole surface generated.



11. If the line AB be conceived to decrease uniformly, as it is moved forward, always in a parallel position, till it terminates in a point, the figure generated by the motion of the decreasing line AB, will be a triangle ABC.



We may, instead of supposing the triangle to be generated by the line AB, decreasing as it advances toward C, suppose it to generated by an infinite series of decreasing ordinates, parallel to AB; each of which may be supposed to be drawn through the distance, between itself and the next one. Thus, let *ab*, *ef*, &c., be a series of decreasing ordinates situated equidistant from each other on the line AC, and let *ab*, be drawn into the position A1 on the line AB, *cd*, into the position *a*1, *ef*, into the



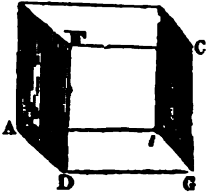
position *c*1, &c., through the whole series. Now, since at the extremity of each of the consecutive ordinates a triangle B1*b*, *b*1*d*, &c., is left without the triangle ABC, which has not been described or passed over by the ordinates, the triangle is therefore not perfectly described by the decreasing ordinates; but if the number of the ordinates are infinitely increased, those spaces become indefinitely small, and hence may be omitted as being of no appreciable value. Moreover, if the motion of the ordinates is reversed, or if they are moved from A toward C, taking AB as the first ordinate. the surface thus described would exceed the triangle itself, in the same amount as it would fall short in the former case; thus, let AB be brought in the position *a*2, *ab*, into the position *c*2, &c., and the small triangles formed above the hypothenuse BC, by this means, would be equal to those falling below it in the former case, and half their sum would be a correction to be added or subtracted in either case; but since this correction is equivalent to the surface generated by half the line AB drawn through the distance A*a*, it follows that the sum of all the ordinates *ab*, *ac*, &c., +half the line AB drawn into the common distance A*a*, generates an area equivalent to the triangle ABC.

But when the number of ordinates are indefinitely increased one-half, AB is infinitely small in regard to their sum, and hence may be omitted.

12. Hence, if an infinite series of equidistant and parallel ordinates to a right line AD, decrease uniformly from AB to AC, then will the surface generated by drawing those ordinates through or into their common distance, be a trapezium.

13. If a surface be drawn through any space not in the direction of a line parallel to the surface, the product will be a solid.

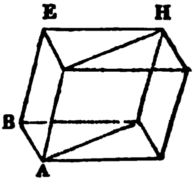
Thus, if a surface ABFD be drawn through the distance DG, till it comes into the position IEGC, it will thereby generate a solid AC.



If the surface is always parallel to its first position, and if any point A, moves through a right line AI, the solid will be a prism, which will also assume a character according to the figure of the generating surface: thus, a rectangular, a triangular, or a polygonal prism, may be generated by the motion of a rectangle, a triangle, or a polygon; and in the same manner may a circular prism, or cylinder, be generated by the motion of a circle, always parallel to its first position, and in a direction perpendicular to such generating surface, If instead of one surface being drawn through the whole distance AB, there be a series of similar surfaces parallel to each other, and if each be drawn through their respective distances, the same solid would thereby be generated.

The same principles may also apply to solids of revolution of any figure about a fixed axis, and the partial revolution of a series of similar figures, one of whose several sides is the common axis.

14. If a plane quadrilateral surface ABGF, be conceived to move uniformly along in a direction perpendicular to itself, and to decrease uniformly in one of its dimensions during its motion, till it terminates in a line; it will, by that means, generate a wedge or a triangular prism, ABEFHD.



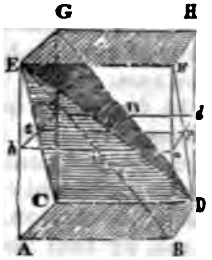
If, instead of the decreasing plane, there be an infinite series of quadrilateral planes equidistant from each other and decreasing in one of their dimensions in consecutive order, and in arithmetical progression till one of them terminates in a line, then the series of planes drawn through their infinitely small distance will generate a wedge or a triangular prism; all of which becomes evident by reference to Art. 11, for the same reasoning will apply here as in that case, since this prism may be conceived to be generated by the perpendicular motion of the triangle, which was there found to be the product of an infinite series of decreasing lines.

15. Hence, if across any plane figures, A, B, C, an infinite number of parallel and equidistant ordinates are drawn, the sum of the ordinates intercepted by each may be regarded as a measure of their surfaces when compared with each other; although lines, however associated, cannot represent surface in absolute terms, yet an infinite number of parallel lines drawn equidistant across two or more surfaces, will represent the ratios of those surfaces to each other, and may hence represent those surfaces in relation to their forms and comparative magnitudes.

Hence, for the purposes of investigation, the properties of geometrical magnitudes and their relations, an infinite series of parallel ordinates drawn across any plane figure, may be regarded as the measure of that figure.

And also, for the same reasons, may an infinite series of parallel and equidistant planes passed through a solid, represent the capacity or value of the solid.

Neither will this mode of investigation lead to any error, seeing, we do not thereby establish any absolute measure for the surface or solidity, in terms of superficial and solid units, but only the relations of certain surfaces or solids through this medium, to certain other surfaces or solids with which they are compared; and since they are compared with each other under the same circumstances, the result of that comparison must hence be correct.

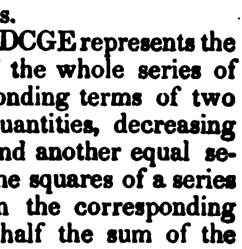


16. Let the rectangular prism AH, whose base ABDC is supposed to be a square, be divided as at Prop. IV. B. I., into the pyramids ABDCE, EFHGD, CDGE, BEFD, through which let a plane *hodc* be passed, cutting the several pyramids in the sections *higs*, *sgnc*, *iopg*, *gpdn*, parallel to the base ABDC. Let hi or $hs=a$, and ia or $dn=b$, then may the section *hode* be expressed by $a^2+2ab+b^2$, that is, the section $higs=a^2$, $(sgnc+iopg)=2ab$, and $gpdn=b^2$; $\sqrt{a^2+ab}$ is a mean proportional between a and $a+b$. Let an indefinite number of planes be passed through the solid parallel to the base, and each section may be expressed in the same manner, but the value of a, it will be seen, is constantly decreasing in arithmetical progression, as we ascend from the base to the vertex E; and hence, represents successively a series in arithmetical progression; the value of b is also increasing in the same order during the successive ascent of the series of planes; but $a+b$ is a constant quantity during the whole change of the relative values of a and b, and hence $a+b$ represents a successive series which is constant. Now, because a^2+2ab, the square of the proportional mean between a, one of the series of arithmeticals, and $a+b$, one of the series of constants, which repsesents a section *hine* through the two pyramids ABDCE, CDFE; it follows that since an infinite series of parallel sections represent the whole of these pyramids, that the sum of the squares of the whole series of geometrical means will represent the whole of the solid ABDCEG. But it has been shown, (Prop. IV. B. I,) that the solid ABDCGE is equal to one-sixth of the product of the sum of the squares of the two bases *plus* four times a middle section drawn into its altitude. Hence, if there be any infinite series of quantities, such that the terms are severally geometrical means between the several terms of a series of arithmeticals, and of a similar series of constant quantities; then will one-sixth of the sum of the square of the first and last terms *plus* four times the square of the middle term drawn into the series, be equal to the sum of the squares of the series.

Hence, if an infinite series of quantities varying in arithmetical progression, be drawn into a similar series of constant quantities, then will the sum of the series of rectangles be equal to the product of the first, *plus* the last term, *plus* four times the middle term drawn into one-sixth of the series.

17. It will appear that since the solid ABDCGE represents the sum of all the, a^2+ab, viz., the squares of the whole series of mean proportionals between the corresponding terms of two other series; if there be a series of quantities, decreasing from z to 0 in arithmetical progression, and another equal series of z a constant quantity, the sum of the squares of a series consisting of mean proportionals between the corresponding terms of the two series, will be equal to half the sum of the squares of the series of z, or equal to half the square of z drawn into the series.

18. It will also appear, that if there be an infinite series of quantities decreasing from z to p in arithmetical progression, and another similar series of constant quantities, z, the sum of the squares of a series of mean proportionals between the corresponding terms of the two former series, will be equal to half the sum of the squares of the series of z, *plus* one-half the sum of a similar series of rectangles of $p\times z$.

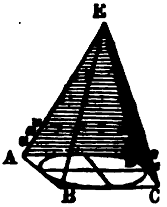


For, let AE be a prism, the length AB or AC, of whose side is equal z, and if we make IG and LO$=p$, and construct the plane LIBD, we shall have the prismoid ABDOCGIL, which may be represented by the sum of the squares of a series of geometrical means between the terms of the series of the decreasing arithmeticals, and those of the constant quantities. But this prismoid may be further divided by the plane GOCB into the wedge ABCDOG, which is equal to half the sum of the squares of the series of z, and the wedge GOLIBC, which is equal to half of a similar series of the rectangles of z into p.

19. Let a series of s, be the series of constant quantities, (and let $s=$AD,) while another series varies from z to p in arithmetical progression, (making $z=$AB, and $p=$GI.) Then will the sum of the squares of the series of geometrical means between the terms of the two former series be equal to half the sum of a similar series of rectangles of $s\times z+$a series of rectangles of $s\times p$.

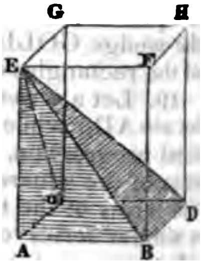
For the prismoid ABCDOGIL equals the series of mean proportionals as before, and the wedge ABDCOG will be equal to half a similar series of rectangles of the series s with a similar series of z, and the wedge IGHLCB will be equal to half the sum of a similar series of s, drawn into an equal series of p.

Hence, a wedge may be conceived to be generated from a series of squares of mean proportionals between a series of constant quantities, *s* or *z*, and an infinite series of lines in arithmetical progression from *z* to *o*, or from *o* to *z*.



20. Let AD, *ad*. *ad* &c., be an infinite series of lines, increasing in arithmetical progression from 0 to *z*, or from E to BA, and the arithmetricals will generate or constitute the triangle ABE, and the sum of their squares will generate or describe the pyramid ABCDE. For if an infinite number of parallel lines or ordinates be drawn across a plane figure, those ordinates will represent the figure in the relation of its magnitude and form; (Art. 12,) but the lines or ordinates, *ad*, *ad*, are infinite in number by hypothesis, and increase from 0 to *z*, or they commence at E and terminate at AB; hence the sum of the ordinates constitute, or are a function of the triangle. Also if an infinite number of parallel planes be passed through a solid, the sum of those planes drawn into their distance, may be regarded as the solid when compared; for the solid consists of an infinite number of parallel planes drawn into their infinitely small distance; (Art. 16) and hence this series of planes would represent the solid in the relation of its magnitude, as the ordinates represent the triangle.

But if CB = AB, then each of the parallel planes, *adc*, *adc*, &c., will be the square of its corresponding ordinate, *ad*, and all the planes will be the sum of the squares of all the ordinates. Hence, the sum of the squares of the series represents the pyramid, ABCDE, in the same manner as the sum of the ordinates represents the triangle, ABE.

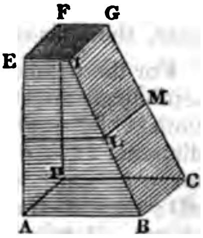


21. If the series of lines be a constant series of *z*, then in the same manner as the series of arithmetricals would describe or generate the triangle ABE, the constant series of *z* would generate the circumscribing rectangle ABFE, and the series of squares of *z*, would generate in like manner the prism ABCDEFHG, circumscribing the pyramid.

If instead of the squares of the several terms of the series, in the proposition generating the pyramid, there be taken the series of circles described on the several terms as diameters, those circles would generate a cone = the cone inscribed in the pyramid. And if instead of the squares constituting the prism, Art. 1, there be taken a similar series of circles, then would the solid generated be a cylinder equal to a cylinder inscribed in the prism. Also, if we take any other figure, which we constitute a similar function of each successive term of the series, we may generate pyramids, prisms, or frusta, of different character, but of the same general species.

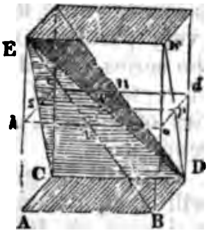
21. If a line be made to increase from p to z, and if its variation be represented by an infinite series of arithmeticals, then the series will truly represent a trapezium, and the series of the squares of the first, may represent a frustum of a pyramid.

22. If a be made to pass successively through all the values from 0 to z, while b is made to pass in like manner through all the values from 0 to p, then the two series drawn into each other will generate a pyramid with a rectangular base.



23. If there are two variable lines a and b, and if a be made to pass in succession from p to z, or from z to p, in an infinite series, while b remains equal to z or s, representing a similar series of constant quantities, then will the solid produced by drawing the corresponding terms of these series into each other represent a prismoid AG; but in this case the prismoid will have two parallel sides ABIE, CDFG. But if, while a is passing from z to p, b at the same time varies successively from z or s to f, then will the solid generated by the series of rectangles of the corresponding terms of the variable quantities, be a prismoid, neither of whose sides would be parallel except the two bases.

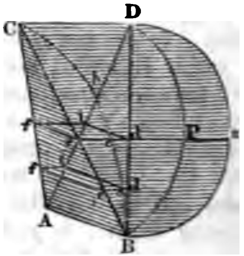
24. If one of the variable magnitudes should be made to pass successively in an infinite series from 0 to z, while the other should pass from z or s to p, then the solid generated by the rectangle of the corresponding terms of the series would be a wedge.



If a be an infinite series of lines, varying from 0 to z in arithmetrical progression, and h be a like series varying from s or z to 0, then if the corresponding terms of the series be drawn into each other, their product will be a triangular pyramid which may be resolved into the two wedges $scng$CD, and $scng$EG; all of which is evident by inspection.

Let the ordinates drawn across the triangle CD, parallel to its base, be a series of ordinates increasing from 0, at the vertice E, to z=CD; and let this series be called a: let the ordinates drawn across the triangle EGD, be a series decreasing in the same time from EG or s to 0; call this series b: then the series represented by a. drawn into that represented by b, will generate the solid EGCD.

25. Let the ordinates drawn across the triangle ABC be a series decreasing in arithmetical progression from z, or AB to 0, and let those drawn across the triangle BCD be a similar series of ordinates increasing from 0 to z, then may the series of proportional means between the corresponding terms, represent a series of equidistant ordinates, drawn across a semi-circle, whose diameter is z=DB, and the rectangle of the corresponding terms may represent a similar series of equidistant ordinates, drawn across such portion of a parabola whose axis is AB, as is intercepted by the curve, and a diagonal from the vertex B to the extremity of the base, the axis of the parabola being equal to z.

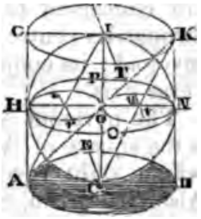


For the ordinates *dn* drawn across a semi-circle are severally mean proportionals, between the abscissæ of the diameter; that is, any ordinate *dn*, is a mean proportional between *d*B and *d*D; now if their be a series of abscissæ *d*B taken in arithmetical progression increasing, then their corresponding abscissæ *d*D, *d*D, &c., will be a series of decreasing arithmeticals; and if BD is equal to CD or AB, then will the abscissæ *d*D, *d*B be severally = to their corresponding ordinates *de*, *ef*, in whatever position they are taken; hence the series of ordinates, across the semicircle is a series of proportional means between the several corresponding terms of the increasing and decreasing series.

Again, it has been shown (Prop. VII, *Sch.* B. I,) that the expression for any ordinate *ec*, drawn across the parabola, CB*h*, between the curve and the diagonal, parallel to the axis, is equivalent to the rectangle *de* × *ef*, and since this is true of every parallel position of the ordinate *ec*; hence the sum of the series of ordinates, is equivalent to the sum of the series of rectangles of the series of variables, = the sum of the series of squares, of a series of ordinates *dn*, &c., across the semi-circle.

Let DB be greater than CD, or AB and the ordinates *dn*, will be equivalent to those drawn across a semi-ellipse, whose major axis is BD, and minor axis CD; but if CD is greater than BD, then will BD be the minor axis of the ellipse, &c.

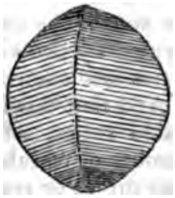
Hence the solid ABCD formed by drawing the corresponding terms of the increasing and decreasing lines into each other is equivalent in its expression to the area CBh of the parabola, and the semi-circle DBn is equivalent to the sum of a series of square roots of an infinite series of parallel ordinates *ec* across the parabola, or of parallel sections *defi* through the solid.



26. The elements contained in the foregoing propositions, may be applied also to solids of revolution; for instead of a prism and its inscribed pyramids in Art. 16, we may substitute a cylinder, and its inscribed cones, &c.; and if planes HPNQ be passed through the cylinder, the cones and their complements, these will be cut in the relation of their magnitudes, respectively. in each section; which will be in all cases, the same as that of the prism, and the inscribed pyramids, of equal base and altitude.

If the semi-circle or semi-ellipse CIN be made to revolve about the axis CI, and by its revolution to produce a sphere, or spheroid, then, because every section described by the parallel ordinate ON, would be proportional to the squares of their circumscribing lines respectively, the sum of the sections described, would be proportional to the sum of the squares of the generating lines; hence the sum of an infinite series of those sections, and consequently the solid generated by the semicircle would be equal to four times a middle section, multiplied by $\frac{1}{6}$ of the series. And any segment or zone of the sphere or spheroid, will be equal to the sum of the bases + four times a middle section × $\frac{1}{6}$ its altitude.

27. If every section *defi* through the pyramid ABCB (Art. 25) should be contracted in one of its dimensions till it becomes a square, and if the edge DB should continue to be a right line, then will the sides BCD, ABD, be plane surfaces, and the side ACD, ABC will become curved, and the whole solid will be equal and similar to a quadrant of a revoloid; and four of such pyramids, would constitute a perfect right revoloid; moreover the sides ABC, and ACD would become semi-circles.



For it has been shown that the square root of every section *defi*, through the solid, is equal to an ordinate drawn across the semi-circle DnB through the same parallel; but the square root of the section, is the side of a square equivalent to the section; hence, if the sides of the sections are severally the ordinates belonging to a semi-circle, then the solid must acquire a form, similar to that of a quadrant of a rectangular revoloid, such as would be formed by passing planes through the axis, bisecting its opposides.

If, when the sections of the solid become squares, its vertices are conceived to be at the extremities of an axis, passing through its centre; then the solid would become an elliptical revoloid; its conjugate axis being = to $\frac{1}{2}$ its vertical or transverse axis.

The pyramid ABCD is equal to $\frac{1}{6}$ of the prism, whose base is the square of AB, and altitude BD; four such solids, or the whole right revoloid is = $\frac{4}{6}$ = $\frac{2}{3}$ the circumscribing prism. Also, the prism, circumscribing the elliptical revoloid, whose base = *defi*, and altitude BD, being = $\frac{1}{4}$ of the former prism, is the prism circumscribing the elliptical revoloid; hence, the elliptical revoloid is = $\frac{2}{3}$ its circumscribing prism.

If from a cylinder of equal base and altitude, two equal cones be taken, one on either base, and of an altitude equal to that of the cylinder, the two remaining portions would be each equivalent to the cylinder; and every section through each of these portions, by planes parallel to the cylinder's base, would be equivalent to a corresponding section through the quadrant of the sphere; each of these portions are = $\frac{1}{6}$ of the cylinder; hence, four of these portions = the sphere, are = $\frac{2}{3}$ the circumscribed prism as found in the Elements of Geometry.

Scholium. We may, from the preceding investigations, draw the following deductions and conclusions.

First, that any series of quantities in arithmetical progression, varying from z to 0, drawn into any series of constant quantities, will produce a quantity whose value is = $\frac{1}{6}$ the sum of the base or maximum product of the variable + four times the value of the product of half the maximum value drawn into the series. Also, that the value of any multiple of this series may in like manner be determined.

Second, that if any series of numbers varying from z to 0, in arithmetical progression be drawn into another similar series, direct or reciprocal, the value of the product is = $\frac{1}{6}$ the sum of the two bases produced + four times a middle base formed by drawing $\frac{1}{2}$ the maximum values of the terms of the series into each other.

And that this is true for any multiple, or power of the variable series, whose exponent is an integer or when, any number of variable quantities are drawn into each other, whether direct or reciprocal, and this is the basis of the *Integral Calculus*, as will appear in the subsequent pages.

CHAPTER II.

ON THE CONSTRTCTION OF QUANTITIES WHOSE ELEMENTS ARE A SERIES OF CONSTANT OR VARIABLE QUANTITIES.

Art. 1. Having proceeded thus far, in analyzing the production of geometrical magnitudes, showing the manner and law of their generation, we are enabled, by having the elements and the law of the production of any magnitudes, to give a geometrical construction of such magnitudes.

We were taught, in the application of algebra to geometry, the mode of constructing integral algebraic quantities or expressions geometrically; we are now to represent a series of quantities, under a single construction; or to construct quantities whose elements are a series, either of constant, or variable quantities.

2. In considering the relations which exist between different quantities, those which, during the whole of any investigation are supposed to retain the same value, are called *constant quantities*; those to which different values are assigned, are called *variable quantities*: constant quantities are usually represented by the former letters of the alphabet, as a, b, c, &c., and variable quantities by the latter, as $uxyz$, &c.

3. When two or more variable quantities are connected in such a manner, that the value of one of them is determined by the value assigned to the other, the former is said to be a function of the other variables.

Thus, in the equation $y=ax+bx^2+c$, where the value of y depends on the value assigned to x; y is said to be a function of x, which is usually expressed by $f'(x)$, $\varphi(x)$, $\psi(x)$, or similar abbreviations.

Also, if an infinite series of equidistant ordinates are drawn across a surface, the sum of those ordinates is a function of the surface. So, also, the sum of an infinite series of planes through a solid may be regarded as a function of the solid, or the solid or surface a function of the planes or ordinates.

4. When any quantity or magnitude as the element of other quantities or magnitudes is variable, the sum of a series of the variable quantity, within its variable limits, may be represented by a dash drawn below or above the letter representing the variable quantity.

Thus $\underline{z}$ or $\bar{z}$ may represent a series of the variable quantity, z. If the incipient value of the variable is z, and the series is decreasing to s, or 0, the dash must be placed below the letter; but if its incipient value is 0 or s, and its terminate value is z, being an increasing series, the dash must be placed above. Thus $\underline{z}$ indicates a series decreasing from z to s or 0; and $\bar{z}$ indicates the series increasing from 0 or s to z.

5. If it is required to express a series of quantities in arithmetical progression from x to s, or from s to x, it may be thus written—$\underline{x} \cdot\cdot s$, or $\bar{x} \cdot\cdot s$; or which is the same, $\underline{x} \cdot\cdot x'$ or $\bar{x} \cdot\cdot x'$, x being the incipient and x' the terminate value of the series.

The condition of questions involving these variables, generally indicate the incipient and terminate values of the increasing or decreasing variables.

6. The expression $\underline{z}$ is equivalent to that of $\bar{z}$, when considered independent of other variables; but the product arising from drawing $\underline{z}$ into $\bar{z}$ is not equivalent to that of drawing $\underline{z}$ into $\underline{z}$; for $\underline{z}z$ indicates that the greatest value of z is drawn into its least value, and consecutively; and $\underline{z}z$ indicates that the greatest value of z is drawn into the greatest, and so on through the series.

A series of the squares of $\underline{z}$ or $\bar{z}$, is represented by $\underline{z}^2$ or $\bar{z}^2$; a series of roots by $\sqrt{\underline{z}}$ or $\sqrt{\bar{z}}$.

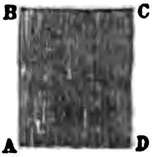
7. When it is designed to express, a series either of constant or variable quantities, without regard to their progression or law of variation, a small capital, of the letter denoting a single term of the series may be used.

Thus in the equation $y=\sqrt{dx}$, if we would express a series of y, it may be written Y, and the equation will be $\mathrm{Y}=\sqrt{d\underline{x}}$; this, unless otherwise restricted, expresses an infinite series of the quantity represented by y. If $y=\sqrt{(dx)}$ is the equation of any figure, then Y, is a function of the surface, or $\mathrm{Y}=\sqrt{(d\underline{x})}\underline{x}$ is the equation to the surface.

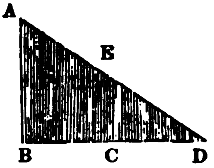
Hence the equation to a surface consists of the equation of the figure considered as a series, drawn into the axis or abscissa.

8. If a be the magnitude AD of any series whose number is n, of lines AB, and whose length is the constant quantity z, then the sum of the lines will be nz; and their magnitude made by drawing them into a surface, will be $\frac{a}{n}zn = az$.

For we have shown that if an infinite series of lines are drawn into their respective distance, the product is a surface; hence, we have the following construction, viz: a rectangle ABDC.



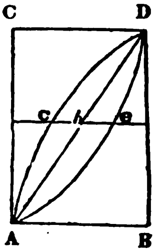
9. Let a be a series of lines $\underline{z}$, constantly decreasing in arithmetrical progression from z to 0, or from the line AB to the point D, while the line BD represents the number or magnitude of the series; and the series of $\underline{z}$ drawn into the quantity a, will be equivalent to the triangle ABD.



If a be a series of lines, AB decreasing from $\underline{z}$ to s, or from AB to EC, then if $AD=a$ = the magnitude of the series, the construction will be the trapezium ABCE.

10. If a be a series of z^2, where z decreases uniformly from AB, to a point D; or from z to 0, then az^2 may be constructed by the exterior space of a parabola $A e$DB, for while $a\underline{z}$ generates the triangle ABD, $a.z^2$ will generate the parabola AeDBA, (Prop. VII, Sch., B. 1,) whose axis is CD.

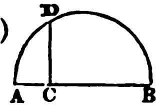
If a be a series of $\sqrt{\underline{z}}$, then may $a\sqrt{z}$ be put under the construction of a semi-parabola, AcDB, whose axis is $DB=a$, and whose base $AB=z$, the series of $\dot{z}$ being a series of ordinates across the triangle ABD, parallel to AB.



12. Let it be required to construct a quantity $d\sqrt{(\underline{z}.\bar{z})}$, or a series of mean proportionals between the corresponding terms of two equal increasing and decreasing series drawn into d, the number of the series.

If d = the line AB, and if z = the same line, the surface generated by drawing d into $\sqrt{(\underline{z}.\bar{z})}$, would be the semicircle ABD.

For the equation to the circle is $y^2=\sqrt{(dx-x^2)}$
or $DC=\sqrt{AC^2+CB^2}$

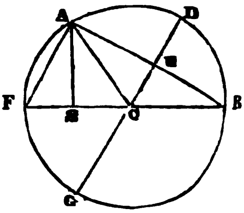


But if z be greater or less than d, then the construction will be a semi-ellipse; which if z is less than d, will have AB for its major axis; but if z is greater than d, AB will be the minor axis.

For the ellipse by its equation, $d^2 : c :: x(d-x) : y^2$ or $d^2y^2 = c^2(dx-x^2)$ is only the circle expanded, or contracted, in the ratio of the major and minor axes.

The equation for the surface of a circle or an ellipse, may more properly be expressed by $d\sqrt{(\underline{x}\bar{x})}$ for a semi-circle or semi-ellipse, and if $x = d$, then the equation will be that of a circle; but if d and x are unequal, the equation becomes that of an ellipse, and d will be the major or minor axis, according as it is greater or less than x, and x will be its conjugate; and its equivalent in either case is $\frac{1}{4}dx \times \pi$, or for the whole circle or ellipse $\frac{1}{4}dx \times \pi$: hence π is a function of $\sqrt{(\underline{x}\bar{x})}$ or $\int \sqrt{(\bar{x}\underline{x})}$.

Hence we have a finite expression for the circles quadrature in algebraic terms, viz: $2d\sqrt{(\underline{x}.\bar{x})}$ or $2x\sqrt{(\underline{x}.\bar{x})} = r^2\pi$, where x is equal to the diameter. And $\pi = (2x \div r^2)\sqrt{(\underline{x}.\bar{x})} =$ the circumference of any circle whose diameter is x; $\bar{x}$ and $\underline{x}$ being series of increasing and decreasing quantities. Hence, $\pi = 2d\sqrt{(\underline{x}x)} \div \frac{1}{4}d^2 = (8 \div d)\sqrt{(\underline{x}\bar{x})}$, or $(8 \div x)\sqrt{(\underline{x}\bar{x})}$.



13. Let ABD be the segment of a circle, the height of the segment DE being equal z; and if the diameter $= x$, then will the chord AB$= \sqrt{(x'z\,;)}$ and if EG $= x - z = x'$, then will the equation to the surface be $2Yz = 2z \sqrt{((\underline{x} \cdot\cdot x')\bar{z})}$. - - - - - - -(1)

Also, let CE $= \frac{1}{2}x - z = u$, then if AB in a decreasing series is drawn into CE, or if $(u\sqrt{(xx)})$ be constructed, it will be equal to the triangle ABC; and the sum of the segment and triangle, $= 2z\sqrt{((\underline{x} \cdot\cdot x')\bar{z})} + u(\sqrt{(xz)}) =$ the sector ACBD. (2)

In the triangle ABF the side $\overline{\text{AF}}$ is equal $2\text{CE} = x - 2z$; and BF : BA : : AF : AS, or $x : \sqrt{xz} : : x - 2z :$ AS the sine of the angle, BCA, or sine of the arc of the segment

$$= \sqrt{(zx)} - 2z\sqrt{(zx)} \div x. \quad - - - - - - - \quad (3)$$

Let AS$=s$, and let the arc ADB$=\pi'$ and the area of the segment, (Prop. XVIII B. IV) will be $(\pi' - s) \times \frac{1}{4}x$, hence the expression $2z\sqrt{((\underline{x} \cdot\cdot x')\bar{z})} = \frac{1}{4}x\pi' - \frac{1}{4}xs$ - - - - - (4)

Therefore the arc ADB of the segment may be expressed

$$\pi' = \frac{2z\sqrt{((\underline{x} \cdot\cdot x')\bar{z})} + s}{\frac{1}{4}x} = 2z\sqrt{(\underline{x} \cdot\cdot x')\bar{z})} + \sqrt{(xz)} - \frac{2z\sqrt{(xz)}}{x} \quad (5)$$

14. If a series of variables consisting of two or more factors, as its elements, are drawn into the number denoting the series, the construction can more conveniently be represented by a solid.

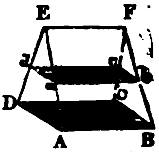
Surfaces properly consist only of factors, equivalent to the second power; cubes of factors equivalent to the third power, or the product of three factors.

But we may, under certain conditions, construct quantities representing solids as surfaces; those whose factors are constant, by any surface arbitrarily chosen, and those which are variable by such curves as yield to the conditions of the expression; and we may thence proceed to construct such solids as would result from drawing such surfaces into a series; or such as would be the result of drawing the variable series of elements of such surface into a constant, or a variable quantity; and the quantity so constructed may, by being put under a superficial construction, become the base of another quantity or magnitude, made by drawing this quantity into another given quantity or into a series; and we may proceed, in this manner, to construct quantities geometrically, converting one construction into a base for the next, and so on.

This may be performed algebraically, and with greater facility, inasmuch as a quantity involving any power of the variable may be assumed as a base for a higher power of the series, and thus the powers and roots of variables may be extended at pleasure; and the laws of variation or their increase or decrease in value, may be determined by geometrical construction and analytical deductions.

15. If it were required to construct a series, a, of the quantity bc, drawn into each other, b and c being constant quantities; here it is evident that, for the purposes of construction, bc may represent either a line or a surface; if the series of bc be represented by a line, then bc, drawn into a, or abc, will represent a rectangle. But, if bc represents a surface, then abc will represent a prism. And for the purposes of investigation, the conditions of the quantities would be similar in either case; for the line would have the same relation to the rectangle generated by drawing the line into the series, as the surface to the prism, generated by drawing the surface into the series.

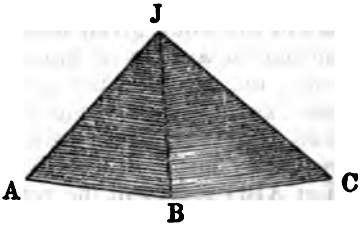
16. Let it be required to construct a series a, of bz or $a.b\underline{z}$, where z is a series decreasing uniformly to o.

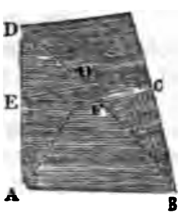


This would evidently be represented under the form of a wedge ABCDEF, whose base ABCD is represented by bz, and the perpendicular ED represents the value of a.

If $\underline{z}$ decrease only to s, then the construction would be the prismoid ABCD $abcd$, whose greater base, ABCD, is represented by bz, and whose lesser base, $abcd$, is represented by bs.

17. If u be also a quantity decreasing uniformly with z, n such ratio as to reach o at he same time, then the expression $a.\underline{u}\underline{z}$ would generate a pyramid ABCJ; which will be its proper construcion; and if $u=z$, or if the expression can be but under he form $a.\underline{z}^2$, then the pyramid will have a square base.





18. Let a series $a.\bar{z}\underline{z}$, be constructed. This quantity, representing the products of the corresponding terms of two series, one increasing from o to z, while the other decreases from z to o, drawn into a, the quantity denoting the number of the series, generates the double wedge ABGD, where AB, or GD, is equal to z, and the perpendicular AD$=a$.

If $\underline{u}$ represent a series of ordinates drawn across the triangle ADG, then au will represent the triangle; and if these ordinates are severally drawn into a series of u, the result would be the same as before.

19. Let it be required to construct $Y^2=p\underline{x}$, which is the equation to the vertical parabolic revoloid, $y=\surd(px,)$ being the equation to the parabola, p being the parameter, is a constant quantity, we have x variable.

First, we may draw p into a series of $\underline{x}$, which gives us the triangle DEA (see diagram to art. 16,)$=p\underline{x}$, p being equal to DA, $x=$the axis of the revoloid$=$DE the altitude; and if this triangle, as a base, is also drawn into x, the axis of the revoloid, we shall have the solid ABCDEF.

Or, more properly, if we first draw the axis x into the series of the variables, we shall have the triangle as before, with a base AD; and if we multiply this by $p=$DC, we shall have the solid described as before, then will any section *abcd* of this solid parallel to the base ABCD, be equivalent to a similar section through a parabolic revoloid, or pyramoid, which it is designed to represent.

The expression for a cube may at all times be constructed on a surface by means of curves; but every different species of solid requires some peculiar construction, according to the equations for the ordinates or sections of the solid, the expressions for which may be transferred to expressions for ordinates to a surface, or equations to some curve; and since any multiple, or power of a series $\underline{x}$, whose exponent is an integer, is known, when the series or root is known; we can hence discover an innumerable variety of curves, which are quadrable; but, in general, in descending powers, it is not certain that the series of roots of a given series may be so. Thus, the circle is equivalent to a series of square roots, of the ordinates to a parabola; the series of the parabala is quadrable, but not the circle, except in certain functions of given quantities; we may get an expression for its value, but it will be under an incommensurable form.

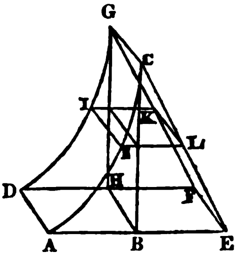
Let ABD represent the vertical segment of a rectangular spherical revoloid; and its equation considered in relation to its figure, will be $y^2 = (x \,..\, x')x$ - - - - (1)

the equation considered as a solid, will $Y^2z = z(\underline{x} .. x')\bar{z}$ (2)
which from its organization will readily be discovered to be cubable, for its value is $\frac{4z}{6}$ $(x'z + 4(\frac{1}{2}x + \frac{1}{2}x')\frac{1}{2}z) = \frac{2}{3}xz^2$
$+\frac{4}{3}x'z^2$, - - - - - - - - - (3)
its convex surface will be $4xz$ - - - - - (4)

Hence, the solidity of a spherical segment will be =
$(\frac{2}{3}xz^2+\frac{4}{3}x'z^2)\frac{1}{2}\pi = \frac{1}{6}xz^2\pi + \frac{1}{3}x'z^2\pi$ - - - - (5)
and its surface will be πxz - - - - - (6)

20. Let it be required to construct $Y^2 = \frac{c^2}{d^2}(dx + x^2,)$ the equation to an hyperbolic revoloid, Y^2 being a series of parallel planes, being the squares of the ordinates to the axis.

Because this equation involves powers higher than the square; that is, because the terms $(dx+x^2,)$ multiplyed by $\frac{c^2}{d^2}$, produces a solid, the series represented by these quantities drawn into a, would be of a higher power than a cube; hence, in order that our construction shall be under a cubic form, a $(d\bar{x}+\underline{x}^2)$ must be represented on a plane. Let a=CB.



Hence, we have $a.\underline{x}^2$=the exterior parabolic surface ABC, AB being the axis of the parabola, and also $d\bar{x}$ = CBE; now, if this is drawn into $\frac{c^2}{d^2}$, we shall have the solid ABECFDG, for the construction of a solid equivalent to the quadrant of an hyperbolic pyramoid or revoloid; every section NIKL through this solid is equivalent to a corresponding section through a hyperbolic pyramoid of the same altitude and equivalent base.

Hence, it will be perceived that the hyperbolic revoloid is cubable; for this, its representative is equal to one-sixth of the product of the sum of the base AEFD *plus* four times a middle section NIKL drawn into the altitude BC. This is also true in relation to the parabolic revoloid.

And since a paraboloid or hyperboloid is to its respective circumscribed revoloid as a circle to its circumscribed square, or as $r^2\pi : 4r^2$, it follows that the paraboloid and hyperboloid, are also cubable in terms of π.

21. It may be perceived that in constructing the quantity $a\underline{x}$, where $\underline{x}$ is a series decreasing to 0, we have a triangle; hence the value of $a\underline{x}$ is $\frac{1}{2}ax$; or the value of $x\underline{x}$ may, in like manner be shown to be $\frac{1}{2}x^2$.

Also, since $\underline{x}^2x$ is equivalent to a pyramid whose base is x^2, and whose altitude is x; hence, $\underline{x}^2x$ is equal to $\frac{1}{3}x^3$.

And if we have a series of $\underline{x}^3$ to be drawn into x or a, we may also construct $\underline{x}^3$ by art. 20, and we shall here have $\underline{x}^3x$ $=\frac{1}{4}x^4$; also $x^nx=\frac{1}{n+1}x^{n+1}$

Hence, the law of the progression of the powers of the variable quantities is manifest, for the value of the continued product of any series of power, of a series of variable quantities drawn into any constant quantity, is equal to the same power of the maximum value of the variable drawn into the same quantity, divided by a number denoted by the index of the power *plus* 1.

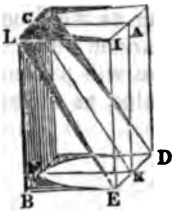
If the variable has a fractional exponent, the same consideration will also apply. Thus, $\underline{x}^{\frac{1}{2}}x$ is equivalent to $\frac{2}{3}x^{1\frac{1}{2}}$, and $\underline{x}^{\frac{1}{3}}x$ is equivalent to $\frac{3}{4}x^{1\frac{1}{3}}$ and $\underline{x}^{\frac{1}{n}}x$ equivalent to $\frac{1}{n+1}x^{\frac{1}{n}+1}$ &c.

Where the product of any power of a variable quantity is made the base of another construction, according to its determinate value. the magnitude constructed on this base, is no longer considered as a function of the variables, entering into the former constructions as elements of this base; but this new construction is subject to the laws of its own organization; and if raised to any power, or multiplied by any number of factors, either as constants or variables, the different powers depend on, or are a function of the base assumed, and not of its original constituents.

Thus. if we have the quantity x^3x, its value is $\frac{1}{4}x^4$; now, if $\frac{1}{4}\underline{x}^4$ is assumed as the base of another construction, as $(\frac{1}{4}\underline{x}^4)x$, or $\frac{1}{4}\overline{x}^4x$ the product is not $=\frac{1}{20}x^5$, but is $=\frac{1}{5}x^5$.

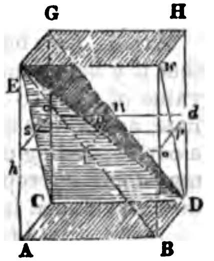
If a series $\underline{uxz}$ are drawn into a, the product is equivalent to $\frac{1}{4}uxza$; the co-fficient being the same as that for the product of $\underline{x}^3x$, which gives $\frac{1}{4}x^4$ as its equivalent.

When the terms of the variable series denoted by x or z, are drawn into each other according to their reciprocal value, or where an increasing series is drawn into a decreasing series, the value of some power of the product of the series may be obtained; and the other powers and roots of this value are subject to peculiar laws for their development. For the continued products of these series, are not subject to the same ratio, or the same order in the variation in their successive changes from one power to another, as is observed in the products of such as are either all increasing or all decreasing. Thus, if we have $x\sqrt{\underline{x}\overline{x}}$, let this quantity be involved till its radical is removed, and we have $x^2\underline{x}\overline{x}$, the value of which may be determined by a construction. Thus, $\underline{x}\overline{x}.x$, which is equivalent to a double wedge, (Art. 24, Chap. I,) or which will be more available for our present purpose, it may be put under the construction of the parabola cut off by a diagonal, (Art. 25, Chap. 1,) the value of which is evidently equal $\frac{1}{4}x^2\times4\times\frac{1}{6}x=\frac{1}{6}x^3$, which is equal to one-half the value of $\underline{x}^2x$; and if this value is again put into a series, and drawn into $\underline{x}$, we shall have $\frac{1}{2}\underline{x}^2.x$, or $\underline{x}^2\bar{x}.x$, whose proper geometrical construction will be the solid CELI, which is the solid of a prism circumscribing a parabolic complemental ungula, which (Prop. XVIII, Cor. 2, B. II.) is equal to one-fifth of the prism CANELI $=\frac{1}{10}x^4$.



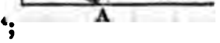
Let there be a series of $\underline{x}^2x$, a series of $\bar{x}^2x$, and two series $\bar{x}\underline{x}x$, and the sum of the series will be equivalent to x^3, as may be represented by the prism AH.

For $\underline{x}^2x$ equals the pyramid ABCDE, $\bar{x}^2x$ equals the pyramid EFHGD, and $2\bar{x}xx$ equals two wedges CDEG completing the prism whose value is x^3.



Let there be a series a of z drawn into a reciprocating series of $\bar{x}$, the terminate value of z being z', and the incipient value of x being 0. Since the product of these series $a.(z\cdot\cdot z')\bar{x}$ gives us for a construction a solid *hino*CD, whose value is equal to $\frac{1}{6}ax(2z'+z)$; and this also represents a segment of a quadrant of a revoloid where z and x increase and decrease in the same arithmetical ratio, equal to a segment of a cylindrical ungula.

Also, if we have a series of the products of reciprocating variables, or $\bar{z}z$ drawn into a series of h or z, its value has been found equal to $\frac{1}{6}hz^2$ or $\frac{1}{6}z^3$, let this be drawn into another series of $\ddot{z}$, and we have the following construction, which is a parabola whose vertice is D; and we have $z\underline{z}z^2$ equal the parabolic ungula ADBC equal $\frac{2}{3}AB\times HA\times\frac{1}{2}BC=\frac{2}{3}(z\times(\frac{1}{2}z\times\frac{1}{2}z)\times\frac{1}{2}z)=\frac{1}{12}z^4$; hence, the series zzz^2 is equivalent to $\frac{1}{12}z^4$.



Let this be drawn into another series of $\bar{z}$, and we shall have $z\underline{z}\bar{z}^3=\frac{1}{6}z(z^3+4(\frac{1}{8}z^3\times\frac{1}{2}z))=\frac{1}{6}z^4+\frac{1}{24}z^5$; if this is drawn into another series of $\ddot{z}$, we shall have $z\underline{z}\bar{z}^4=\frac{1}{6}z(z^3+4(\frac{1}{16}z^4\times\frac{1}{2}z))=\frac{1}{6}z^4+\frac{1}{48}z^6$; hence the law of progression is manifest.

If we have a series of $zz^2\bar{z}^3$, its value will be $\frac{1}{6}z(2z^3+\frac{1}{4}z^4)=\frac{1}{3}z^4+\frac{1}{24}z^5$. Hence, the law of the progession of this series will easily be discovered.

Scholium. The results obtained by the preceding notations and geometrical constructions, are similar to those obtained by the integral calculus, the same principles as here used serving as the foundation of that science; and in order that a comparison between the two modes of notation may be instituted, we will present some of the elementary principles of the calculus, as the subject of the next chapter.

CHAPTER III.

DIFFERENTIAL AND INTEGRAL CALCULUS.

Art. 1. As a basis of the deferential *calculus*, we may premise that if y be a function of x, and if a change takes place in the value of $f(x)$ so that x becomes $x+h$, x being quite indeterminate, and h any quantity whatever, either positive or negative, a corresponding change must take place in the value of y, which may then be represented by y'. If the quantity $f(x+h)$ be now developed in a series of the form

$$f(x)+\mathrm{A}h+\mathrm{B}h^2+\mathrm{C}h^3+\ldots\ldots$$

which is always practicable, in which the first term is the original function $f(x)$, and the other terms ascend regularly by positive and integral powers of h, and A, B, C. &c., are independent of h; then the co-efficient of the simple power of h in this series is called *the first differential co-efficient of y or $f(x)$*. This is the fundamental definition of the differential calculus.

2. Let us now examine the change which takes place in the function for any change that may be made in the value of the variable on which it depends.

Let us take, as a first example,

$$u = ax^2,$$

and suppose x to be increased by any quantity h. Designate by u' the new value which u assumes, under this supposition, and we shall have

$$u' = a(x+h)^2,$$

or by developing

$$u' = ax^2+2axh+ah^2.$$

If we subtract the first equation from the last, we shall have

$$u'-u=2axh+ah^2;$$

hence, if the variable x be increased by h, the function will be increased by $2axh+ah^2$.

If both members of the last equation be divided by h, we shall have

$$\frac{u'-u}{h} = 2ax+ah, \quad (1)$$

hich expresses the ratio of the increment of the function to at of the variable.

The value of the ratio of the increment of the function to at of the variable is composed of two parts, $2ax$ and ah. If now, we suppose h to diminish continually, the value of e ratio will approach to that of $2ax$, to which it will become ual when $h=0$. The part $2ax$, which is independent of h, therefore the *limit* of the ratio of the increment of the funcon to that of the variable. The term, *limit of the ratio* degnates the ratio at the time h becomes equal to 0. This ra- is called the *differential co-efficient* of u regarded as a funcon of x.

3. Then let y be a function of x, such that

$$y=ax^2$$

Let x become $x+h$ and y become y'

$$y'=a(x+h)^2$$

panding $$=ax^2+2ax\,.\,h+ah$$

This, it will be perceived, is a series of the required form, e first term ax^2 is the original function y, and the other rms ascend by integral and positive powers of h; hence, cording to our definition, $2ax$ the co-efficient of the simple wer of h in this series, is the first differential co-efficient of or $f\,(x)$

4. Again, let

$$y=x^3$$

Let x become $x+h$ and y become y'

$$y'=(x+h)^3$$

panding $$=x^3+3x^2\,.\,h+3x\,.\,h^2+h^3$$

Here, therefore, $3x^2$, the co-efficient of the simple power of is the first defferential co-efficient of x^3.

5. Again, let

$$y=ax^3+bx^2+cx+d$$

Let x become $(x+h)$ and y become y'

$\therefore\ y'=a(x+h)^3+b(x+h)^2+c(x+h)+d$

panding $= ax^3+3ax^2h+3axh^2+h^3+bx^2+2bxh+bh^2+cx+ch$ d, arranging according to powers of h,

$$=(ax^3+bx^2+cx+d)+(3ax^2+2bx+c)h+(3ax+b)h^2+h,$$

series of the required form, for the first term is ax^3+bx^2+cx d, the original function, and the succeeding terms ascend gularly by powers of h.

Hence, $3ax^2+2bx+c$ the co-efficient of the simple power of in the development of y' is the first differential co-efficient of or ax^3+bx^2+cx+d.

6. We have now to introduce a notation by which this ratio ay be expressed. For this purpose we represent by dx the st value of h, that is, *the value of* h *which cannot be diminhed according to the law of change to which* h *is subjected,*

without becoming 0; and let us also represent by du the corresponding value of u: we then have

$$\frac{du}{dx} = 2ax. \quad (2)$$

The letter d is used merely as a characteristic, and the expressions du, dx, are read, *differential* of u, *differential* of x.

It may be difficult to understand why the value which h assumes in passing from equation (1) to equation (2), is represented by dx in he first member, and made equal to 0 in the second. We have represented by dx the *last* value of h, and this value forms no appreciable part of h or x. For, if it did, it might be diminished without becoming 0, and therefore would not be the last value of h. By designating this last value by dx. we preserve a trace of the letter x, and express at the same time the last change which takes place in h, as it becomes equal to 0.

7. Let us take as a second example,

$$u = ax^3.$$

If we give to x an increment h, we shall have,

$$u' = a(x+h)^3 = ax^3 + 3ahx^2 + 3ah^2x + ah^3.$$

hence, $$u' - u = 3ahx^2 + 3ah^2x + ah^3,$$

and the ratio of the increments will be

$$\frac{u'-u}{h} = 3ax^2 + 3ahx + ah^2,$$

and the *limit* of the ratio, or *differential co-efficient,*

$$\frac{du}{dx} = 3ax^2.$$

In the function

$$u = nx^4, \text{ we have } \frac{du}{dx} = 4nx^3.$$

And if $$y = f(x)$$

the first differential co-efficient of y is denoted by the symbol $\frac{dy}{dx}$, thus in the above examples (Arts. 4 & 5.)

$$y = ax^2$$

$$\frac{hy}{dx} = 2ax$$

$$y = x^3$$

$$\frac{dy}{dx} = 3x^2$$

$$y = ax^3 + bx^2 + cx + d$$

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

manner if $u=f(z)$ the first differential co-efficient of u) will be represented by $\frac{du}{dz}$.

e.–Since a constant quantity is not susceptible of change, anifest that it can have no differential co-efficient, or if $\frac{dy}{dx}=0$.

find the first differential co-efficient of any power of a simple algebraic quantity.

$y=x^n$

x become $x+h$

$y'=(x+h)^n$

anding by the binomial

$\cdot nx^{n-1}h+\frac{n(n-1)}{1.2}x^{n-2}h^2+\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}h^3+\&c.$

$\frac{y}{x} = nx^{n-1}$

n this it is manifest that

first differential co-efficient of any power of a simple aic quantity is found by multiplying the quantity by the of the power, and then diminishing the exponent by unity.

x. 1. $y = x^7$ $\quad \frac{dy}{dx} = 7x^6$

2. $y = ax^{p+q}$ $\quad \frac{dy}{dx} = a(p+q)x^{p+q-1}$

3. $y = a+x^{-q}$ $\quad \frac{dy}{dx} = -qx^{-(q+1)}$

4. $y = a+7x^{-\frac{m}{n}}$ $\quad \frac{dy}{dx} = -7\frac{m}{n}x^{-\left(\frac{m}{n}+1\right)}$

INTEGRAL CALCULUS.

object of the Integral Calculus is to discover the pri- function from which a given defferential co-efficient has erived.

primitive function is called the *integral* of the proposed itial co-efficient, and is obtained by the application of ferent principles established in finding differential co- its and by various transformations.

When we wish to indicate that we are to take the integral of a function, we prefix the symbol S. Thus if

$$y = ax^4$$

We know that $dy = 4ax^4dx$

If then, the quantity $4ax^3\,dx$ be given in the course of any calculation, and we are desirous to indicate that the primitive function from which it has been derived is ax^4, we express this by writing

$$S\,4ax^3\,dx = ax^4$$

The characteristic S signifies *integral* or *sum*. The word *sum*, was employed by those who first used the differential and and integral calculus, and who regarded the integral of $x^m dx$ as the *sum* of all the products which arise by multiplying the mth power of x, for all values of x, by the constant dx.

When constant quantities are combined with variable quantities by the signs + or — they disappear in taking the differential co-efficients, and therefore they must be restored in taking the integral.

Thus, if $y = ax^3 + b$

or, $y = ax^3 - b$

or, $y = ax^3$

In each three cases equally

$$dy = 3ax^2\,dx$$

Hence in taking the integral of any function it is proper always to add a constant quantity, which is usually represented by the symbol C. Thus, if it be required to find the integral of a quantity such as

$$dy = 3ax^2\,dx$$

$$y = S\,3ax^2\,dx$$

$$= ax^3 + C$$

where C may be either positive, negative, or 0. We cannot determine the value of C in an abstract example, but when particular problems are submitted to our investigation, they usually contain conditions by which the value of C can be ascertained.

By reversing the principles established for finding the differential co-efficients, or differentials of functions, we shall obtain an equal number of rules for ascending to the integrals from the derived functions. Recurring therefore to these we shall perceive that

1. *The integral of the sum of any number of functions is equal to the sum of the integrals of the individual terms, each term retaining the sign of its co-efficient.* Thus, if

$$dy = 4ax^3\,dx + 3bx^2\,dx - 2bx\,dx + dx$$

$$y = S\,4ax^3\,dx + S\,3bx^2\,dx - S\,2\,bx\,dx + S\,dx + C$$

II. Since, if

$$y = ax^m$$

$$dy = max^{m-1}\,ax$$

it is manifest that

The integral of a function raised to any power is obtained by adding unity to the exponent of the function, and by dividing the function by the exponent so increased, and by the differential of the function.

Ex. 1.

$$dy = ax^n\,dx$$

$$y = \frac{ax^n + 1}{n+1} + C$$

Ex. 2.

$$dy = \frac{a}{x^n}dx$$

$$= ax^{-n}\,dx$$

$$y = \frac{ax^{-(n-1)}}{-(n-1)}$$

$$= -\frac{a}{(n-1)\,x^{n-1}} + C$$

Ex. 3.

$$dy = (a + x)^n\,dx$$

$$y = \frac{(a + x)^{n+1}}{n+1} + C$$

$$dy = \frac{1}{(a + x)^n}\,dx$$

$$= (a + x)^{-n}\,dx$$

$$y = -\frac{1}{(n-1)\,(a + x)^{n-1}}$$

This rule applies to all functions of the form

$$dy = (a + bx^n)^m\,cx^{n-1}\,dx$$

for these can all be reduced to the form $az^m\,dz$.

Scholium.

The coincidence of the results obtained by the notation, used in Chap. II, with those determined by the *Calculus*, indicates that the same principles serve as the foundations of both; which, as was observed at the close of Chapter I, are the following, viz., that when any series of quantities vary in arithmetical progression, the sum of the series as well as their powers, or products, with a constant quantity, or with another arithmetical series, may be measured by the sum of the products of the maximum values + the minimum values + four times the products of their medium values multiplied by $\frac{1}{6}$ of the magnitude, representing the number of the series.

Any series of numbers which are not subject to these conditions, cannot be made the subjects of accurate numerical determination or calculation, by the *Calculus* ; hence, we may always determine, by construction, whether any geometrical magnitude, or algebric quantity, is susceptible of accurate development numerically in terms of given quantities. One advantage possessed by the investigation of geometrical subjects, by the principles contained in Chapter II, of this subject, is, that the notation there used, is more elementary than that of the *Calculus*, and expresses the conditions of geometrical subjects in a more obvious and inteligent manner, and is more direct in its results.

Hence, instead of pursuing the method of defferentation, we express the conditions of quantities depending on variable factors according to their several conditions, or organization, and proceed to integrate or sum up the series of functions, and determine the magnitude of the production, according to the principles therein contained. Hence $S\,x^2$ expresses definitely the function of the series of squares of a series of variables, and if this series be drawn into a constant quantity x, and integrated, we shall have according to our notation $\frac{1}{3}x^3$.

If this is made equal to u, or if we have $u = \frac{1}{3}x^3$, its differential is $\frac{du}{dx} = x^2$, which, being again integrated by the calculus, we have again $\frac{1}{3}x^3$, showing similar results, from different considerations. The notation by the calculus is arbitrary, but by this inductive ; and hence, by its intimate connection with the principles of the calculus, may serve to render that subject more obvious.

There are cases, however, for which the calculus is more particularly adapted than the notation here referred to ; such as drawing tangents to, and rectifying curve-lines. This is accomplished by that science in a manner the most elegant and complete ; but in relation to surfaces and solids, nothing can be more complete or satisfactory, than the discussions which we have introduced.

CHAPTER IV.

On the centres of surfaces and solids, the method of finding them, &c.

Article 1. The centre of surfaces and solids is susceptible of two considerations, that of magnitude and distance. Hence, we have the centre of *aggregation* and the *virtual centre.*

2. The *centre* of *aggregation* or *centre* of *magnitude* of any plane surface, is that point through which, if a line is drawn in any direction in the plane, it shall divide the surface equally.

3. The centre of *magnitude* of any solid is that point, through which if a plane is passed in any direction, the plane shall divide the solid equally.

4. The *virtual centre,* or *centre* of *gravity* of any plane is that point through which, if a line be drawn in any direction in the plane, the sum of all the points, on one side, drawn into their distances from the line, shall be equal to the sum of all the points on the other side drawn into their respective distances from the line.

5. The virtual centre of any solid is that point through which if a plane be passed in any direction, the sum of the points, on one side, drawn into their distances from the plane, shall be equal to the sum of all the points on the other side, drawn into their distance from the plane.

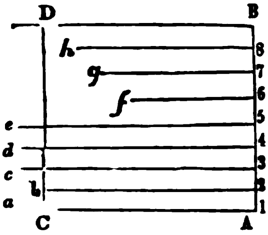
6. The virtual centre between any two points is evidently in the right line connecting the two points, and equidistant from each.

Hence the virtual centre of a right line, is in the middle of that line, or is in the centre of magnitude of the line.

7. In a system of points, their virtual centre has respect to the magnitude of the lines drawn from the several points to such centre.

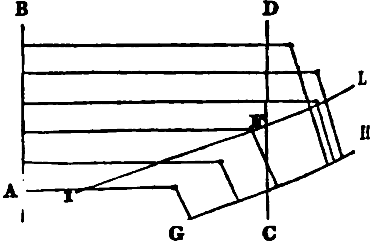
But the centre of magnitude, has regard only to the number of points.

8. If there be drawn from a system of points, lines perpendicular to a given base, the sum of those lines divided by their number, will give their average length ; which is = to the distance of the virtual centre of the system of points from the base.



Let AB be a given base line, and let *a*, *b*, *c*, *d*, *e*, &c., be a system of points; if perpendiculars *a*1, *b*2, *c*3, &c., be drawn from the points to meet the base line AB, and if the sum of these lines be divided by their number, the quotient will give their average length, or the distance AC, from which, if a line CD be drawn, parallel to AB, it shall pass through the virtual centre of the system of points.

For if the line CD is drawn, according to the conditions expressed in the proposition, the portions of the lines cut off beyond the line CD, are sufficient to extend those which fall short of that line, to the line CD. Hence, the line CD passes through the virtual centre of the system, for the lines drawn from the points on both sides of the line are equal.

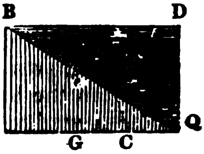


Hence, if two base lines AB, GH are drawn, not parallel to each other, and if the system of points are connected to both of the bases, and we proceed, as in the proposition to draw CD parallel to AB, and IL parallell to GH, the intersection F, of the lines CD, and IL will be the virtual centre of the system.

If the base line AB, should pass through the system of points, so as to leave one portion of them on one side, and another portion on the other, then the lines connecting the points to the base must be estimated on one side positive, and on the other negative; and the virtual centre will fall on one side or the other, as the positive or negative signs predominate.

9. If, instead of a system of points, the virtual centre of a system of parallel lines is required. Let a base line be drawn parallel to those lines, and because the lines are to each other, as the number of equidistant points, arbitrarily taken in each; hence, if we take the product of the several lines into their respective distance from the base, the sum of these products divided by the sum of the lengths of the lines, will determine the distance from the base, through which, if a line is drawn parallel to the base, it shall pass through the virtual centre.

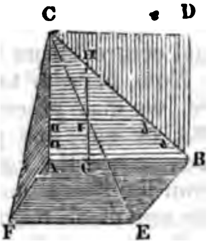
10. The virtual centre of a surface may be found by supposing parallel ordinates drawn across its surface, and assuming iose ordinates, as representing the surface, and computing ie distance of the virtual centre of the ordinates from a iven base ; and if from the properties of the figure, it cannot e readily discovered what part of the line passing through ie centre, is occupied by the centre, then another base may e assumed, making an angle with the former, and if ordinates re supposed to be drawn across the figure, parallel to this ase, another line may be found parallel to this base, passing irough the centre ; hence, the intersection of these two lines ill be the centre required.



11. If AQ be a base line drawn through ay point, as suppose the vertex of any ody, or figure QBD, and if a denote any rdinate EF of the figure, d = AG, its disnce from the base line AQ, and S = the ım of all the ordinates, or the whole figure BD ; then the distance IC of the virtual centre from AQ, is enoted by the (sum of all the ad) ÷ S.

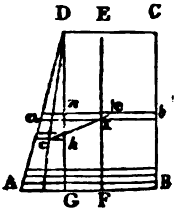
PROBLEM I.

To find the centre of a triangle.



Through the vertice C of the triangle raw a base line CD, parallel to AB, let ı indefinite number of ordinates ab, or parallel to DC, be conceived to be drawn cross the triangle each of which we ıay conceive to be drawn into their reective distances be, &c., from the line D, which may be represented by a sees of $\underline{x}$; hence, if the series of $\underline{z}$, whose ıagnitude is x, be drawn into the series of $\underline{x}x$, the product may e represented by a pyramid, whose base ABEF is = zx nd whose altitude is AC, or x ; hence, $\frac{1}{3}$AB × BD × BD = $\frac{1}{3}$ B × BD^2 or $xz\underline{x}$, $=\frac{1}{2}x^2z$, = the sum of the products, but (Art.) the sum of the products ÷ S, the sum of the series of orinates, or series of z, whose measure is $\underline{z}x$, is equal to the istance of the centre, from the line CD. The sum of the ries of ordinates may be expressed by half the rectangle of B × BD, or $\frac{1}{2}$ zx. Hence, by (Art. 11.) $\frac{\frac{1}{3}AB^2 \times AD^2}{\frac{1}{2}AB \times HD}$ $\frac{2}{3}$AC = $\frac{2}{3}$DB or $\frac{1}{3}z^2x \div \frac{1}{2}zx = \frac{2}{3}x$, that is the virtual cene is some where in a line ab $\frac{2}{3}$ of the distance from the vertice C of the triangle to the base AB, in like manner, we shall find it also in a line GH, $\frac{1}{3}$ of the distance from the vertice B toward the side CA ; hence, it must be in the intersection of the two lines.

Problem 2. *Let it be required to find the virtual centre of a trapezium,* ABCD.



First, let DC, parallel to AB, be taken as the base and imagine an indefinite number of equidistant ordinates *ab*, drawn across the figure, parallel to the base, and if these are severally drawn into their respective distances from the base DC, we shall have constructed a wedge, whose base is equal to AB, drawn into EF, and whose altitude is equal to EF.

Let z = AB, and z' = DC, then will the series of ordinates be represented by $\underline{z}..z'$ let EF $= x$, then will the series of decreasing distances be represented by $\underline{x}$.

Then, we have the solid generated by the production of $\underline{x}\times(z..z')=\frac{1}{6}x(2z+z')x=\frac{1}{6}x^2(2z+z')$, let this be divided by $\frac{1}{2}(z+z')\,x$, and we have $\frac{x\,(2z+z')}{3(z+z')}$ = the distance of the virtual centre I from the line DC.

Now, in order to find in what part of the line *Ab* is the centre, we may proceed as before to construct a quantity on CB, DG, or any line parallel to CB, consisting of ordinates EF, &c., across the figure, drawn into their respective distances from the bases, and dividing the quantity thus constructed by the sum of all the ordinates, or the area of the trapeziums.

Or, we may divide the trapezium into the triangle ADG, and the parallelogram GDCB, and proceed to find the virtual centres of each of these figures ; and it is then evident, that if the distance between the two points, thus found, is divided in the alternate ratio of the two figures, the result will determine the centre required.

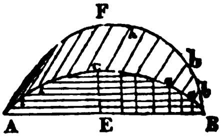
Thus, the distance *hc* of the triangle, from the side DG is $\frac{1}{3}$ *a*H, (Prob. 1,) and the distance *ne* of the centre of the parallelogram from the line DG is evidently $= \frac{1}{2}$ *nb*.

Let t = the area of the triangle, and p = that of the parallelogram, then as *ce* the distance of the two centres, is to $t+p$, so is p, to the distance *e*I, of the vertical centre of the trapezium from the point *e*.

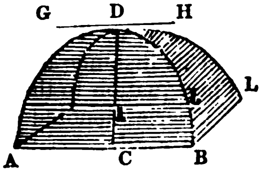
Any rectilinear figure may, if necessary be divided into triangles, and the virtual centres determined for each, and then finding the common centre of every two of these, till they are all reduced to one only, which will be the virtual centre of the whole.

Problem 3. To find the virtual centre of a segment of a circle ABD.

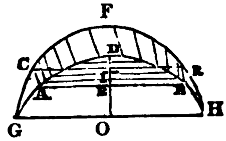
Let parallel ordinates *ab*, &c., be drawn across the segment parallel to its base AB, and let each of these ordinates be conceived to be drawn into their respective distances from AB; then by their properties we shall have generated an ungula ABCF; divide the solidity of this ungula by the area of the segment, and the quotient is the distance EF of the virtual centre from the chord AB.



Or we may let GH be the origin, passing through the vertice D of the segment, parallel to AB, and if the ordinates *ab*, &c., are drawn into their distances from this line GH, or the vertice D, the product will be the ungula ABDL, which is the compliment of the former ungula, and the distance DI of the centre will be found by dividing the solidity of the ungula by the area ABD.

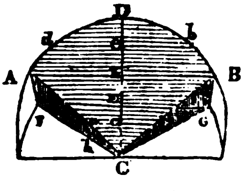


If we make the origin at the centre of the circle, we shall have the ungula ADBRCF, which is a segment of the ungula GDHF, and the solidity of this segment, divided by the surface of the segment ABD, will give the distance OI, of the centre required.



Problem 4, Let it be required to find the virtual centre of a sector of a circle CADB.

Let C be the origin, and if an indefinite number of equidistant ordinates *db*, parallel to the chord AB, are drawn into their respective distances C*e*, &c., we shall have produced the sectoral ungula, FADBGC, which if we divide by the surface, CADB, we shall have the distance CI of the centre.



Let r = CD the radius of the circle, and the chord AB = c then will rc = the convex surface AFDGB of the ungula, and $\frac{1}{3}r^2c$ = the solidity of the ungula. Let π' = the arc ADB, then we have the area CADB $= \frac{1}{2}\pi' r$; hence the distance CI $= \dfrac{\frac{1}{3}r^2c}{\frac{1}{2}\pi' r} = \dfrac{2rc}{3\pi'}$,

Resolving this into a proportion, we have, $3\pi : 2r :: c :$ the distance CI. Hence, generally, the distance of the virtual centre of any sector of a circle is = the fourth proportional to three times the arc of the sector, twice the radius, and the chord of the arc.

Let CE $= \surd(r^2 - \frac{1}{4}c^2,)$ hence we have $\frac{1}{3}r^2c - \frac{1}{2}c^3 =$ the solidity of the pyramid ABGFC; subtract this from the sectoral ungula, and we have $\frac{1}{12}c^3$ for the solidity of the segment ABGFD, divide this by a, the area of the segment ABD and we have the distance of the virtual centre of the segment from the centre C of the circle $= \dfrac{\frac{1}{12}c^3}{a}$.

Problem. 5. To find the virtual centre of the parabola.

Let its distance be estimated from the vertex.

The equation to the parabola is $y^2 = \surd px$, if this be drawn into a series denoted by x, the surface of the parabola will become $y\, x \surd dx$.

Let this be drawn into a series x, and we have $x\underline{x} \surd p\underline{x}$, and since the quantity p will not affect the result, it may be omitted, and we have by removing the radical sign $\underline{x}x^{1\frac{1}{2}} = \frac{2}{5}x^{\frac{5}{2}}$; let this be divided by $x\underline{x}^{\frac{1}{2}} = \frac{2}{3}x^{\frac{3}{2}}$ and we have

$\dfrac{\frac{2}{5}x^{\frac{5}{2}}}{\frac{1}{3}x^{\frac{3}{2}}} = \frac{3}{5}x =$ the distance of the virtual centre from the ertex.

Problem 6. To find the distance of the virtual centre of a semi-parabola from the axis.

Its equation considering the origin at the extremity of the axis is $y = (d - x)^2 + 2x(d - x)$; $d =$ the maximum value of $x =$ the axis; or its equation may be expressed $\underline{x}^2 + 2\underline{x}\overline{x}$. This being put into a series, whose measure is a, represented by the base, we shall have its area, which may be expressed $a\underline{x}^2 + 2a\underline{x}\,\overline{x} = d\text{Y}$, which being integrated, we have $\frac{1}{3}ax^2 + \frac{1}{3}a\underline{x}^2 = \frac{2}{3}a\underline{x}^2$.

Let the expression $a\underline{x}^2 + 2a\underline{x}\overline{x}$, be drawn into $\underline{z}$, z being = the base, and we have $az\underline{x}^2 + 2az\underline{x}\overline{x}$, let this be integrated, and we have $4(\frac{1}{8}zx^2 + \frac{1}{4}zx^2)\frac{1}{6}a = \frac{1}{6}a(1\frac{1}{2}zx^2) = \frac{1}{4}azx^2$.

Hence, $\dfrac{\frac{1}{4}azx^2}{\frac{2}{3}ax^2} = \frac{3}{8}a =$ the distance of the virtual centre from the axis.

Prob. 7. Let it be required to find the virtual centre of a solid.

If the solid be a pyramid, its solidity may be expressed $\underline{x}^2x$.

If we estimate the centre in relation to its distance from the vertex, we may draw this quantity into another series of $\underline{x}$, and we have $\underline{x}^3x$, let this be divided by $\underline{x}^2x$, which represents the base of the latter construction, and we have

$\frac{\underline{x}^3x}{\underline{x}^2x}=\frac{\frac{1}{4}x^4}{\frac{1}{3}x^3}=\frac{3}{4}x$, that is the vertical centre is $\frac{3}{4}$ the distance from the vertex of the pyramid to the base.

Problem. 8. To find the virtual centre of a vertical segment, of a spherical revoloid, or the virtual centre of a segment of a sphere.

The equation of the revoloid is $4y^2 = 4dx-x^2$, let this be drawn into a series of $\underline{x}$, and we have $4x\underline{x}\ (d-x)$, the segment, x being the altitude of the segment, and d the diameter of the sphere, the value of this is

$$\tfrac{1}{6}(2dx^2-2x^3)+\tfrac{1}{6}(4dx^2-2x^3) = dx-\tfrac{2}{3}x^3.$$

Let the former series be drawn into another series of $\underline{x}$, and its value will be $4x(dx^2+\bar{x}\underline{x}^2 = \frac{2}{3}dx^3-\frac{1}{2}x^4$;

hence, $\frac{\frac{2}{3}dx^3-\frac{1}{2}x^4}{dx^2-\frac{2}{3}x^3}=\frac{4dx-3x^2}{6d-4x}$ = the distance of the virtual centre of the segment from the vertex.

This construction will serve for a segment of a sphere, a spheroid, a spherical, or an elliptical revoloid.

Problem. 9. To find the virtual centre of a vertical parabolic revoloid pyramoid, or of a parabolic conoid.

The equation to the revoloid is $y^2=px$, let this be put into a series, whose measure is the axis, and we have its solidity $xp\underline{x}$; let this, as a base, be also drawn into $\underline{x}$, and we have $px\underline{x}^2$ equivalent to $\frac{1}{3}px^3$;

$xp\underline{x}$ is equivalent to $\frac{1}{2}px^2$;

hence $\frac{\frac{1}{3}px^3}{\frac{1}{2}px^2}=\frac{2}{3}x=\frac{2}{3}$. the distance from the vertex to the base.

Problem. 10. To find the virtual centre of an hyperbolic conoid, pyramoid or vertical revoloid.

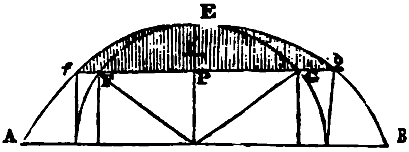
Its equation is $4y^2=4\frac{c^2}{d^2}\ (dx+x^2)$: since $4\frac{c^2}{d^2}$ is a constant quantity, and a factor of the whole expression, it may be omitted, without affecting the result ; then the expression will become $dx+x^2$. Let this be drawn into a series of the form $x(d+\underline{x})_{-}$, which expresses the segment, whose value is, $\frac{1}{6}(dx+x^2)+\frac{1}{6}(\frac{1}{2}dx+\frac{1}{4}x^2)=\frac{1}{6}x(3dx+2x^2)=\frac{1}{6}(3dx^2+2x^3)$

Let the former series be drawn into another series of $\underline{x}$, and there will be produced $x(d + x)\underline{x}^2 = \frac{1}{6}x(dx^2 + x^3) + \frac{1}{6}x \times (\frac{1}{4}dx^2 + \frac{1}{8}x^3)4 = \frac{1}{6}x(2dx^2 + 1\frac{1}{2}x^3) = \frac{1}{6}(2dx^3 + 1\frac{1}{2}x^4)$, hence, we have $\frac{\frac{1}{6}(2dx^3 + 1\frac{1}{2}x^4)}{\frac{1}{6}(3dx^2 + 2x^3} = \frac{4dx + 3x^2}{6d + 4x}$ the distance of the centre from the vertex.

Problem. 11. To find the virtual centre of the arc of a circle.

The virtual centre of an arc of a circle, is the same in reference to the centre of the circle, as that of the segment of a revoloidal curve, whose conjugate diameter is the same as that of the circle. and the base of whose segment is equal to the given arc of the circle.

For, if a revoloidal curve AEB circumscribe the semi-circle DEH, and if any chord FG, be produced to fg, so as to meet the curve, the ordinate fg will be equal to the arc FEG. Draw across the segment fEgf, equidistant ordinates, perpendicular to fg, and they will represent the distance of the several points in the arc, from the line fg, since they are supposed to be drawn from points equidistant from each other on the line fg, or the arc FEG; and, since if there is an infinite number of ordinates, they may be regarded as the area of the segment, it therefore, follows that if this area is divided by the arc, the quotient is the distance of the centre, from the line FG; also, if the area fEgHD is divided by the arc FEG or line fg, the quotient is the virtual centre of the arc fEg from the axis DH.



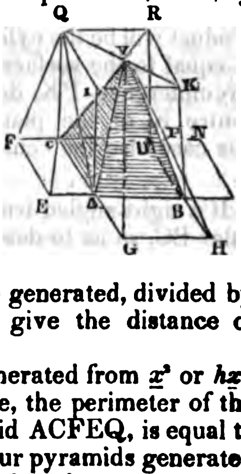
Let c = the chord FGa = the arc FEG, and r = CE, then will cr = area DfEgH, (Prop. III, *Cor.* 4, B. III.,) hence, we have $\frac{cr}{a}$ = the distance EI of the centre.

This is also the virtual centre of the segment fEgf of the revoloidal curve, and it may also be shown that if a series of equidistant ordinates to the axis CE, are drawn through the revoidal surface, or any segment of it parallel to fg, the series of ordinates so constructed, drawn into their several distances from any given line, parallel to such ordinates, will determine the virtual centre of the segment $fgef$, or of the arc FEG, by proceeding as before.

Art. 12. If it be required to find the virtual centre of the arc of an ellipse, a parabola, or an hyperbola, it may be done in a similar manner, by taking a portion of the surface of the elliptic, parabolic, or hyperbolic revoloid, and proceeding as for the arc of the circle, which also gives the virtual centre of the segment of the revoloidal surface pertaining thereto.

Problem 3. To find the virtual centre of the surface of a solid.

Let the proposed surface be the convex surface of a pyramid ABCDV; and because any portion of the convex surface, included between any two sections, by planes parallel to the base, is proportional to the portion of a vertical triangle through the pyramid included between the same planes; it follows that the vertual centre of the convex surface, is the same as that of the vertual triangle; hence the same process will determine both. If it is required to find the virtual centre of the whole surface of a pyramid, including its base, we have only to imagine an infinite number of ordinates to be drawn across the several triangular sides parallel to their several bases, and also a similar series of parallel ordinates across the base, and if each of these ordinates are severally drawn into their respective perpendicular distances from the vertex of the given pyramid, we shall have produced, as many new pyramids AEFCQ, whose bases ACFE, ABHG, &c, are, severally equal to the bases of the sides of the pyramid multiplied by IA, the distance of the base from the vertex, as the given pyramid has sides, and also a prism ABDCQIKR, formed by drawing every line in the base, or the whole surface of the base into the distance of the base from the vertex. And the sum of the imaginary solids so generated, divided by the whole surface of the pyramid, will give the distance of the virtual centre from the vertex.



Let AB$=x$ and if the pyramid is generated from $\underline{x}^2$ or $h\underline{x}^2$, or if the base of the pyramid is a square, the perimeter of the base will be $4x$; and since each pyramid ACFEQ, is equal to $\frac{1}{3}$ the prism ABCDQRKI: hence the four pyramids generated by the series drawn into the four sides of the base are $= \frac{4}{3}$ the prism, and if $h =$ the altitude IA of the pyramid, $hx^2+\frac{4}{3}hx^2= \frac{7}{3}hx^2 =$ the sum of the four pyramids + the prism ABDCQIKR; the surface of the given pyramid is $= x^2+2x\sqrt{(h^2+\frac{1}{2}x^2)}$

Hence we have,
$$\frac{\frac{7}{3}hx^2}{x^2+2x\sqrt{(h^2+\frac{1}{4}x^2}} = \frac{\frac{7}{3}hx}{x+2\sqrt{h^2+\frac{1}{4}x^2}}$$
equal the distance of the virtual centre from the vertex.

BOOK VII.

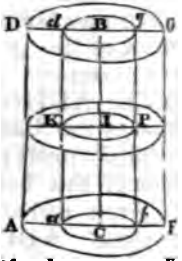
CHAPTER V.

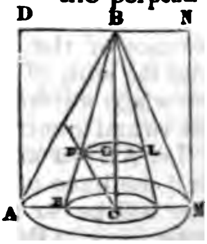
ON THE RELATIONS OF LINES, SURFACES, AND SOLIDS, GENERATED BY MOTION.

THE capacity of any solid generated by the motion of a surface perpendicular to itself, is measured by the generating surface drawn into the distance moved; which distance is always equal to the distance passed through by the virtual centre of such surface.

If the motion of the generating surface is such, as that it always maintains a parallel position, and moves in a direction perpendicular to itself, the proposition is sufficiently manifest.

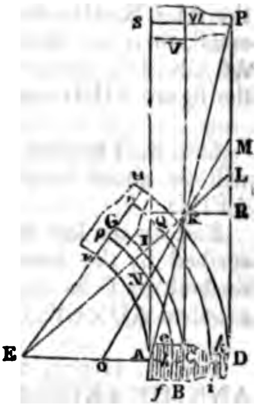
Let now the rectangle ACBD revolve about the side BC, which remains fixed, and the product will be the cylinder DF, whose solidity is equal to the surface ACBD drawn into the circumference PK, described by the virtual centre K, of the plane, which centre is in this case also the centre of aggregation.





If a right-angled triangle ABC revolve about the perpendicular BC, so as to describe the cone ABD, this is also measured by the triangle ABC drawn into the circumference FL, described by the virtual centre of the triangle. The virtual centre of the triangle we have shown to be situated at the point F, on the line BE, from the vertex bisecting the base at a distance from B $=\frac{2}{3}$ its length. Let the surface ABC be multiplied by the circumference described by the centre F; and since the radius FG$=\frac{2}{3}$EC, hence the circumference FG$=\frac{2}{3}$ the circumference EC, and because the triangle ABC$=\frac{1}{2}$ its circumscribing rectangle ADBC, which generates a cylinder ADNM, the generating surface of the triangle ABC drawn into the circumference, is equal to one-third the cylinder generated by the rectangle, or one-third the rectangle drawn into the circumference EC, as it ought to be.

And in general, let any plane figure be revolved about any line or axis without the figure, but always in the same plane, and the solid generated will be measured by the generating surface drawn into the arc described by the virtual centre of the surface.



Let AFHD be a solid generated by the plane ABD; through C, the virtual centre of which, draw DCAE, perpendicular to the axis of rotation, and meeting HGFE in E, let an indefinite number of parallel ordinates, *ef*, *ik*, &c., be drawn across the generating surface, parallel to the axis about which it revolves; and the solid generated is equal to all of those ordinates, drawn into the distances passed through by each; viz., the ordinate drawn across the point A×the arc AF+the ordinate *ab* drawn through *c*×the arc CG+, &c., through the whole series. And because EA, EC, ED, &c., are as the arcs AF, CG, DH, &c. Hence E*h*×*ef*, and E*l*×*ik*, &c., are as *ph*×*ef*, and *rl*×*ik*, and because EC drawn into all the *ef*, *ik*, &c., is equal to all, the E*h*×*ef*, E*l*×*ik*, &c., it follows that CG×all the *ef*, *ik*, &c., is equal to all the *ph*×*ef*, *rl*×*ik*, &c.; or that the solid ABDHF is equal to the generating surface ABD*e* drawn into the line described by the virtual centre of the surface.

Cor. 1. Hence, if any curve or any line be made to revolve about any axis exterior to such curve, but in the same plane, the surface described by its motion will be equal to the line or curve drawn into the distance passed through by the virtual centre of such line or curve.

For, let the perimeter of the figure generating the solid above, be the generating line, and let us suppose its virtual centre the same as before; let every point in this perimeter be reduced to the line AD by means of perpendiculars thereto; and the figure generated by its revolution about the axis, is equal to all the *ph*, *rl*, &c., described by every point; but we have seen that all the *ph*, *rl*, &c., are as all the E*h*, E*l*, &c.; and since the sum of all the E*h*, E*l*, &c., is equal to as many times EC, therefore the sum of all the *ph*, *rl*, &c., is equal to as many times CG, or equal to ABD*e*×CG, that is, the surface described by the perimeter ABD*e*, is equal to ABD*e* drawn into the line described by its virtual centre C.

Cor. 2. From E draw EIKL, cutting the upright prismatic figure erected on the given base ABD, so as that any perpendicular AI may be equal to the corresponding arc AF. Then will the figure AILD be equal to the figure AFHD.

For, by similar figures, all the AF, CG, DH, &c., are as all the AI, CK, DL, &c., each to each; and as one of each are equal, therefore they are all equal, each to each; viz., all the AI, CK, DL, &c., equal to all the AF, CG, DH, &c.; that is, the figure AILD equal to the figure AFHD.

Cor. 3. Through K draw MKNO; then the figure ANMD will be equal to the figure AIKLD, or equal to the figure AFHD.

For, by the last corallary, AFMD is equal to the figure described by the base AD, revolving about O, till the arc described by C be equal to CK; which, by the proposition, is equal to AD×CK, or AD×CG.

Cor. 4. Hence, all the upright figures AQKRD, AIKL, ANKMD, AKPD, &c., of the same base, and bounded at the top by lines or planes cutting the upright sides, and passing through the extremity K, of the line CK, erected on the virtual centre of the base, are equal to one another; and the value of each will be equal to the base drawn into the line CK.

Hence, also, all figures described by the rotation of the same line or plane about different centres or axes, will be equal to one another, when the arcs described by the virtual centre are equal. But if those arcs be not equal, the figures generated will be as the arcs. And in general, the figures generated, will be to one another, as the revolving lines or planes drawn into the arcs described by their respective virtual centres.

Cor. 5. Moreover, the opposite parts NIK, MLK, of any two of these figures, are equal to each other.

Cor. 6. The figure ASPD is to the figure APD, as AS to CK; for, by similar triangles, they will be as AD to AC.

For ASPD is equal to AD×AS, and APD equal to AD ×CK.

Cor. 7. If the line or plane be supposed to be at an infinite distance from the centre about which it revolves, the figure generated will be an upright surface or prism, the altitude being the line described by the virtual centre; so that the base drawn into the said line will be equal to the base drawn into the altitude, as it ought for all upright figures, whose sections parallel to the base are all equal to each other.

Scholium. If a right line, or parallelogram, revolve about a line perpendicular to the length, there will be described a ring either superficial or solid; and as the virtual centre of the describing line, or parallelogram, is also the centre of magnitudes, it follows, therefore, that such surfaces or solids, are equal to the generating magnitude drawn into the distance passed through by the centre of magnitude.

When the centre of rotation is in the end of the line, the line will describe a circle whose radius is the said describing line, and whose circumference is double the circumference described by the virtual centre; consequently, the radius drawn into half the circumference, will be the area of the circle.

If a semi-circle revolve about a diameter, and describe the surface of a sphere, then will the surface of the sphere be equal to the revoloid arc × by the circumference described by the virtual centre $= \frac{1}{2}\pi \times \frac{cr}{\frac{1}{2}\pi} \times \pi = 2r\pi =$ the circumference into the diameter.

And for the solidity of the sphere, we shall have the distance of the virtual centre equal $\frac{d^3}{12a}$, where d is the diameter, and a the area of the segment; let twice this distance be multiplied by π, and also by a; and we have $\frac{2d^3\pi}{12} = \frac{d^3\pi}{6}$ the solidity of the sphere, as before found in the Elements of Geometry.

For the solidity of the parabolic spindle: putting $b =$ the base, and $a =$ the altitude, or axis of the generating parabola.

We have found that $\frac{2}{5}a$ is the distance of the centre of gravity from the base, and consequently $\frac{16}{5}a\pi =$ the line described by the centre of gravity; but $\frac{2}{3}ab$ is $=$ the revolving area; therefore $\frac{16}{5}a\pi \times \frac{2}{3}ab \div$ the $\frac{32}{15}a^2b\pi$ will be the content, which is $\frac{8}{15}$ of the circumscribed cylinder.

For the paraboloid. Making the notation as in the last example, and making $n =$ the area of a circle, whose diameter is 1; $\frac{3}{8}b$ will be the distance of the centre of gravity of the semi-parabola from the axis, consequently $\frac{3}{8}b \times 8n \times \frac{2}{3}ab = 2ab^2n =$ the solidity $=$ half the circumscribed cylinder.

MENSURATION.

Having, in the elementary parts of the work, introduced such subjects of mensuration as depend on principles therein discussed, it only remains for us now to present the higher branches of the subject, or such subjects in mensuration, as depend on the higher branches of geometry.

The subject of mensuration admits of three general divisions: lines, superficies, and solids; but since the mensuration of lines is so intimately connected with that of surfaces, we shall make but two general divisions, termed superficies, and solids.

PART I.

MENSURATION OF SUPERFICIES.

PROBLEM I.

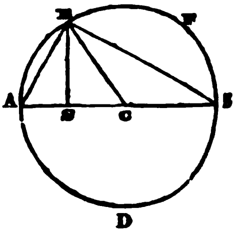
To find the area of a segment of a circle.

CASE I.

When the arc, sine, and radius are given.

RULE.—Multiply the difference between the arc of the segment and its sine by half the radius. (Prop. XIII, B. IV.)

Let π' = the arc, s = the sine, r = the radius, and A the area of the segment; and $A = \frac{1}{2}\pi' r - \frac{1}{2}sr$.



Ex. 1. What is the area of a segment AE, whose arc AE is 2,09438, and whose sine ES = 1,73205, the radius = 2. ?

$$\left.\begin{array}{r} 2.09438 \\ -1.73205 \end{array}\right\} \tfrac{1}{2}r = .36233 \times 1 = .38233$$

the area of the segment AE.

Ex. 2. What is the area of the segment EADB, whose arc EADB = 4.18878 and sine ES = .86602 ?

In this example because the sine is considered negative, by Trigonometry, the arc being greater than that of a semi-circle, we shall have by the rule

$$\left.\begin{array}{r} 4.18878 \\ + .86602 \end{array}\right\} \tfrac{1}{2}r = 1.66138 = \text{the area required.}$$

Ex. 3. What is the area of the segment whose arc is 6.9813, and whose sine is 6,4278, the radius being 10?

Scholium. If the arc and radius, or the sine and radius only are given, the other parts may be taken from the table of natural sines, and the area of the segment calculated by the rule. The arc of any segment less than a semi-circle may be found approximately by formula 3, (Prop. IX, B. IV.) $\frac{1}{2}x = \sqrt{(4vr+\frac{1}{4}s^2)} - \frac{1}{2}s$. Where v is the versed sine or height of the segment, r the radius, and s the sine of the half arc, or the $\frac{1}{2}$ chord of the segment, x being = the arc of the segment.

It will appear that this gives the value of the arc to a great degree of exactness when the segment is small.

Let us see how near the truth this comes for a semi-circle.

In this case, the sine and versed sine are each equal to the radius, which suppose = 1.

Whence we have $\frac{1}{2}x = \sqrt{4\frac{1}{4}} - \frac{1}{2} = 1,56155$
And $x = 3.12310$
the true number being 3.14159

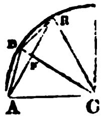
The difference of which is .01849, the error in a segment = the semi-circle.

Let us assume that the error in any smaller segment, is proportional to the sixth power of v or v^6, then if we correct this by deducting $v^6\times.01849$ therefrom, the result will, in this case be correct, and if our hypothesis is correct, it will give the proper result for any smaller arc.

CASE II.

When the arc, chord, versed sine, and radius are given.

Rule.—Multiply the difference between the chord and arc by half the radius, to which add half the product of the chord and versed sine.



Investigation. Let ABD be a segment; this is composed of the segments AD + DB + triangle ADB, but the segment AD = DB = (arc AD — AF) $\times\frac{1}{2}r$; segment AD+DB = (arc ADB — chord AB) $\times\frac{1}{2}r$ and triangle ADB = $\frac{1}{2}$(AB×DF).

Ex. What is the area of the segment ABD, whose arc ADB, is 10,4719, and whose chord AB=10, the versed sine DF or height of the segment =1.3398?

$\left.\begin{matrix}10,4719\\ -10\end{matrix}\right\} 5+\frac{10\times 1,3398}{2} = .9058 =$ the area of the

CASE III.

When the radius of the circle, and the degrees of the arc only are given

RULE.—Find by Trigonometry, or by the table of natural sines, the sine of the given arc for a circle whose radius is 1, observing that the same sine answers for an arc and complement, multiply this by the given radius, which gives the sine of the given arc. Then say, as 180° is to the given arc, so is π to π', so is the semi-circumference of a circle whose diameter is 2, to the length of the given arc; then proceed as in Rule 1.

Ex. 1. What is the area of the segment AE, (see diagram to Case 1,) whose arc AE = 60°; the radius EC being 1?

$$\sqrt{(EC^2 - SC^2)} = ES = \sqrt{(r - \tfrac{1}{2}r)} = \sqrt{(\tfrac{1}{2})} = .86602 = s$$

And $180° : 60° :: \pi : \pi' :: 3.14159 : 1.04719$ = the arc AE $= \pi'$. Hence, $(1.04719 - .86602) \times \frac{1}{2} = .09058$ = the area of the segment AE.

Ex. 2. What is the area of the segment EBF, whose arc EFB is 120°, the other quantities remaining the same as before?

The sine of the arc AE is also the sine of the arc EFB. The arc EFB = 2 arc AE = 2.09438; hence $(2.09438 - 86602) \times \frac{1}{2} = .61418$ = the area of the segment EBF.

Ex. 3. What is the area of the segment EADB, whose arc EADB is 240°, the other quantities remaining the same?

As $180° : 240 :: 3.14159 : 4.18878$ = the arc EADB.

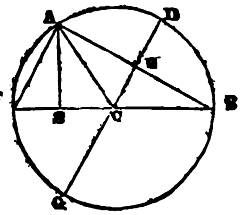
Since the segment EADB is greater than a semi-circle, its sine, ES, is considered negative by Trigonometry, we have $(4.18878 + ,86602) \times \frac{1}{2} = 2.52745$ = the area of the segment EADB.

Ex. 4. What is the area of the segment ADBFE, whose arc is 300°?

Scholium. The difference of the segments EBF, and the segment AE is equal to the sector ACE; the segment ADC − segment EBF = sector ACE + triangle ECB.

CASE IV.

When the chord of the segment, its height or versed sine and radius are given.



RULE.—As the radius is to half the chord, so is twice the difference of the versed sine and radius, to the sine of the arc of the segment; divide this by the radius, reducing it to the sine of an arc of a circle, whose radius is 1.

Then in the table of natural sines, take out the arc answering to that sine in degrees, and proceed as in Rule III, to find its length; then proceed as in rule 1st., to find the area of the segment.

Investigation. In the right angled triangle FAB, we have FA=2CE, and because the triangle ASB is similar to FAB or CEB—hence, CB : EB : : FA : AS.

Or without finding the sine AS, of the arc ADB, proceed to take out from a table of natural sines the arcs AB, DB, answering to AE, BE, the sines of those arcs respectively; and after finding their lengths as in Rule III, proceed by Rule II, to find the area.

Ex. 1. What is the area of a segment ABD, whose chord AB=17,3205, and whose height ED=5, the radius being 10?

In this example we have FB=20, AB=17,3205 and AF=(10 — 5)2=10, to find AS.

Or, we may make the triangle CEB, whose side CB=10, EB=8.6602, and CE = 5, and FA=2CE to find AS, by the rule.

Hence, 10 : 8.6602 : : (10 — 5)2 : 8.6602 = the sine; and 8.6602÷10=.86602 = the tabular sine: the arc answering thereto is that of 60°, but this segment being greater than a quadrant, the arc must be the supplement of 60°=120°. Then 180° : 120° : : 3.14159 : 2.09439 = the length of the arc of 120° in a circle whose radius is 1.

Hence, 2.09439×10=20.9439 = the arc of the given segment.

Then $\left.\begin{matrix}20.9439\\-8.6602\end{matrix}\right\}5, = 61.418 =$ the area of the segment ABD.

Or, taking the same example, having found the length of the arc ADB=20.9439, we have by Rule II.

$\left.\begin{matrix}20.9439\\-17.3205\end{matrix}\right\}5+17.3205+¼=61.418 =$ the area the same

Scholium. If two of the following parts, viz: the chord, versed sine, and radius are given, the other may be found by the formulæ (in mensuration *El. Geom.* Prop. XIII.)

Scholium 2. The triangle ACB is = the triangle AFC.

$$EB \times CE = \tfrac{1}{2} CB \times AS.$$

That is, the product of the sine of an arc × its cosine = ½ sine of twice the arc × radius.

CASE V.

When the chord and radius only are given.

Rule.—Divide half the chord by the radius, and the quotient will be the sine of half the arc of the segment; find the arc corresponding thereto in the table of natural sines, in degrees, and multiply it by 2; then find its length as in Rule 3, and multiply it by the radius.

Take the versed sine $= ED = CD' - \surd(CA - AE.)$ Then proceed as in Rule 2 to find the area of the segment.

Ex. Taking the same example as in the last rule, having found the arc = 20.9439, we have

$$ED = 10 - \surd(100 - 69{,}899905404) = 5.$$

$$\text{Hence, } \left.\begin{array}{r}20.9439\\ -17.3205\end{array}\right\} 5 + 173205 \times 2.5 = 61.418.$$

Examples for Practice.

Ex. 1. What is the area of a segment whose arc is 3.14159 and whose sine is .87785, the radius being 10?

Ex. 2. Required the area of the segment whose chord is = 12, the radius = 10. Ans. 16.35.

Ex. 3. What is the area of the segment whose height is 2, the chord being 20? Ans. 26,8804.

Ex. 4. What is the area of a segment of a circle whose arc is 110° the radius being 1?

Ex. 5. Required the area of the segment whose height is 5, the diameter being 8. Ans. 33.0486.

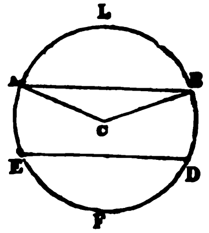
PROBLEM II.

To find the area of a circular zone AEDB, *or the space included between two parallel chords* AB, ED, *and two arcs* AE, BD.

Rule. Multiply the two arcs AE, BD, of the zone by ½ the radius, to which add the sum of the products of the sines of the external segments, with ' the radius, if on different sides of the centre, or add the difference of those products, if on the same side. (Prop. XXIII Schol. Formula 3, B. IV.)

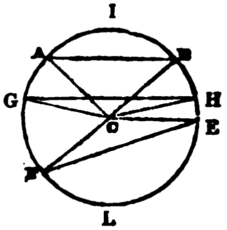
$$A = \tfrac{1}{2}(rx'-rs')-\tfrac{1}{2}(rx-rs,)$$

Ex. What is the area of a zone ABDE, the sum of whose arcs AE+BD is=6,28318 the sine of the arc ALB=2,57178, and the sine of the arc EFD = 2,62386; the radius being = 3 ?.



$$\left.\begin{array}{r}6,28314\\+262386\\+257178\end{array}\right\}\times\tfrac{3}{2}=17,31817$$ the area required.

Scholium. The same rule will apply to any portion ABEF of the circle included between two chords, that are not parallel. Hence, if the sum of the axes AF, AE in this figure, is equal to the sum of AE, BD in the last, and if the arcs AIB, FLE in this are respectively = ALB, EFD in that, then the portion AFEB in this will be = the zone AEDB in that.



Ex. 2. What is the area of a zone AGHB whose two chords are on the same side of the centre, the sum of the arcs of the zone being 2,09436, the sines of the arcs on each side being 2,59806, and — 2,95440, the radius being = 3 ?

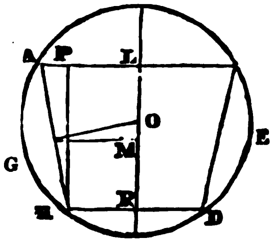
$$\left.\begin{array}{r}2,09436\\+2,59806\\-2,95440\end{array}\right\}\times\tfrac{3}{2}=2,60703$$ the area required.

Note. Other rules have been given for finding the areas of circular segments and zones in Mensuration, El. Geom; formulæ are there given for finding such data as are required, for the elements of the area.

Examples for Practice.

Ex. 1. Required the area of the zone ABEDHGH, the greater chord AB = 136 feet, the less chord HD = 68 feet, and the distance LP = 248 feet.

Ans. 55655.1965159 sq. feet.



Ex. 2. Suppose the zone to have its parallel chords equally distant from the centre of the circle O, each chord AB and HD = 12.49 feet and their distance LP = 10 feet; required the area of the zone. Ans. 148.86672 sq. feet.

Ex. 3. Supposing the circular zone ABEDHGA, having its greater parallel chord = 40 yards, being equal to the diameter of the circle, the less chord = 20 yards, and their distance LP = 17.310508 yards; required the area of the zone.

Ans. 592.08244 sq. yards.

Ex. 4. Required the area of the zone ABEDHGA, the parallel chords AB, and HD, being 16 feet and 12 feet, and their distance HP = 14 feet. Ans. 253.0792 sq. feet.

Ex. 5. The zone, whose parallel chords AB = 40, HD = 30, and the breadth = 35; required the area of the zone.

Ans. 1581.745.

Ex. 6. Suppose the two parallel chords AB and HD = 80 feet and 60 feet, and the perpendicular distance from each other = 70 feet; it is required to find the distance of the greater chord AB from the centre at O; and also to find the radius of the circle.

The distance OP . . . . . = 30 feet = 1st Ans.
The radius of the circle OF = 50 feet = 2nd Ans.

Ex. 7. Required the area of the zone ABEDHGA, whose arcs AGH, BED are together = 160°, the arc AOB being 110°, and the radius of the circle 10 feet.

Ex. 8. Required the area of the portion ABEF, included between the two oblique chords AB, FE, (see diagram to Scholium above,) whose arc AF = 60°, BE = 30°, and AIB = 110°, the radius being 20.

PROBLEM III.

To find the circumference of a circle approximately.

CASE I.

When the radius sine, and cosine of any small arc is given.

Rule 1. Divide $1\frac{1}{2}$ times the product of the radius and sine by the sum of the radius, and half the cosine, which will give the length of the arc, multiply this by the number of such arc in the whole circumference, and the product is the circumference of the circle; the accuracy of which depends in the smallness of the arc.

Let s = the sine,
c = the cosine,

and $\pi' =$ the arc corresponding to these functions,
$\pi =$ being the semi circumference, and r the radius of the circle $= 1$.

$\pi' = \frac{\pi}{n}$; n being the number of parts each $= \pi'$ that the semi-circle is supposed to be divided into. Then (Prop. IX, B. IV. Formula 1.)

$$\pi' = \frac{\frac{3}{2}rs}{r+\frac{1}{2}c}$$

Let $\pi' = \frac{\pi}{20000}$

$$\begin{aligned} \text{Then } s &\doteq \ ,000157079632033 \\ c &= \ ,999999987462994 \\ \text{and } \tfrac{3}{2}rs &= \ ,000235619448050 \\ r+\tfrac{1}{2}c &= 1,499999993831497 \end{aligned}$$

$$\text{Hence } \frac{\frac{3}{2}rs}{r+\frac{1}{2}c} = \ ,000157079632676$$

and, $000157079632676 \times 20000 = 3,1415926535,2 =$ the circumference of the circle, whose radius is 1, which is true to the last figure, which should be 8 instead of 2.

Scholium. Other formula may be found for determining the circumference of a circle at Prop. IX B. XIV, viz., formulæ 2 and 3, to which the student is referred ; at which place, will also be found some important trigonometrical formulæ for finding the value of the sines and cosines, and other functions of the circle.

RULE 2. Divide 6 times the product of the radius and versed sine of a small arc, by the sum of the sine + 4 times the sine of half the arc, which will give the length of the arc, which, multiplied by n, the number of times this arc is contained in the circle, gives the circumference.

$\pi' = \frac{6A'}{a+2b}$ (Prop. XVII,) which may be expressed

$$\frac{6r \,.\, versin \,.\, \pi'}{sin.\ \pi' + 4\ sin.\ \frac{1}{2}\pi'} = \frac{6rv}{s+4s'}$$

Ex. Having the sine and cosine of an arc=to $\frac{1}{100}$ part of a quadrant$=$ $,0157073173118 = s$
and $,9998766324816 = c$; it is required from these data to find the arc π' the radius being 1.

The versed sine $= 1 -$ $,9998766324816$
$=$ $,0001233675183 = v$

Hence, $6rv = 0007402051098$

And by Trigonometry we shall find $s' =$ $,0078539008887$.

Hence, $s + 4s' =$ $,0471229208666$.

And ,0007402051098÷,0471229208666 $= \frac{6rv}{s+4s'}$ $=$,015707963267,5 $= \pi'$ which is true to the last figure, which should be 9 instead of 5.

Scholium. The expression $\frac{6A'}{a+4b}$ may also be put under the form $\frac{6rs}{\text{Cos. } \pi' + 4 \text{ Cos. } \frac{1}{2}\pi'}$ or $\frac{6rs}{c+4c'}$

Ex. 2. Let it be required to find the value of π' to twenty decimal places; for this purpose let π' be an arc of $\frac{1}{10000}$ part of a quadrant, or 1 minute, according to the French centesimal divisions of the circle. The sine of this are to 21 decimal places, according to Legendre's Trigonometry, is

$=.00015707963203352556 3 = s$

and its cosine$=.99999998766299452400 5 = c$

And by the trigonometrical formula $\cos. \theta = \sqrt{\frac{1 + \cos. \theta}{2}}$,

we have $\cos. \frac{1}{2}\pi'$ $=.99999999691574867619 5$.

Hence we have $6rs = .00015707963203352556 3$,

and $6+4c'$ $=.59999999753259892278 5$.

Therefore, $\frac{6rs}{c+4c'} = \pi'$ $=$ 00015707963267948 96,5, which is true to the last figure, which should be 6 instead of 5.

Scholium. Since there is no limit to the smallness of the arc, which may be taken, and since its sine and cosine may be calculated to any number of decimal places whatever, it therefore follows, that there is no limit to the accuracy with which the circle's circumference may be calculated by this method.

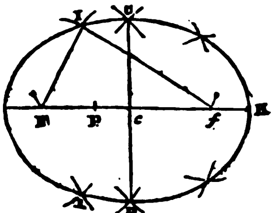
The circumference of the circle, as found by M. DeLagney, to 128 decimal places, by a method furnished by the calculus, is as follows

The circumference of a circle whose diameter is 1, is
3.1415926535897932384626433832795028841971693993751
0582097494459230781640628620899862803482534211706
798214808651 32723066470938446 + or 7—.

The series has more recently been extended to 154 decimal places. We might proceed by this method to verify the results obtained by the calculus, and extend the number of decimals much farther, were it worth the labor; but since we have the result already, extended beyond what is practically useful, the labor may be reserved for those who have leasure and inclination to pursue it. It may be shown that, however far, the circumference should be developed in terms of the diameter, the expression would never terminate, or in other words, the circumference and diameter of a circle are incommensurable in terms of each other. See notes.

PROBLEM IV.

1. *To describe an Ellipse.*

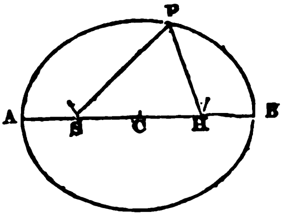


Let TR be the major axis, CO the minor axis, and *c* the centre. With the radius T*c* and centre C, describe an arc cutting TR in the points F, *f*; which are called the two foci of the ellipse.

Assume any point P in the major axis; then with the radii PT, PR, and the centres F*f*, describe two arcs intersecting in I; which will be a point in the curve of the ellipse.

And thus, by assuming a number of points P in the major-axis, there will be found as many points in the curve as you please. Then with a steady hand, draw the curve through all these points.

Otherwise with a Thread.



Take a thread of the length of the axis-major AB, and fasten its ends with two points in the foci, SH. Then stretch the thread, and it will reach to P in the curve: and by moving a pencil round within the thread, keeping it always stretched, it will trace out the ellipse.

There are various instruments used for the construction of this and the other conic sections. But we have not room, consistantly with our plan, to describe them here.

PROBLEM V.

In an ellipse having either three of the following parts given, viz., the major or minor-axis, the ordinate, or abscissæ, to find the fourth.

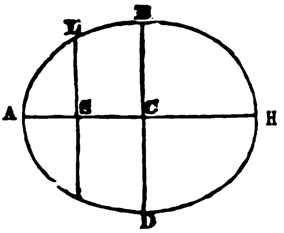
CASE I.

To find the ordinate.

When the major axis, the minor axis and abscissæ, are given.

Rule. As the major-axis is to the minor-axis, so is the square root of the product of the two abscissæ to the ordinate.

Ex. 1. In the ellipse of ABHD the major-axis AH = 70, the minor-axis BD = 50, and the two abscissæ AS = 14, HS = 56, it is required to find the length of the ordinate LS.



$$AH : BD :: \sqrt{(AS \times HS)} : LS,$$

viz., $70 : 50 :: \sqrt{(14 \times 56)} : 20$, the length or the ordinate required.

Ex. 2. If the major, and minor-axes, of an ellipse are 80 and 60, the abscissæ AS = 16, what is the length of the ordinate?

Ans. 24.

CASE II.

To find the two abscissæ.

When the major, and minor axes, and ordinate are given.

Rule. As the axis-minor is to the axis-major, so is the square root of the difference of the squares of the semi-minor axis and ordinate, to the distance between the ordinate and centre; which distance, added to and subtracted from, the semi-axis major will give the two abscissæ.

Ex. 1. The major-axis AH = 70, its conjugate BD = 50, and the ordinate LS = 20; required the two abscissæ AS, HS.

$BD : AH :: \sqrt{((\frac{1}{2}BD)^2 - LS^2)} : CS$, viz.,

$50 : 70 :: \sqrt{(25^2 - 20^2)} : 21$, the distance from the centre to the ordinate.

Hence, $\frac{1}{2}AK \pm SC = (70 \div 2) \pm 21 = 35 \pm 21 = 56$ and $14 = AS, HS$, the two abscissæ.

2. What are the two abscissæ AS, HS, the ordinate LS = 24, and axes AH BD = 80 and 60?

Answer 16 and 64.

3. The major-axis AH = 36, its conjugate BD = 24, and ordinate LS = 8; required two abscissæ HS, AS.

Answer $18 \pm 3\sqrt{2} = 18 \pm 4{,}2426408 = 22{,}2426408$ and $13{,}7573592$.

CASE III.

To find the major axis.

When the minor axis, ordinate, and abscisses, are given.

Rule. From the square of half the minor axis, subtract the square of the ordinate; then extract the square root of the remainder. Next add this root to the semi minor axis, if the less abscissa be given, but subtract it if the greater abscissa is given, reserving the sum or difference. Then say as the square of the ordinate, is to the rectangle of the abscissa and minor axis, so is the reserved sum or difference to the major axis.

Ex. 1. In the ellipse ABHD, there are given the minor axis BD = 50, the ordinate LS = 20, and the less abscissa AD = 14; required the major axis AH.

First. $\surd[(\frac{1}{2}BD)^2 - LS^2] = \surd(16^2 - 20^2) = \surd225 = \surd(5^2 \times 3^2) = 5 \times 3 = 15$, the square root of the difference of the semi-conjugate axis, and the ordinate.

Then $\frac{1}{2}BD + 12 = 25 + 15 = 40$, the sum.

Secondly. $LS^2 : BD \times HS :: 40 : AH$, viz., $20^2 : 50 \times 14 :: 40 : 70 = AH$, the major axis required.

Ex. 2. If the minor axis BD = 40, the ordinate CS = 16, and the less abscissa AS = 36; what is the length of the major axis AH. Ans. 180.

CASE IV.

To find the minor axis.

When the major axis, ordinate, and absbissæ, are given.

Rule. As the square root of the product of the two abscissæ is to the ordinate, so is the major axis to the minor axis.

Ex. 1. The major axis AH = 180, the ordinate HS = 16, and the greater abscissa HS = 144; required the length of the conjugate axis AD.

Here AH − AS = 180 — 144 = 36 = HS, the less abscissa.

Then $\surd(AS \times BS) : LS :: AH : BD$, viz., $\surd(144 \times 36) : 16 :: 180 : 40$, the conjugate axis AH.

Ex. 2. The major axis AB = 70, the ordinate LS = 20, and the abscissa HS = 14; required the conjugate axis BD. Ans. 50.

PROBLEM VI.

To find the area of an ellipse.

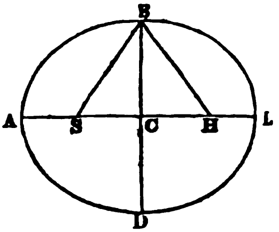
CASE I.

When the major and minor axes are given.

Rule. Multiply the product of the semi axes by $\pi = 3,14159$ or the circumference of a circle whose diameter is 1, and this product will be the area.

Ex. 1. Required the area of an ellipse ABLD, whose axes are AL = 70, and BD = 50.

$\frac{1}{2}$AL × $\frac{1}{2}$BD × π = 70 × 50 × 3,14150 = 2748 . 9, the area of the ellipse required.



Ex. 2. What is the area of the ellipse whose major axis is 23, and the minor axis = 18 ? Ans. 339.2928.

Ex. 3. The major and minor axes being 61,6, and 44 respectively, required the area of the ellipse.
Ans. 2128.7481,6.

Ex. 4. What is the area of an ellipse, whose axes are 25 and 19 ? Ans. 373,06381.

Ex. 5. What is the area of an ellipse whose axes are 23, and 17 respectively ? Ans. 307,09042.

Scholium. If there be two or more concentric ellipses FGHK, *fghk*, the area of the inner one subtracted from that of the outer one, will be the area of the elliptical ring included between them.

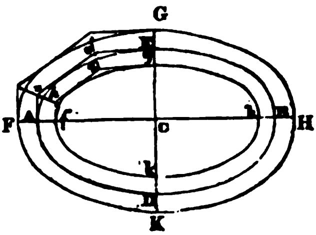
Hence, also as for a circular ring (Mensuration *El. Geom.*) so with the elliptical ring, its area is equal to the difference of the rectangles of the semi axes of the inner one and outer one, multiplied by $\pi = 3,14159$.

Let the ellipse be taken, whose axes are 25 and 19 ; 23 and 17, in the last two examples ;

and we have $(\frac{25}{2} \times \frac{19}{2} - \frac{23}{2} \times \frac{17}{2}) \times \pi = (12,5 \times 9,5 - 11,5 \times 8,5)\pi = (118,75 - 97,75)\pi = 21\pi = 65,97339 =$ he area of the ring F*f*, G*g*, H*h*, K*k*.

Let the results be taken from the two examples referred to, and the area of the ring will be found to agree with this, viz., the area of the outer ellipse is there found - - =373,06381
The area of the inner one - - - =307,09042

The difference is =65,97339
the same as found above



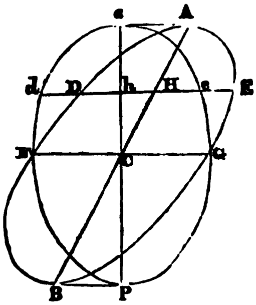
CASE II.

When any two conjugate diameters are given.

Rule. Multiply continually together any two semi conjugate diameters, the sine of their included angle, and π. (Prop. IV, Cor. 5, B. I.)

Ex. The two conjugate diameters AB, FG, of the ellipse ADFBGEA being 32 and 28, and their included angle 77° $34\frac{1}{4}'$; required its area.

The sine of 77° $34'\frac{1}{4}$ is ,9765625 ; therefore, 9765625 × 16 × 14 × 3,14159 = 687.225 = the area.



PROBLEM VII.

To find the area of the segment of an ellipse, cut off by an ordinate to any diameter.

CASE I.

When the ordinate is perpendicular to either of the principal axes.

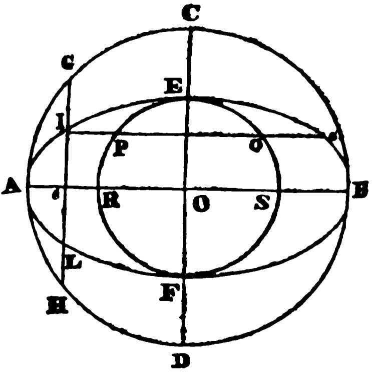
Rule. Find the corresponding segment of a circle of the same height, described on the same axis, to which the cutting line or base of the segment is an ordinate.

Then, as this axis is to its conjugate, so is the circular segment to the elliptical segment.

Or find the area of a circular segment, whose versed sine or height is equal to the quotient of the height of the elliptic segment divided by its axis. Then multiply continually together, this segment and the two axes of the ellipse, for the area of the segment required.

Ex. What is the area of an elliptic segment ILA, cut off by the line IL, parallel to, and at the distance of $7\frac{1}{2}$ from the minor axis EF, the axes being 35 and 25 ?

$17\frac{1}{2} - 7\frac{1}{2} = 10$ the height Ae of the segment.



Then $2\sqrt{(Ae \times Be)}$ = GH the corresponding ordinate or chord to a segment of the circumscribing circle$=2\sqrt{(10\times25)}$ $= 15{,}8113883 \times 2 = 31{,}6227766$.

Let the semi chord Ge be divided by the semi diameter, and we shall have the corresponding sine of the arc GA of a circle, whose radius is $1 = 15{,}8113883 \div 17{,}5 = .903508$, corresponding to which, is the arc of $64° 37\frac{1}{2}'$; hence the arc GAH $= 64° 37\frac{1}{2}' \times 2 = 129° 15'$.

Then AB $\times \pi = 35 \times 3{,}14159$ = the circumference of the circle ACBD = 109, 95565.

And $360° : 129° 15' :: 109{,}95565 : 39{,}4771$ = the arc GAH of the segment ; and by Problem I, Case II,

$$(39{,}4771 - 31{,}6227)\ \tfrac{1}{2}r + 31{,}6227 \times 10 \div 2 =$$
$$7{,}8544 \times 8{,}75 + 31{,}6227 \times 5 = 68{,}726 + 158{,}1135 =$$
$$226{,}8395 = \text{the area of the circular segment GHA.}$$

Then $35 : 25$, or $7 : 5 :: 226{,}8395 : 162, 171$ = the area of the elliptic segment ILA.

Scholium 1. If the area of the segment ILFBE had been required, the circular arc GCBDH should have been taken instead of the arc GAH.

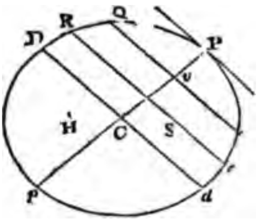
2. If the segment I*o*E, whose base is parallel to the major axis, is required, it may in like manner be found from its relation to the segment POE of the inscribed circle.

CASE II.

When the base of the elliptic segment is oblique to the axes.

Rule. Divide the abscissa Pv by its diameter Pp, and find a circular segment whose versed sine or height is the quotient. Then multiply continually together the area thus found, and the two axes, for the elliptic segment. Or multiply continually together the circular segment, the diameter Pp, to which the base of the segment is a double ordinate, its conjugate diameter Dd, and the sine of their included angle, for the area of the elliptic segment.

Ex. The principal axes of an ellipse being 35 and 25, it is required to find the area of a segment QqP, whose base Qq is an ordinate to the diameter Pp, whose length is 33, it being divided by the ordinate into the two abscissæ Pv = 7, and pv = 26.



FA ÷ AB = 7 ÷ 33 = $.2121\frac{7}{33}$= the versed sine or height of the segment.

The area of a segment corresponding to this height in a circle, whose diameter is 1, is .12162866.

Hence, .12162869 × 25 × 35 = 106.4251 = the segment QqP.

PROBLEM VIII.

To find the circumference of an ellipse.

First, find the area of an elliptic ring included between an interior and exterior concentric ellipse, whose axes are severally the axes of the given ellipse + n, and the same axes — n ; then divide this area by the average distance between the exterior and interior curves, and the quotient will be the circumference of the given ellipse. (Prop. VIII, B. IV.)

Ex. Let it be required to find the circumference AEBDA of an ellipse, whose axes AB, ED are 24 and 18. (See diagram to Scholium, Prob. VI.)

Let us assume two other exterior and interior concentric ellipses, whose axes FH, GK are = AB + n, and ED + n ; and fh, gk, are = AB — n and ED — n ; and if n = 2, we shall have the area of the ring, as found in examples under

Scholium Prob. VI$=65,97339$. Let this be divided by the average distance as found in Proposition VII, B. IV$=\frac{9899+1}{2}$ and we have 66,3115 for the elliptical circumference AEBDA.

Scholium 1. If great accuracy is not required, the following approximating rules from Hutton's Mensuration may be used.

Rule 1. Multiply the sum of the semi axes by π, or 3,1416, and the product will be the circumference *nearly*.

Ex. Required the circumference of an ellipse, whose axes are 24 and 18.
$(12 + 9) + 314159 = 21 \times 3,14159 = 65,9735$ equal the circumference *nearly*.

Or Rule 2. Multiply the square root of half the sum of the squares of the two axes by π, and the product will be nearly = the circumference.

Ex. Taking the same example as before, we have $\sqrt{\frac{24^2 + 17^2}{2}} \times 3,14159 = 66,6433 =$ the circumference *nearly*.

It will be observed by comparing the last two results, that the former one is nearly as much in defect, as the latter is in excess; hence, if we take half the sum of the two we shall have the circumference of the ellipse more accurately.

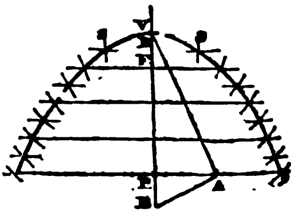
Thus $+ \left.\begin{matrix} 65,9736 \\ 66,6433 \end{matrix}\right\} \div 2 = 66,3084$, which is very near the truth.

PROBLEM IX.

To construct a parabola; having given any ordinate PQ *to the axis, and abscissa* VP.

First, find the focus F thus; bisect PQ in A; draw AV, and AB perpendicular to it; take VF = PB, and F will be the focus.

Arithmetically. Divide the square of the ordinate by four times the abscissa, and the quotient will be focal distance VF.



Then, in the axis, produced without the vertex V, take VC = VF ; draw several double ordinates SRS ; then with the radii CR, and the centre F, describe arcs cutting the corresponding ordinates in the points S.

Draw the curve through all the points of intersection, and it will be the parabola required.

PROBLEM X.

Of any abscissa X, *its ordinate* y, *and latus rectum, or parameter* p ; *having two given, to find the third.*

CASE I.

To find the latus rectum.

Divide the square of the ordinate by its abscissa, and the quotient will be the latus rectum.

Or, take a third proportional to the abscissa and ordinate, for the latus rectum.

That is, $p = y^2 \div x$.

EXAMPLE.

If the abscissa be 9, and its ordinate 6

Then $6 \times 6 \div 9 = 36 \div 9 = 4 =$ the latus rectum.

CASE II.

To find the abscissa.

Divide the square of the ordinate by the latus rectum, and the quotient will be the abscissa.

That is, $x = y^2 \div p$.

EXAMPLE.

If the ordinate be 6, and the latus rectum 4.

Then $6 \times 6 \div 4 = 36 \div 4 = 9 =$ the abscissa.

CASE III.

To find the ordinate.

Multiply the latus rectum by the abscissa, and the square root of the product will be the ordinate.

That is $y = \sqrt{px}$.

EXAMPLE.

The absciss being 9, and the
Then $\sqrt{(9 \times 4)} = \sqrt{36} = 6 =$ th

PROBLEM XI.

Of any two abscisæ A, B, *taken upo*
their two ordinates a, b ; *having an*
fourth.

The abscissæ are to one another a
dinates. That is, as any one abscis
ordinate, so is any other abscissa to tl
and conversely. Or, as the root of o
nate, so is the root of any other absc

$$\text{Hence}\begin{cases}\sqrt{A} : \sqrt{B} :: a : a\sqrt{\frac{B}{A}} = \\ \sqrt{B} : \sqrt{A} :: b : b\sqrt{\frac{A}{B}} = \end{cases}$$

$$\text{And}\begin{cases}a^2 : b^2 :: A : \frac{Ab^2}{a^2} = B \\ b^2 : a^2 :: B : \frac{Ba^2}{b^2} = A\end{cases}$$

Ex. 1. If an abscissa $= 9$ corresp
required the ordinate whose abscissa
Here $\sqrt{9} : \sqrt{16} :: 6 : 6 \times 4 \div 3$

Ex. 2. Required the abscissa corr
6, the ordinate belonging to the abs
Here $8^2 : 6^2 :: 16 : 9 =$ the absci

PROBLEM XI

To find approximately the length of a
off by an ordinate to

When the abscissa and ord

Rule. To the square of the ordin
square of the abscissa, and twice th
will be the length of the curve, *nearl*

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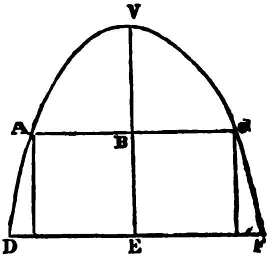
* See Hutton's Men

Ex. The abscissa VH = 2, and the ordinate AB = 6, required the length of the curve EVE.

Here $2\sqrt{(AB^2 + \frac{4}{3}VB^2)} = 2\sqrt{[6^2 + (2^2 + \frac{4}{3})]} = 2\sqrt{\frac{124}{3}} = \frac{4}{2}\sqrt{31 \times 3} = \frac{4}{3}\sqrt{93} = 9.6436508 \times \frac{4}{3} = 12.8582$, the length of the curve AVC, *nearly*.



Examples for Practice.

Ex. 1. What is the length of the parabolic curve AVC, whose abscissa VB = 2, and the ordinate AB = 8?

Ans. 17.4356.

Ex. 2. Required the length of the parabolic curve DAVCF, when the abscissa VE = 16, and the ordinate DE = 12.

Ans. 42.142615.

Ex. 3. Required the length of the parabolic curve DAVCF, when the abscissa VE = 8, and the ordinate DE = 16.

Ans. 36.951.

PROBLEM XIII.

To find the area of a parabola, when the base and height are given.

Rule. Multiply the base by the height, and two-thirds of the product will be the area.

Ex. Required the area of the parabola AVCA, the abscissa VB = 2, and the base, or ordinate, AC = 12.

Here $\frac{2}{3}(AC \times VB) = \frac{2}{3}(12 \times 2) = 16$, the area of the parabola AVCA required.

Examples for Practice.

Ex. 1. What is the area of a parabola DAVCFD, whose abscissa VE = 10, and the double ordinate DF = 16.

Ans. 106⅔.

Ex. 2. Required the area of a parabola DAVCFD, whose base or ordinate DF = 15, and the abscissa VE = 22? Ans. 220.

Ex. 3. What is the area of a parabola AVCA, the base or ordinate AC = 20, and the height or abscissa VB = 6. Ans. 80.

PROBLEM XIV.

To find the area of parabolic frustum, or zone of a parabola, or of the space included between two parallel ordinates.

The two ordinates, and their distance being given.

Rule. To the sum of the squares of the two ordinates, add their product, divide the result by the sum of the two ordinates, the quotient multiplied by two-thirds of the altitude of the frustum, will give the area.

Ex. Required the area of the parabolic frustum ACFDA, the two parallel ordinates DF, and AC = 10, and 6, and the distance BE = 4.

$$\left(\frac{(DF^2 + AC^2) + (DF \times AC)}{DF + AC}\right) \times \tfrac{2}{3}BE$$

$$= \left(\frac{(10^2 + 6^2) + (10 \times 6)}{10 + 6}\right) \times 4 \times \frac{2}{3} = \frac{136 + 60}{16} \times \frac{8}{3}$$

$$= \frac{196}{8 \times 2} \times \frac{8}{3} = \frac{98}{3} = 32\tfrac{2}{3},$$ the area of the frustum ACFDA.

Examples for Practice.

Ex. 1. What is the area of the parabolic frustum ACFDA, whose two ordinates DF and AC = 10 and 6, and the distance BE = 3? Ans. 24½.

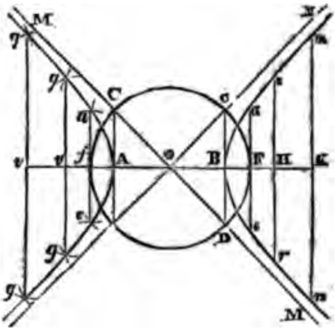
Ex. 2. The greater end of the frustum DF = 30, the less end AC = 20, and their distance BE = 15; required the area. Ans. 380.

Ex. 3. The greater end of the frustum DF = 20, the less end AC = 10, and their distance BE = 12. Ans. 186⅔.

OF THE HYPERBOLA.

PROBLEM XV.

To construct or describe a hyperbola.



Let *o* be the centre of the hyperbola, or the middle of the transverse AB; and BC perpendicular to AB, and equal to half the conjugate.

With the centre *o*, and radius C*o*, describe the circle, meeting AB produced in F and *f*, which are the two foci of the hyperbola.

Then assuming several points *vv*, &c., in the transverse produced, with the radii A*v*, B*v*, and centres *f*, F, describe arcs intersecting in the several points *g*, *g*, &c., through which points draw the hyperbolic curve.

If straight lines *o*M, *o*N, be drawn from the point *o*, the middle of the transverse diameter, through C, and D, the extremities of the conjugate, they will be the *asymptotes* of the hyperbola, the property of which is to approach continually to the curve, but not to meet it, until they be infinitely produced.

PROBLEM. XVI.

In an hyperbola to find the transverse axis or conjugate axis, or ordinate or abscissa.

CASE I.

To find the ordinate.

When the transverse axis, conjugate axis, and the abscissa are given.

Rule. As the transverse axis is to the conjugate axis, so is the square root of the product of the two abscissæ to the ordinate.

Note. In the hyperbola, the less abscissa added to the axis' gives the greater abscissa.

Ex. If the transverse axis AB = 24, the conjugate axis CD = 21, and the less abscissa BH = 8, what is the length of the corresponding PH.

Here $AB : CD :: \sqrt{[(AB + BH) \times BH]} :: PH$, viz. $24 : 21 :: \sqrt{[(24 + 8) \times 8]} : 14$, the length of the corresponding ordinate PH, required.

Examples for Practice.

Ex. 1. The transverse axis $AB = 60$, the conjugate axis $CD = 36$, and the less abscissa $BH = 20$, required the corresponding ordinate PH. Ans. 24.

Ex. 2. The transverse diameter $AB = 50$, the conjugate diameter $CD = 40$, and the greater abscissa $AH = 64$; required the ordinate PH. Ans. $\frac{32}{5}\sqrt{14}$.

Ex. 3. Required the length of the ordinate MK, whose transverse axis $AB = 609$, the conjugate axis $CD = 588$, and the less abscissa $BK = 116$. Ans. 280.

CASE II.

To find the two abscissæ.

When the transverse axis, the conjugate axis, and the ordinate, are given.

Rule. As the conjugate axis is to the transverse axis, so is the square root of the sum of the squares of the ordinate and semi-conjugate to the distance between the ordinate and centre, or half the sum of the abscissæ. Then will the sum of this distance and the semi-transverse be the greater abscissa, and their difference the less.

Ex. The transverse axis $AB = 24$, the conjugate axis $CD = 21$, and the ordinate $PH = 14$; required the two abscissæ AH, and BH.

Here $CD : AB :: \sqrt{[PH^2 + (\frac{1}{2} CD)^2]} : HO$, viz.

$21 : 24 :: \sqrt{(14^2 + 10.5^2)} : 20$.

Then the two abscissas AH and $BH = HO \pm \frac{1}{2}AB = 20 \pm 12 = 32$ and 8.

Examples for Practice.

Ex. 1. The transverse axis $AB = 60$, the conjugate axis $CD = 36$, required the two abscissæ AH, and BH, corresponding to the ordinate $PH = 24$.

Ans. $AH = 80$, and $BH = 20$.

Ex. 2. The transverse axis $AB = 120$, the conjugate axis $CD = 72$, and the ordinate $MK = 48$, required the two abscissæ AK and BK. Ans. $AK = 160$, and $BK = 40$.

PROBLEM XVII.

To find the length of any arc of an hyperbola approximately beginning at the vertex.

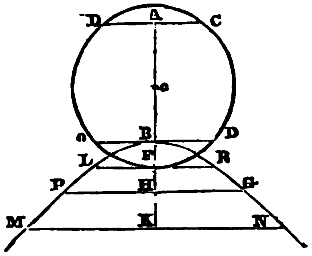
When the transverse and conjugate axis, the ordinate, and abscissa, are given.

RULE. *First.* Add 21 times the the square of the conjugate to 19 times the square of the transverse, and multiply this sum by the abscissa; to this product add 15 times the transverse, multiplied by the square of the conjugate, and call this quantity the dividend.

Secondly. Add 21 times the square of the conjugate to 9 times the square of the transverse, and multiply this sum by the abscissa; to this product add 15 times the transverse, multiplied by the square of the conjugate, and call this quantity the divisor.

Thirdly. Then divide the dividend by the divisor, and multiply the quotient by the ordinate for half the length of the curve, or multiply the quotient by twice the ordinate for the length of the whole curve, *nearly.**

Ex. 1. Required the length of the hyperbolic curve PLBRG to the abscissa BH=2.1637, and the ordinate PH=10; the two axes AB and CD=80 and 60.



Here $\dfrac{(21CD^2+19AB^2)\times BH+(15AB\times CD^2)}{(21CD^2+\ 9AB^2)\times BH+(15AB\times CD^2)}\times 2PH$

$=\dfrac{[(21\times 60^2)+(19\times 80^2)]\times 2.1637+(15\times 80\times 60^2)}{[(21\times 60^2)+(\ 9\times 80^2)]\times 2.1637+(15\times 80\times 60^2)}\times 2\times 10$

$=\dfrac{(75600+121600)\times 2.1637+4320000}{(75600+\ 57600)\times 2.1637+4320000}\times 20$

$=\dfrac{426681.64+4320000}{288204.84+4320000}\times 20=\dfrac{4746681.64}{4608204.84}\times 20=1.03005\times 20$

$=20.601$, the length of the whole curve PLBRG required.

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* Hutton's Mensuration.

Ex. 2 Required the length of the hyperbolic arc PLBRG, the abscissa BH = 20, the ordinate PH = 24, and the two axes AB and CD = 60 and 36. Ans. 62.652.

PROBLEM XVIII.

To find the area of an hyperbola.

When the transverse axis, conjugate axis, and the abscissa, are given.

Rule. To the product of the transverse axis and abscissa, add $\frac{5}{7}$ of the square of the abscissa, and multiply the square root of the sum by 21; to this product add 4 times the square root of the product of the transverse axis and abscissa; then multiply this sum by 4 times the product of the conjugate axis and abscissa, and divide this last product by 75 times the transverse axis, the quotient will give the area of the hyperbola, *nearly.* *

Ex. 1. Required the area of the hyperbola PBGP, whose abscissa BH = 10, the transverse and conjugate axis AB and CD = 30 and 18.

Here $\frac{\{21\sqrt{[(AB\times BH)+\frac{5}{7}BH^2]}+4\sqrt{(AB\times RH)}\}\times 4CD\times BH}{75\,AB}$

$= \frac{\{21\sqrt{[(30\times 10)+(\frac{5}{7}\times 10^2)]}+4\sqrt{(30\times 10)}\}\times 4\times 18\times 10}{75\times 30}$

$= \frac{(21\sqrt{371\frac{3}{7}}+4\sqrt{300})\times 8}{25} = \frac{(21\sqrt{\frac{2600}{7}}+40\sqrt{3})\times 8}{25} =$

$\frac{8}{25}\left(\frac{21}{7}\times\frac{10}{1}\sqrt{(26\times 7)}+40\sqrt{3}\right) = \frac{8}{25}\times(30\sqrt{182}+40\sqrt{3}) =$

$\frac{8\times 10}{25}\times(3\sqrt{182}+4\sqrt{3}) = \frac{16}{5}\times(40.4722128+6.9282032) =$

151.681328, the area of the hyperbola PBGP required.

Ex. 2. What is the area of the hyperbola MBNM, the abscissa BK = 25, the transverse and conjugate axis AB and CD = 50 and 30? Ans. 805.090844.

PROBLEM XIX.

To find the area of any mixtilineal figure by means of equidistant ordinates, terminated by a curve on one side, and a right line as a base on the other.

Rule. To the sum of the first and last ordinates add 4 times the sum of all the even ordinates, and twice the sum of all the odd ordinates, rejecting the first and last; and $\frac{1}{3}$ of this result, multiplied by the common distance of the ordinates, will give the area, *very nearly*. Prop. XV, B. IV.

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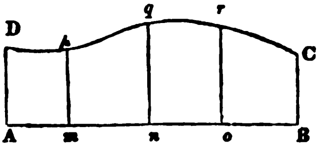
* Hutton's Mensuration.

Scholium. This rule is absolutely true for a parabola, if the ordinates are parallel to its axis. And if the distances between the ordinates is small, it is approximately true for any other curve.

Ex. 1. Required the area of an irregular figure, bounded on one side by a curve line at five equidistant ordinates, the breadths being AD=8.2, $mp = 7.4$ $nq=9.2$, $or = 10.2$, BC = 8.6; the length of the base AB = 39, and the common distance of the ordinates A*m*, *mo*, *no*, *o*B, each = 9.75.

Here $\frac{1}{3}[(AD+BC)+4(mp+or)+2nq] \times 9.75 = \frac{1}{3}[(8.2+8.6)+4(7.4+10.2)+(9.2\times2)] \times 9.75 = \frac{1}{3}(16.8+70.4+18.4)\times9.75=(105.6\div3)\times9.75=343.2$, the area of the space ADCBA required.



Ex. 2. Required the area of an irregular space AD*qn*A bounded on one side by a curve line, and divided by three equidistant ordinates perpendicular to the base A*n*, the ordinates being AD=8, $mp=6$, and $nq=10$, the length of the base A*n*=14, and the common distance A*m*, *mn*, each equal 7.
Ans. 98.

Ex. 3. The abscissa of a parabola being 2, and the base or ordinate 12, required the area of the parabola.

Here, by taking three ordinates, of which the first and last are each nothing, the middle one being the abscissa=2, and the common distance=6; hence the area of the parabola=16 =Ans.

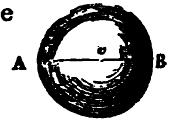
MENSURATION OF SOLIDS.

PROBLEM I.

To find the solidity of a sphere, spheroid, a spherical or an elliptical revoloid.

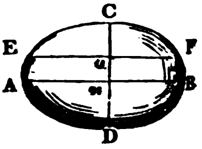
Rule.—Multiply a central conjugate section by the vertical axis, and take two-thirds of the product for the solidity.

Ex. 1. What is the solidity of a sphere, whose diameter is 10 feet?



$31.4159\times\frac{1}{2}=78.5397$ = to a central section; hence, $78.5397\times10\times\frac{2}{3}=523.931$ cubic feet the solidity.

Ex. 2. What is the solidity of a prolate spheroid, ACBD, whose vertical or fixed axis, AB, is 10, and its revolving axis, CD, is 5?

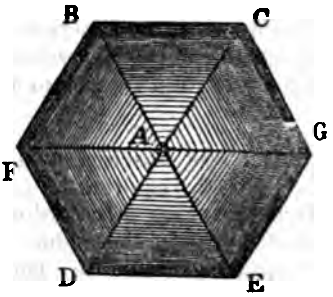


$3.14159\times5\times1\frac{1}{4}=19{,}63494$ = a central conjugate section.

Hence, $19.63494\times10\times\frac{2}{3}=130.9329$, the solidity required.

Ex 3. Required the solidity of a spherical hexagonal revoloid, BCGEDF, whose vertical axis is ten feet.

By referring to the table of Polygons. (*Mensuration El. Geom.*) we find the area of a hexagon, circumscribed about a circle whose diameter is 10, is 17,320508; hence, $17{,}320508\times10\times\frac{2}{3}=115{,}47005$, the solidity required.



Ex. 4. Required the solidity of an elliptical rectangular revoloid, whose vertical axis is 48 inches, and conjugate axis is 36 inches.

$36\times36\times48\times\frac{2}{3}=41472$ cubic inches.

Ex. 5. What is the solidity of an elliptical rectangular revoloid, whose vertical axis is 36 inches, and whose conjugate is 48 inches? Ans. 55296 cubic inches.

Ex. 6. What is the solidity of an oblate spheroid, whose revolving axis = 48, and whose conjugate or fixed axis = 36 inches? Ans. 43429.4784 cubic inches.

Ex. 7. Required to find the solid content of the earth, supposing its circumference to be 25000 miles.

Ans. 263859375000 cubic miles.

Ex. 8. What is the solid content of a sphere, whose diameter AB = 25 feet?

Ans. 8181.25 cubic feet.

Ex. 9. Required the solidity of a sphere, whose circumference is 18.6 feet. Ans. 108.665413272 cubic feet.

PROBLEM. II.

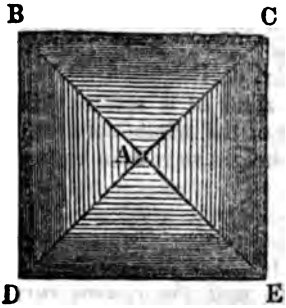
To find the surface of a sphere or of a spherical revoloid.

Rule.—Multiply the perimeter of its central conjugate section by the vertical axis, and the product is the whole surface.

Ex. 1. What is the surface of a sphere whose diameter is 10? Ans. $31.4159\times10=314.159$, the surface required.

Ex. 2. Required the surface of a rectangular spherical revoloid, BCED, whose vertical axis is 10 feet.

$10\times4\times10=400$ square feet, the surface required.



Ex. 3. Required the surface of a ball, whose diameter AB = 1 inch. Ans. 3.1416 square inches.

Ex. 4. How many square inches will cover a globe of 12 inches in diameter? Ans. 452.3904 square inches.

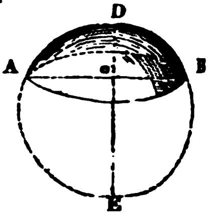
Ex. 5. Required the superficies of the terraqueous globe, supposing the diameter AB = 7958 miles. And if only one-fourth part of its surface be dry land, and two acres sufficient to produce food for one person; how many persons can live on the earth at one time.

Ans. { 198956786.5824 sq. miles, the surface of the globe
49739196.6456 sq. miles, dry land.
15916542927 persons can live on the earth.

PROBLEM III.

To find the solidity of any segment or zone of a sphere.

Rule.—To half the sum of the areas of the two bases multiplied by the altitude, add the solidity of a sphere whose diameter is equal to the altitude of the segment or zone.

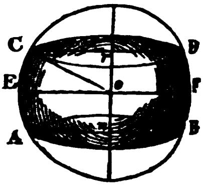


Ex. 1. What is the solidity of a spherical segment ABD, whose base is 10 and whose height *o*D is 2?

$$10\times2=20.$$

and $\frac{1}{6}\pi D^3=\frac{1}{6}\times3.14159\times2\times2\times2=4.18876$.

hence $20+4.18876=24,18876$ = the solidity of the segment.



Ex. 2. What is the solid content of a zone EFDCE, whose height *or* = 30 inches, the greater diameter EF = 60 inches, and the less diameter AB = 40 inches? Ans. 75398.4 cubic inches.

Ex. 4. Required the solidity of the middle zone of a sphere ABDCA, the diameter of the whole sphere EF = 80 inches, the height *nr*=64 inches.

Ans. 233070.5408 cubic inches.

PROBLEM IV.

To find the convex surface of any segment or zone of a sphere, or spherical revoloid.

Rule.—Multiply the perimeter of a middle section of the whole sphere or revoloid, perpendicular to the vertical axis, by the height of the segment or zone.

Scholium. This is the same as the rule given in the Elements of Geometry for a spherical segment or zone, viz: its convex surface is there said to be equal to the height of the segment or zone, multiplied by the circumference of the sphere. The same rules as there given for segments and sectors of a sphere, will answer also for segments and sectors of right revoloids.

Ex. 1. What is the convex surface of a segment of a right revoloid, whose height is 2 feet, the perimeter of a central conjugate section of the whole revoloid being 40 feet?

$40\times2=80$, the surface required.

Ex. 2. Required the convex surface of a zone of a rectangular right revoloid whose height is 6 feet, the whole altitude of the revoloid being $10\frac{1}{2}$ feet. Ans. 252 feet.

Ex. 3. Required the convex surface of a segment of a hexagonal right revoloid, whose height is $5\frac{1}{2}$ feet, the axis of the revoloid being 10 feet. Ans. 190.5255825 square feet.

PROBLEM V.

To find the solidity of a sector of a spherical or right revoloid.

Rule.—Multiply its convex surface by one-third the semi-axis of the revoloid.

Ex. What is the solidity of a revoloidal sector, whose convex surface is 10 square feet, the axis of the revoloid being 10 feet?

$$10\times\frac{1}{3}\times5=16\frac{2}{3}\text{, the solidity.}$$

PROBLEM VI.

To find the solidity of a segment or zone of a spherical revoloid.

Rule.—Find the solidity of the revoloidal sector having the same convex surface; find also, the solidity of the pyramid having the same base as the segment, and whose vertice is in the centre of the revoloid: subtract the solidity of the pyramid from that of the sector, which will give the solidity of the segment, if the segment is less than a semi-revoloid; and add the solidity of the pyramid to that of the sector, if the segment be greater than a semi-revoloid.

Ex. 1. What is the solidity of a segment of a rectangular revoloid whose convex surface is 40 square feet, the axis of the revoloid being 10 feet?

Here, the sector will be found $= 40\times\frac{1}{3}=66\frac{2}{3}$ solid feet, and the height of the segment will be found $= 1$ foot; hence, $(5^2—4^2)=36 =$ the base of the segment; and $36\times4\times\frac{1}{3}= 48$ $=$ the solidity of the pyramid.

Therefore, $66\frac{2}{3}—48=18\frac{2}{3}$ cubic feet the solidity required.

Ex. 2. Required the solidity of the segment of an octagonal revoloid, whose convex surface is 100 feet, the axis of the revoloid being 10 feet.

PROBLEM VII.

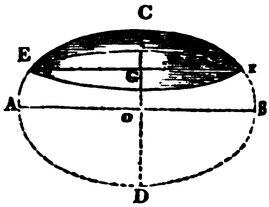
To find the solidity of a segment of a spheroid, made by a plane parallel to either axis.

Scholium. Since the segment of a spheroid is the segment of a sphere, expanded or contracted in the ratio of the major and minor axes ; hence, we have the following.

Rule. Find the solidity of a corresponding segment of the same altitude, from a sphere described on the same axis as that of the segment ; then as this axis is to its conjugate, so is the spherical segment to the spheroidal segment, if the segments base is parallel to the fixed axis. Or as the square of this axis is to the square of its conjugate, so is the spherical segment to the spheroidal segment, if the base is circular or parallel to the revolving axis.

Note. The same will also apply to the segment of an elliptical revoloid compared with a corresponding segment of a spherical or right revoloid.

Ex. 1. In the prolate spheroid ACBD the fixed axis AB=50. the revolving axis CD=30, required the solidity of the segment EFCE, its height EG=6, the base being parallel to the fixed axis AB.



The solidity of a spherical segment, whose height is CG, the diameter being CD=1470.2688.

Hence, by the rule,

CD : AB : : 1470.2688 : EFCE ;

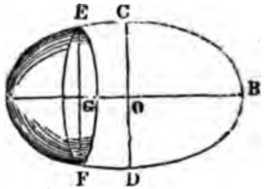
or 30 : 50 : : 1470.2688 : 2450.448 the solidity of the segment EFCE.

Ex. 2. In an *oblate* spheroid, whose revolving axis AB=50, the fixed axis CD=30 ; required the solid content of the segment EFCE, whose height = 5, its base being perpendicular to the revolving axis. Ans. 1099.56.

Ex. 3. Required the solidity of the segment EFCE of the *prolate* spheroid ACBDA, the fixed axis DB=48, the revolving axis CD=38. and the height of the segment CG=16, the base being perpendicular to the revolving axis.

Ans. 13883.8878.

Ex. 4. Required the solidity of the segment EAFE of a *prolate* spheroid, the height $AG=5$ inches, its base being parallel to the revolving axis, which is 30 inches, its fixed axis being 50 inches.



The solidity of a spherical segment, whose altitude is AG of a sphere, whose axis is AB, is 1832.6 cubic inches.

$AB^2 : CD^2 : : 1832.6 : EAFE$;

or $2500 : 900 : : 1832.6 : 659.736$ the solidity of the segment EAFE.

Ex. 5. Required the solid content of the segment of the *prolate* spheroid EAFE, its base being parallel to the revolving axis ; the height $AG=1$, the fixed axis AB—10, and the revolving axis $CD=6$. Ans. 5.2778′.

Ex. 6. The fixed axis CD of an *oblate* spheroid being 30, the revolving axis $AB=50$, and the height of the segment=6, its base being parallel to the revolving axis ; required the solidity of the spheroidal segment. Ans. 4084.08.

PROBLEM VIII.

To find the solid content of the middle frustum of a spheroid.

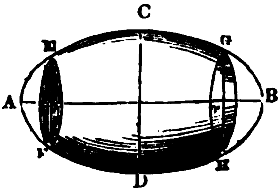
CASE I.

When the ends are circular, or parallel to the revolving axis.

Rule. To twice the square of the middle diameter, add the square of the diameter of one end ; multiply this sum by the length of the frustum, and the product again by .2618 (*which is one-third of* .7854,) for the solidity of the middle frustum.

Scholium. This, and the following rule is derived from the principles contained in scholium page 156, this volume.

Ex. 1. Required the solidity of the middle frustum FCGHDFE of a *prolate* spheroid, the middle diameter $CD=30$, the diameter of each circular end EF or $GH=18$, and the length $cr=40$.



Here $(2CD^2+GH^2) \times cr \times .2618 = [(30^2 \times 2)+ 18^2] \times 40 \times .2618 = (1800+324) \times 40 \times .2618 = 22242.528$ the solidity of the middle frustum ECGHDFE required.

Ex. 2. What is the solidity of the middle frustum ECGHDFE of an *oblate* spheroid, having the less diameters of the circular ends EF and GH, each equal 40; the middle or greater diameter $CD=50$, and the length $cr=18$?

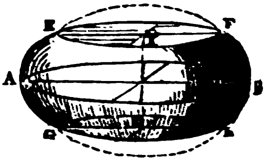
Ans. 31101.84.

CASE II.

When the ends are elliptical, or perpendicular to the revolving axis.

Rule. To twice the product of the major and minor axes of the middle section, add the product of the major and minor axis of one end; then multiply this sum by the length of the frustum, and the product again by .2618, for the solid content of the middle frustum.

Ex. 1. In the middle frustum EFHGE of an *oblate* spheroid, the major and minor axes of the middle or greater elliptic section AB are 50 and 30, and the major and minor axis at one end EF are 40 and 24, the height $IK=9$; required the solid content of the middle frustum.



Here $(50\times30\times2)+(40\times24)\times9\times.2618=(300+960)\times 2.3562=9330.552$, the solidity of the frustum EFHGE required.

Ex. 2. In the middle frustum EFHGE of an *oblate* spheroid, the two axes of the middle ellipse are 50 and 30, and those of each end are 30 and 18, the height of the frustum $IK=40$; required the solid content of the frustum EFHGE.

Ans. 37070.88.

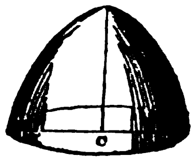
PROBLEM IX.

To find the solidity of a paraboloid or a vertical parabolic revoloid.

Rule. Multiply the area of the base by half the height.

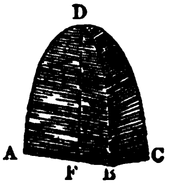
Ex. 1. If the diameter of the base of a paraboloid be 12 feet, and height 22 feet, what is the solidity?

Ans. 1243.



Ex. 2. If the sides AB, CB of the rectangular base of a parabolic semi-revoloid or pyramoid ABCD are each = 10 inches, and the altitude FD=18 inches, required its solidity.?

Ans. 1800 cubic inches.



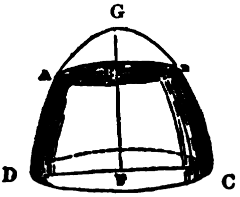
PROBLEM X.

To find the solidity of the frustum of a parabolic conoid, or or paraboloid, or of a vertical parabolic revoloid.

Rule. Multiply the sum of the areas of the two ends by half their distance.

Ex. 1. What is the solid content of the frustum of a paraboloid, the greater diameter DC = 30, the least diameter AB=24, and the altitude EF=9 ?

Ans. 5216.6268.



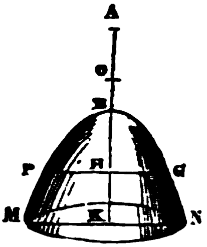
Ex. 2. What is the content in *wine gallons* of a cask in the form of two equal frustums of a paraboloid ; the length=2EF =40 inches, the bung diameter DC=32 inches, and the head diameter AB=24 inches ; the gallon containing 231 cubic inches ? Ans. 108.768 gals.

PROBLEM XI.

To find the solidity of a hyperbolic conoid, or otherwise called a hyperboloid.

Rule. To the square of the radius of the base, add the square of the diameter in the middle, between the base and top; multiply this sum by the altitude, and the product again by .5236, for the solidity of the hyperboloid. (Art. 22, Chap. II, B. V.)

Ex. 1. What is the solidity of an hyperboloid MBNM, whose altitude KB=10, the radius of its base MK=12, and the middle diameter PG=6$\sqrt{7}$?



Here $(MK^2 + PG^2) \times KB \times .5236$ $= [12^2 + (6\surd 7)^2] \times 10 \times 5.236$ $= [144 + (6^2 \times 7)] \times .5236 =$ $(144 + 252) \times 5.236 = 2073.456$, the solidity of the hyperboloid MBNM required.

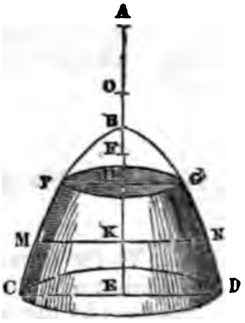
Ex. 2. Required the solidity of the hyperboloid MBNM, whose altitude KB=50, the radius of its base MK=52, and the middle diameter PG=68. **Ans. 191847.04.**

PROBLEM, XII.

To find the solidity of the frustum of a hyperbolic conoid, or hyperboloid.

Rule. To four times the square of the middle diameter, add the sum of the squares of the greatest and least diameters; then multiply this sum by the altitude of the frustum, and that product again by .1309, (*it being the sixth part of* .7854,) for the solidity.

Ex. 1. Required the solidity of the frustum of the hyperbolic conoid PGDCP, the height EH=7, the greatest diameter CD=48, the middle diameter MN=38, and the least diameter PG=27.



Here $(4MN^2 + CD^2 + PG^2) \times EH \times$ $.1309 = [(38^2 \times 4) + 48^2 + 27^2] \times .1309 =$ $(5776 + 2304 + 729) \times 7 \times .1309 = 8809$ $\times 7 \times .1309 = 8071.6867$, the solid content of the frustum PGDCP required.

Ex. 2. Required the solidity of the frustum of a hyperbolic conoid PGDCP, whose greatest diameter CD = 10, the least diameter PG=6, the diameter MN=8½, and the altitude EH=12. **Ans. 667.59.**

Ex. 3. A cask, in the form of two equal frustums of a hyperbolic conoid, having its bung diameter CD=32 inches, its head diameter PG = 24 inches, and the diameter in the middle, between the bung and head $MN = \frac{8}{5}\surd 310$, the length of the cask 2 EH = 40 inches; required the content in *wine gallons.*

Ans. 24998.7584 cubic inches=108.219, &c. gallons.

PROBLEM XIII.

To find, by a general rule, the solidity of any solid, frustum, or segment, produced by the revolution of any conic section, or of any revoloid circumscribing such solids.

General Rule. To the sum of the ends, add four times a section equidistant therefrom, and multiply this sum by one-sixth of the length.

Scholium. This rule is true for any solid or segment, which is generated by any multiple or power of a series of numbers in arithmetical progression. (See Book V, Chap. 1 & 2.)

Ex. 1. What is the solidity of a sphere, whose diameter is 2? The area of its central section, or of its great circle is 3,14159.

Hence $3{,}14159\times4\times\frac{1}{3}=4{,}18878$ the solidity.

Ex. 2. What is the solidity of a zone of a spheroid, whose two bases are 10 and 5 square inches, and whose central section parallel to the bases is 9 square inches, the height of the zone being 18 inches? Ans. 153 cubic inches.

PROBLEM XIV.

To find the solidity of a circular spindle, produced by the revolution of a circular segment about its base or chord as an axis.

Rule. From $\frac{1}{3}$ of the cube of half the axis, subtract the product of the central distance into half the revolving circular segment, and multiply the remainder by four times 3.14159.

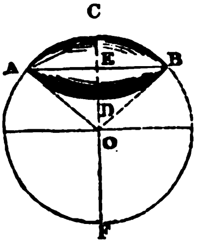
If a=the area of the revolving circular segment,

l=half the length or axis of the spindle,

c=the distance of the axis from the centre of the circle to which the revolving segment belongs;

The solidity$=(\frac{1}{3}l^3-\frac{1}{2}ac)\times4\times3.14159$.

Ex. Let a circular spindle ACBD be produced by the revolution of the segment ABC, about AB. If the axis AB be 140, and CP half the middle diameter of the spindle be 38.4; what is the solidity?



The area of the revolving segment is	3791
The central distance OE	44.6
The solidity of the spindle	374402

PROBLEM XV.

To find the surface of a circular spindle.

Rule. Subtract the product of the revolving arc, multiplied by the central distance, from the product of the length of the spindle into the radius, and multiply the remainder by twice 3.1416, and this product will be the surface.

Ex. Required the surface of a circular spindle ACBDA, whose length AB = 40, and middle diameter CD = 30, the length of the arc ACB being $53\frac{1}{3}$.

The height of the revolving segment$=15$, the radius of the circle$=20\frac{5}{6}$, the central distance $OE=5\frac{5}{6}$.

$(AB\times OC—ACB\times OE)\times 3{,}1416\times 2$

$=(40\times 20\frac{5}{6}—53\frac{1}{3}\times 5\frac{5}{6})\ 6.2832=3281.22\frac{2}{3}$, the surface of the spindle ACBDA required.

PROBLEM XVI.

To find the solidity of the middle frustum of a circular spindle.

Rule. From the square of half the axis of the whole spindle subtract $\frac{1}{3}$ of the square of half the length of the frustum; multiply the remainder by this half length; from the product subtract the product of the revolving area into the central distance; and multiply the remainder by twice 3.14159.

If $L=$half the length or axis of the whole spindle,

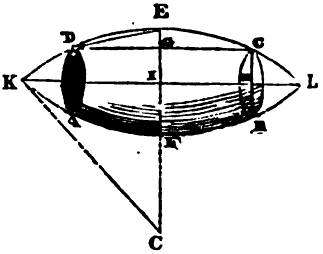
$l=$half the length of the middle frustum,

$c=$the distance of the axis from the centre of the circle,

$a=$the area of the figure which, by revoling, produces the frustum;

The solidity$=((L^2—\frac{1}{3}l^2)\times l—ac)\times 2\times 3.14159$.

Ex. If the diameter of each end of a frustum of a circular spindle be 21.6, the middle diameter 60, and the length 70; what is the solidity?



The length of the whole spindle is	79.75
The central distance	11.5
The revoling area	1703.8
The solidity	136751.5

Scholium. The middle frustum of a circular spindle may be resolved into a cylinder, whose two bases are AD, CB, and a ring described by the revolution of the segment CDE about the axis KL; this ring may also be resolved into a cylindric segment, whose base is the segment DCE, and whose altitude is =the inner diameter of the ring, and a circular spindle formed by revolving the segment DCE about its chord DC. (Prop. XI, Cor. 3, B. III.) Hence its content may be calculated accordingly.

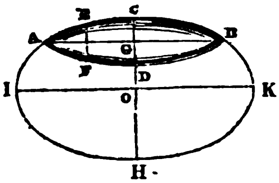
PROBLEM XVII.

To find the solidity of an elliptic spindle.

Rule. First find the solidity of a circular spindle, generated from a segment, whose height CG, is the same as that of the elliptical segment, generating the elliptic spindle, the radius of the circle being CO.

Then, as the length of the circular spindle is to that of the elliptic spindle, so is the solidity of the circular spindle to that of the elliptic spindle.

Ex. If half the middle diameter CD of an elliptic spindle is 38.4, and its axis AB=200, its central distance OG being 44.6, what is its solidity?



The solidity of a circular spindle having the same middle diameter, and the same central distance, we have found (Prob. XIV,)=374402, but its length is 140, therefore by the rule.

140 : 200 : 374402 : 534860 the solidity required.

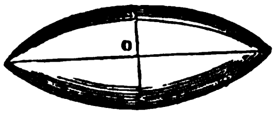
PROBLEM XVIII.

To find the solidity of a parabolic spindle, produced by the revolution of a parabola about a double ordinate or base.

Rule. Multiply the square of the middle diameter by $\frac{8}{15}$ of the axis, and the product by .7854.

Ex. If the axis of a parabolic spindle be 30, and the middle diameter 17, what is the solidity?

Ans. 3631.7

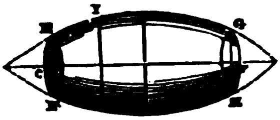


PROBLEM XIX.

To find the solidity of the middle frustum of a parabolic spindle.

Rule. Add together the square of the end diameter, and twice the square of the middle diameter; from the sum subtract $\frac{2}{5}$ of the square of the difference of the diameters, and multiply the remainder by $\frac{1}{3}$ of the length, and the product by .7854.

If D and d=the two diameters, and l=the length ;
The solidity$=(2D^2+d^2-\frac{2}{5}(D-d)^2)\times\frac{1}{3}l\times.7854$.



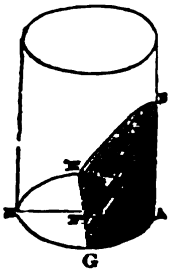
Ex. If the end diameters of a frustum of a parabolic spindle be each 12 inches, the middle diameter 16, and the length 30 ; what is the solidity ?

Ans. 5102 inches.

PROBLEM XX.

To find the convex surface of a cylindric ungula.

Rule. From the product of the diameter and sine, subtract the product of the arc and cosine, and multiply the difference by the altitude divided by the versed sine.



Let h=the altitude AD,
v=the versed sine AF,
d=the diameter AB
a=the arc EAG,
s=the right sine FG,
c=the cosine of the half arc.

Then $\frac{ds-ac}{v}\times h$=the convex surface.

Scholium 1. When F is the centre of the base ; then $v=s=d$, $c=o$; and then the rule becomes dh, viz., the surface is= the product of the diameter into the height.

2. When AF excedes $\frac{1}{2}$AB, then ac must be added, and the expression becomes $\frac{ds+ac}{v}\times h$=the surface.

Ex. 1. What is the curve surface of an ungula EGDA, whose base is half the base of the cylinder and height, AD= 10, the radius FG=10 ? Ans. 100.

Ex. 2. Given the diameter AB = 100 the height AD=140, and the versed sine AF=10, required the curve surface.

Ans. 5962,738.

Scholium. The same considerations will apply to cylindric ungulas, as for circular spindles, taking GE as the axis of the spindle, and AD the circumference of a middle section. (Prop. IV, Cor. 1, B. III.)

PROBLEM XXI.

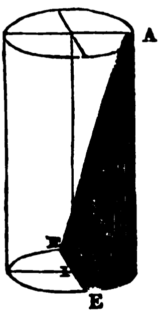
To find the solidity of a cylindric ungula.

CASE I.

When the base of the ungula is = half the base of the cylinder

RULE. Multiply the square of the radius of the base by the altitude of the ungula, and take $\frac{2}{3}$ the product for the solidity.

Scholium. This rule is absolute, without reference to the circle's quadrature, (Prop. VI, Cor. 5, B. III,) ($s=\frac{2}{3}r^2h$, Formula 6, Page 92.)



Ex. What is the solidity of the cylindric ungula AEFC, whose base EFC is half that of the cylinder, the diameter EF being 6, and the altitude CA of the ungula being 16, $3^2\times 16\times\frac{2}{3}=96$ the solidity required.

CASE II.

When the base of the ungula is greater or less than half that of the cylinder.

Subtract the product of the area of the base by the difference between the radius and the versed sine or height, from *one-twelfth* of the cube of the chord of the base, if the versed sine be less less than the radius, *otherwise add this product,* multiplying this result by the altitude of the ungula, and *divide this product by the versed* sine.

If a=the area FEC of the segment forming the base of the ungula, r=the radius, FI v=the versed sine CI, c = the chord EF, and h=the altitude AC.

Then will the solidity of any cylindric ungula=

$$\left(\tfrac{1}{12}c \pm (r \backsim v)\ a\right)\frac{h}{v}$$

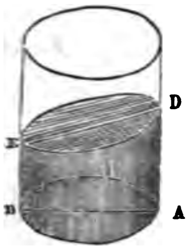
Ex. Given the diameter HC 50 inches, the altitude AC of the ungula=120 inches the versed sine IF=10 inches, required the solidity of the ungula.

The chord EF will be found=40 inches. The area of the base 279.56.

Whence, by the rule, AF being less than $\frac{1}{2}$HC, we have

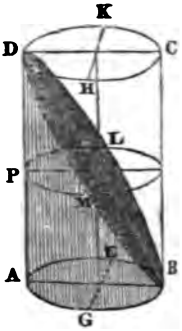
$$\left(\tfrac{1}{12}c^3-(r-v)a\right)\frac{h}{v}=136709\tfrac{1}{4}$$ cubic inches, the solidity of the ungula EFCA.

Scholium. When the section passes obliquely through the opposite sides of the cylinder, the content of the ungula may be found by multiplying the sum of the greatest and least heights of the ungula by the area of the base, and its surface may be found by multiplying ½ the sum of the greatest and least heights by the perimeter of the base.



Hence, the ungula ABD is equal to half the cylinder ABCD, both in its surface and solidity.

2. The complement LMHKD, DKHCB of any ungula LMPD, BGAED, may be found by subtracting the solidity of the ungula from a portion of the cylinder of equal base and altitude.



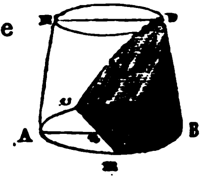
PROBLEM XXII.

To find the solidity of a conical ungula, cut from the cone, or frustum, by a plane parallel to the side of the cone.

Rule. Multiply the area of the base by the diameter of the base of the frustum, and divide the product by the difference of the diameters of the two bases; from this quotient subtract four-thirds of the product of the less diameter by the square root of the product of the less diameter, and difference of the diameters. Multiply the remainder by one-third of the height, and the product will be the content.

Let a=the area of the base cmB,
D=AB the diameter of the base of the frustum,
d=ED the diameter at the top,
h=the height.
Then will the solidity of the ungula=

$$\left(\frac{aD}{D-d}-\tfrac{4}{3}d\times\sqrt{d(D-d)}\right)\tfrac{1}{3}h$$



Ex. If the diameter AB=30 inches, the diameter ED=19.2 inches, and the height od = 18 inches, what is the content of the ungula ? Ans. 1606.41.

Scholium. Other formulæ may be found in the general scholia, at pages 191, &c., for cylindric and conic ungulas; it is unnecessary to extend the problems here.

PROBLEM XXIII.

To find the solidity of a cylindrical ring.

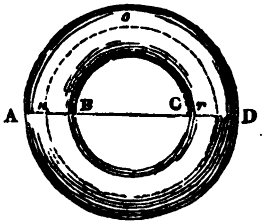
Rule. Multiply half the sum of the inner and outer circumferences by the area of a section of the ring.

Scholium. This rule answers for all rings, the virtual centres of whose sections are in the centre of magnitudes of such sections.

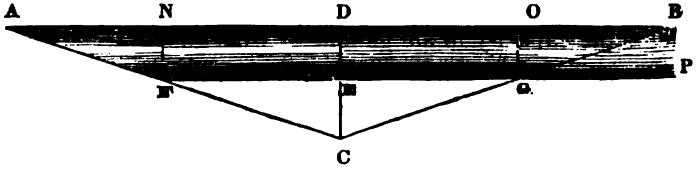
For all other rings see Prop. XI. and Corollaries B. III.

What is the solidity of the cylindric ring AD, whose inner diameter BC is 10 inches, and whose outer diameter AD is 20 inches?

Ans. 925.436 inches.

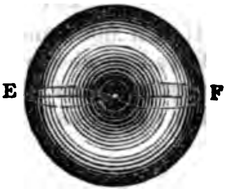


Scholium. This ring is equivalent to a segment ABGFA of a cylinder, whose section DE is equal to that of the ring, and whose length FG = the inner circumference, and AB = the outer circumference.

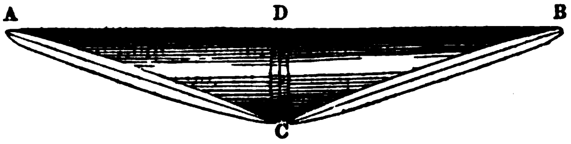


Ex. 2. What is the solidity of the cylindric ring EF, whose inner diameter is 0, and its outer diamer EF 10 inches?

Ans. 308,478 inches.



Scholium. This ring is equivalent to the segment ABC of a cylinder, whose length AB is=the outer circumference of the ring.



GAUGING OF CASKS.

Art. 1. Gauging of casks is a practical art; and since casks are not commonly constructed in exact conformity with any regular mathematical figure, the subject does not admit of being treated in a very scientific manner; by most writers on the subject, however, they are considered as nearly coinciding with one of the following forms:

1.	The middle frustum	of a spheroid,
2.		of a parabolic spindle.
3.	The equal frustums	of a paraboloid,
4.		of a cone.

The *second* of these varieties agrees more nearly than any of the others, with the forms of casks, as they are commonly made. The first is too much curved, the third too little, and the fourth not at all, from the head to the bung.

2. Rules have already been given, for finding the capacity of each of the four varieties of casks. As the dimensions are taken in *inches*, these rules will give the contents in cubic inches. To abridge the computation, and adapt it to the particular measures used in gauging, the factor .7854 is divided by 282 or 321; and the quotient is used instead of .7853, for finding the capacity in ale gallons or wine gallons.

$$\text{Now}\ \frac{.7854}{282}=002785, \text{ or } .0028 \text{ nearly};$$

$$\text{And}\ \frac{.7854}{231}=.0034$$

If then .0028 and 0034 be substituted for .7854, in the rules referred to above; the contents of the cask will be given in ale gallons and wine gallons. These numbers are to each other nearly as 9 to 11.

PROBLEM I.

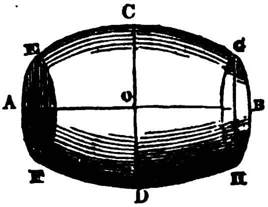
To calculate the contents of a cask, in the form of the middle frustum of a spheroid, being a cask of the first variety.

Rule. Add together the square of the head diameter, and twice the square of the bung diameter; multiply the sum by $\frac{1}{3}$ of the length, and the product by .0028 for ale gallons, or by .0034 for wine gallons.

If D and d= the two diameter, and l=the length;

The capacity in inches= $(2D^2+d^2)\times\frac{1}{3}l\times.7854$.

And by substituting .0028 or .0034 for 7854, we have the capacity in ale gallons or wine gallons.



Ex. What is the capacity of a cask of the first form, whose length AB is 30 inches, its head diameter EF 18, and its bung diameter CD 24?

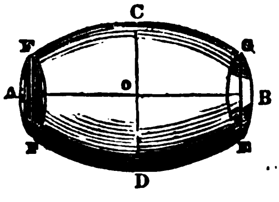
Ans. 41.3 ale gallons,
or 50.2 wine gallons.

PROBLEM II.

To calculate the contents of a cask, in the form of the middle frustum of a parabolic spindle, being a cask of the second variety.

Rule. Add together the square of the head diameter, and twice the square of the bung diameter, and from the sum subtract ⅖ of the square of the difference of the diameters; multiply the remainder by ⅓ of the length, and product by .0028 for ale gallons, or .0034 for wine gallons.

The capacity in inches= $(2D^2+d^2-\frac{2}{5}(D-d)^2)\times\frac{1}{3}l\times.7854$.



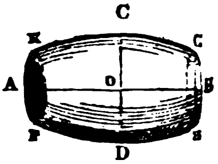
Ex. What is the capacity of a cask of the second form, whose length AB is 30 nches, its head diameter BF = 18, and its bung diameter CD=24?

Answer 40.9 ale gallons,
or 49.7 wine gallons.

PROBLEM III.

To calculate the contents of a cask, in the form of two equal frustums of a paraboloid, being a cask of the third variety.

Rule. Add together the square of the head diameter, and the square of the bung diameter; multiply the sum by half the length, and the product by .0028 for ale gallons, or .0034 for wine gallons.



The capacity in inches= $(D^2+d^2)\times\frac{1}{2}l\times.7854$.

Ex. What is the capacity of a cask of the third form, whose dimensions are, as before, 30, 18, and 24?

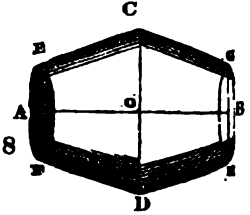
Ans. 37.8 ale gallons, or 45.9 wine gallons.

PROBLEM IV.

To calculate the contents of a cask, in the form of two equal frustums of a cone.

RULE. Add together the square of the head diameter, the square of the bung diameter; and the product of the two diameters; multiply the sum by $\frac{1}{3}$ of the length, and the product by .0028 for ale gallons, or .0034 for wine gallons.

The capacity in inches= $(D^2+d^2+Dd)\times\frac{1}{3}l\times.7854$.



Ex. What is the capacity of a cask of the fourth form, whose length AB is 30, and its diameters EF and CD = 18 and 24?

Ans. 37.3 ale gallons, or 45,3 wine gallons.

Scholium. In the preceding rules, it is supposed that the cask corresponds to the different varieties, whereas, it is seldom that a cask perfectly coincides with either; but for the greater certainty of the truth, when accuracy is required, the following rules, the demonstration of which will be found in Hutton's Mensuration, are to be preferred.

PROBLEM V.

To calculate the contents of any common cask from three dimensions.

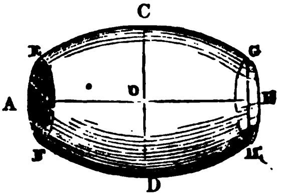
RULE. Add together

25 times the square of the head diameter,

39 times the square of the bung diameter, and

26 times the product of the two diameters.

Multiply the sum by the length, divide the product by 90, and multiply the quotient by .0028 for ale gallons, or .0034 for wine gallons.



The capacity in inches=

$(39D + 25d^2 + 26Dd)\times\frac{l}{90}\times.7854.$

Ex. What is the capacity of a cask whose length is 30 inches, the head diameter 18, and the bung diameter 24?

Ans. 39 ale gallons,
or $47\frac{1}{3}$ wine gallons.

PROBLEM VI.

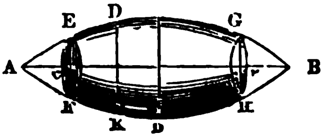
To calculate the contents of a cask from four dimensions, the length, the head and bung diameters, and a diameter taken in the middle between the head and the bung.

Rule. Add together the squares of the head diameter, of the bung diameter, and of double the middle diameter; multiply the sum by $\frac{1}{6}$ of the length, and the product by .0028 for ale gallons, or .0034 for wine gallons.

If D = the bung diameter, d = the head diameter, m = the middle diameter, and l=the length;

The capacity in inches=

$(D^2+d^2+\overline{2m^2})\times\frac{1}{6}l\times.7854.$



Ex. What is the capacity of a cask, whose length cr is 30 inches, the head diameter EF=18, the bung diameter CD = 24, and the middle diameter IK $22\frac{1}{4}$?

Ans. 41 ale gallons,
or $49\frac{2}{3}$ wine gallons.

Scholium. In making the calculations in gauging, according to the preceding rules, multiplications and divisions are frequently performed by means of a *Sliding Rule*, on which are placed a number of logarithmic lines, similar to those on Gunter's Scale.

Another instrument commonly used in gauging is the *Diagonal Rod.* By this, the capacity of a cask is very expeditiously found, from a single dimension, the distance from the bung to the intersection of the opposite stave with the head. The measure is taken by extending the rod through the cask, from the bung to the most distant part of the head. The number of gallons corresponding to the length of the line thus found, is marked on the rod. The *logarithmic* lines on the gauging rod are to be usad in the same manner, as on the sliding rule.

ULLAGE OF CASKS.

Art. 2. When a cask is *partly* filled, the whole capacity is divided, by the surface of the liquor, into two portions; the *least* of which, whether full or empty, is called the *ullage*. In finding the ullage, the cask is supposed to be in one of two positions; either *standing*, with its axis perpendicular to the horizon; or *lying*, with its axis parallel to the horizon. The rules for ullage which are *exact*, particularly those for lying casks, are too complicated for common use. The following are sufficiently near approximations. See Hutton's Mensuration.

PROBLEM VII.

To calculate the ullage of a standing cask.

Rule. Add together the squares of the diameter at the surface of the liquor, of the diameter of the nearest end, and of double the diameter in the middle between the other two; multiply the sum by ¼ of the distance between the surface and the nearest end, and the product by .0028 for ale gallons, or .0034 for wine gallons.

If D=the diameter of the surface of the liquor,
d=the diameter of the nearest end,
m=the middle diameter, and
l=the distance between the surface and the nearest end;

The ullage in inches$=(D^2+d^2+2m^2)\times\frac{1}{4}l\times.7854$.

Ex. If the diameter at the surface of the liquor, in a standing cask, be 32 inches, the diameter of the nearest end 24, the middle diameter 29, and the distance between the surface of the liquor and the nearest end 12; what is the ullage?

Ans. 27⅘ ale gallons, or 33¾ wine gallons.

PROBLEM VIII.

To calculate the ullage of a lying cask.

Rule. Divide the distance from the bung to the surface of the liquor, by the whole bung diameter, find the area of a circular segment, whose versed sine is the quotient in a circle, whose diameter is 1, and multiply it by the whole capacity of the cask, and the product by 1¼ for the part which is empty.

If the cask be not half full, divide the depth of the liquor by the whole bung diameter, and find the area of the segment, multiply, &c., for the contents of the part which is full.

Ex. If the whole capacity of a lying cask be 41 ale gallons, or 49⅔ wine gallons, the bung diameter 24 inches, and the distance from the bung to the surface of the liquor 6 inches, what is the ullage? Ans. 7⅞ ale gallons, or 9½ wine gallons.

OF THE

SPECIFIC GRAVITY OF SOLIDS AND FLUIDS.

The specific gravities of bodies are their relative weights, contained under the same given magnitude as a cubic foot, or a cubic inch, &c.

A table of the Specific Gravities of Bodies, and the weight of a cubic foot of each, in ounces, avoirdupois.

Platinum,		Common stone, . .	2520
Rolled, . .	22666	Loom, . . .	2160
Hammered, .	20335	Clay, . . .	2160
Pure gold		Brick, . . .	2000
Hammered, .	19360	Ivory, . . .	1825
Cast, . .	19256	Sand, . . .	1520
Gold 22 car. fine cast,		Coal, . . .	1250
	17484	Sulphuric acid, .	1840
Mercury, .	13596	Nitrous acid, .	1550
Lead, . . .	11351	Nitric acid, . .	1217
Silver, cast, .	10474	Human blood, .	1054
Copper, . .	8788	Cow's milk, . .	1031
Soft steel,		Box-wood, . .	1030
Hammered, .	7839	Sea-water, . .	1028
Cast, . .	7832	Vinegar, . . .	1026
Hard steel,		Tar, . . .	1015
Hammered, .	7817	Common water, .	1000
Cast, . .	7815	Red wine, . .	990
Bar iron, . .	7787	Linseed oil, . .	932
Tin, . .	7290	Proof spirits at 510,	923
Cast iron, . .	7208	Olive oil, . .	913
Zinc, . .	6860	Alcohol, pure, .	792
Granite, . 3500 to	4000	Æther, . . .	726
Flint glass, . .	3329	Air, . . .	1¼

Note. The several sorts of wood are supposed to be dry. Also as a cubic foot of water weighs just 1000 ounces, avoirdupois, the numbers in this table express not only the specific gravities of the several bodies, but also the weight of a cubic foot of each, in avoirdupois ounces; and hence, by proportion, the weight of any other quantity, or the quantity of any other weight, may be known as in the following problems.

PROBLEM I.

To find the magnitude of any body from its weight.

Rule.—As the tabular specific gravity of the body is to its weight in avoirdupois ounces, so is one cubic foot, or 1728 cubic inches, to its content in feet, or inches respectively.

Ex. 1. Required the solid content of an irregular block of common stone, which weighs 1 cwt. or 1792 ounces.

Here, as 2520 oz : 1792 ox : : 1728 cubic inches : to its solid content.

Or,

As 5 oz. : 256 oz : : 24 cubic inches : $1228\frac{4}{5}$ cubic inches, the solid content required.

Ex. 2. How many cubic feet are there in a ton weight of dry oak? Ans. $38\frac{138}{185}$ cubic feet.

Ex. 3. What is the solid content, and diameter of a cast iron ball, that weighs 42 pounds, its specific gravity being 7208?

$$\text{Ans.}\left\{\begin{array}{l}\text{Solidity} = 161.095 \text{ cubic inches.}\\ \text{Diameter} = 6.75 \text{ inches.}\end{array}\right.$$

PROBLEM II.

To find the weight of a body from its magnitude.

Rule.—As one cubic foot, or 1728 cubic inches, is to the solid content of the body, so is its tabular specific gravity to the weight of the body.

Ex. 1. Required the weight of a block of marble, whose specific gravity being 2700, the length = 63 feet, the breadth and thickness each = 12 feet; this block being the dimensions of one of the stones in the walls of Balbeck.

Here, as 1 cubic foot : 63 × 12 × 12 (= 9072 cubic feet) : : 2700 oz. : $683\frac{7}{16}$ tons weight, almost equal to the burthen of an East India ship.

Ex. 2. What is the weight of a block of dry oak, which measures 10 feet long, 3 feet broad, and $2\frac{1}{2}$ feet deep?

Ans. $4335\frac{15}{16}$ pounds.

Ex. 3. What is the weight of a leaden ball, $4\frac{1}{4}$ inches in diameter, its specific gravity being 11351?

Ans. 16 lbs. 7 oz.

Ex. 4. Required the weight of a cast iron shell, 3 inches thick, its external diameter being 16 inches, and its specific gravity 7208.

$$\text{Ans.}\left\{\begin{array}{l}\text{Solidity} = 1621.0656 \text{ cubic inches.}\\ \text{Weight} = 2 \text{ cwt. } 3 \text{ qr. } 9 \text{ lb. } .5 \text{ oz.}\end{array}\right.$$

PROBLEM III.

To find the specific gravity of a body heavier than water.

Rule.—Weigh the body both *in water* and *out of water*, by a *hydrostatic balance*, and take the difference of these results, which will be the weight lost *in water*.

Then say, as the weight lost *in water*, is to the weight of the body *in air*, so is the specific gravity of water, to the specific gravity of the body.

Ex. 1. A piece of stone weighed ten pounds *in air*; but, *in water*, only 6¾ pounds; required the specific gravity.

Here, as 10 — 6¾ (= 3¼) : 10 : : 1000 : to the specific gravity of the body.

Or,

As 13 lbs. : 40 lbs. : : 1000 oz. : 3077 oz. = Ans.

Ex. 2. A piece of copper weighs 36 oz. *in air*, and only 31.904 oz. *in water*; required the specific gravity of copper.
Ans. 8788 ounces.

Eq. 2. Required the specific gravity of a piece of granite stone which weighs 7 lbs. *in air*, and 5 lbs. *in water*.
Ans. 3500 ounces.

PROBLEM IV.

To find the specific gravity of a body lighter than water.

Rule.—Fasten to the lighter body, by a slender thread, another body heavier than water, so that the mass compounded of the two may sink together. Weigh the heavier body, and the compound mass, separately, both *in water* and *out of it*, then find how much each loses *in water*, by subtracting its weight *in water* from its weight *in air*.

Then say, as the difference of these remainders is to the weight of the lighter body *in air*, so is the specific gravity of water to the specific gravity of the lighter body.

Ex. 1. Suppose a piece of elm weighs 12 lbs. *in air*, and that a piece of metal, which weighs 18 lbs. *in air*, and 16 lbs. *in water*, is affixed to it, and that the compound weight is 6 lbs. *in water*; required the specific gravity of the elm.

Here 18 — 16=2 pounds, the metal lost *in water*;
and (18+15) — 6 = 33 — 6=27 pounds, the compound lost *in water*.

Then 27 — 2=25 pounds, the elm lost *in water*.

As 27 — 2 (=25 lbs.) : 15 lbs. : : 1000 oz. : to the specific gravity of the elm.

Or,

As 1 lb. : 3 lbs. : : 200 oz. : 600 oz. = Ans.

Ex. 1. A piece of ash weighs 20 lbs. *in air*, to which is affixed a piece of metal, which weighs 15 lbs. *in air*, and *in water*, 13¼ lbs.; and the compound, *in water*, weighs only 8¼ lbs; required the specific gravity of the ash.

Ans. 800 ounces.

Ex. 2. Suppose a piece of fir weighs 11 lbs. *in air*, and a piece of steel being affixed which weighed 16 lbs. *in air*, and *in water* 14 lbs.; and the compound *in water* weighs only 5 lbs.; what is the specific gravity of the fir?

Ans. 550 ounces.

Ex. 4. A piece of cork weighing 20 lbs. *in air*, had a piece of granite fixed to it, that weighed 120 lbs. *in air*, and 80 lbs. *in water;* the compound mass weighed 16⅔ lbs. *in water;* what was the specific gravity of the cork?

Ans. 240 ounces.

QUESTIONS FOR EXERCISE.

1. Having a rectangular marble slab, 58 inches by 27, I would have a square foot cut off parallel to the shorter edge; I would then have the like quantity divided from the remainder, parallel to the longer side; and this alternately repeated, till there shall not be the quantity of a foot left; what will be the dimensions of the remaining piece?

Ans. 20.7 inches by 6.086.

2. Given two sides of an obtuse angled triangle, which are 20 and 40 poles; required the third side, that the triangle may contain just an acre of land? Ans. 58.876 or 23.099.

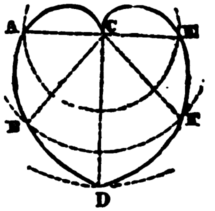
3. The ellipse in Grosvenor-square measures 840 links across the longest way, and 612 the shortest, within the rails; now the walls being 14 inches thick, what ground do they inclose, and what do they stand upon?

Ans. { inclose 4*a*. 0*r*. 6*p*.
stand on 1760¼ sq. feet

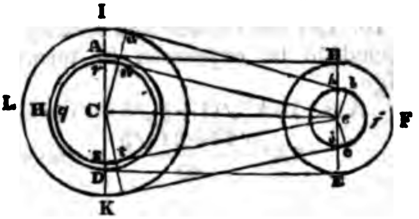
4. What is the length of a chord, which cuts off one-third of the area, from a circle whose diameter is 289?

Ans. 278.6716.

5. What is the area of the heart CABDFEC, the axis CD = 10 inches. (See article on spirals, page 140.)
Ans. 104.7193 inches.



6. There are two pulleys AHD, BFE, the diameters AD, and BE are each = 20 inches, and the distance Cc is 4 feet; the pulley BFE is put in motion around its axis by a belt ABFEDH, passing round AHD. Now if the pulley ILK on the same axis with AHD, is 40 inches in diameter, what must be the diameter *hi* of a corresponding pulley *hfi*, around which the belt I*lfe*KL may pass, so as to be of the same length and tension as that of the belt ABFEDH, and what will be the ratio of the angular velocity of the two pulleys.



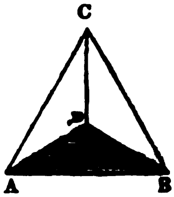
Draw C*a*, *cb* perpendicular to *al*, and from *c* draw *cn* perpendicular to C*a*; *cl* and *cn* will be parallel to *al*, and hence will be = *al*: with the radius C*n* describe a circle *nrqst*, and the tangent *nc* will be = the tangent *al*, $= \sqrt{(Cc^2 - (IC - hc)^2)}$.

The arc *rn* is = arc I*a* — arc *hl*.

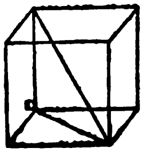
It is required from these data to find the diameter *ih*.

Scholium. In the solution of this problem, it will be necessary to express the arc *rn* in terms of its functions; the mode of conducting the solution will be left for the student.

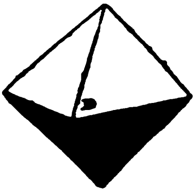
7. Required an expression for the superficies, and also for the solidity of a tetrædron ABCD, in terms of its linear edge, AB=A.
Ans. $A^2\sqrt{3}$ = the surface.
$\frac{1}{12}A^3\sqrt{2}$ = the solidity.



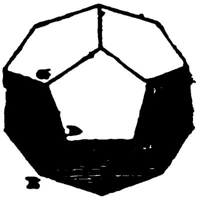
8. Required expressions for the surface and solidity of a regular hexædron or cube in terms of its edge. Ans. $6A^2$,= its surface.
A^3,= its solidity.



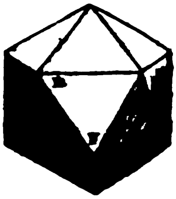
9. How will you express the surface and solidity of an Octædron in terms of its edge? Ans. $2A^2\sqrt{3}$ = the surface.
$\frac{1}{3}A^3\sqrt{2}$ = the solidity.



10. Let the surface and solidity of a dodecædron be expressed in terms of its edge.
Ans. $15A^2\sqrt{(1+\frac{2}{5}\sqrt{5})}$ = the surface.
$5A^3\frac{\sqrt{47+21\sqrt{5}}}{40}$ = the solidity



11. Express in terms of its edge, the surface and solidity of an Icosædron.
Ans. $5A^2\sqrt{3}$ = the surface.
$\frac{5}{6}A^3\frac{\sqrt{7+3\sqrt{5}}}{2}$ = the solidity.



12. The diameter of the Winchester bushel was 18½ inches, and its depth 8 inches: what must the diameter of a bushel be when its depth is 7½ inches? Ans. 19.1067.

13. Of what diameter must the bore of a cannon be, which is cast for a ball of 24 lbs. weight, so that the diameter of the bore may be 1-10 of an inch more than that of the ball, and supposing a 9 lb. ball to measure 4 inches in diameter?
Ans. 5.646 inches.

14. Suppose the ball on the top of St. Paul's church is 6 feet in diameter, what did the gilding of it cost, at 3½*d*. per square inch? Ans. £237, 10*s*. 1*d*.

15. What will the gilding of a right rectangular revoloid, whose axis is 4 feet, come to at 5 cents per square inch?
Ans. $720.

16. A silver cup, in form of the frustum of a cone, whose top diameter is 3 inches, its bottom diameter 4, and its altitude 6 inches, being filled with beer, a person drank out of it till he could see the middle of the bottom; it is required to find how much he drank?

Ans. 42.899844 cubic inches = .152127 ale gallons, or 1 gill and ½ nearly, the quantity required.

17. Two persons would divide between them, by a plane perpendicular to the base, a hay rick, in the form of a paraboloid, whose altitude is 40, and the diameter of its base 30 feet; it is required to find the difference between the solidities of the parts, supposing the altitude of the section to be 28 feet. Ans. 11265.75803, the difference required.

18. In the construction of a railroad, having contracted for the sum of $1000 to excavate a certain section, the area of whose conjugate sections in 11 different places, taken at equal distances of 3 rods each, including the ends, are as follows, viz: the first, 150 square feet, the second, 160. the third, 165, the fourth, 172, the fifth, 190, the sixth, 210, the seventh, 224, the eighth, 240, the ninth, 202, the tenth, 108, and the eleventh 0. After having disposed of the materials to be excavated at 10 cents per cubic yard, to be delivered on an adjoining section, I afterward received an offer to have the whole labor of excavation and delivery performed for 30 cents per yard; shall I gain or lose by my contract if I accept of the offer, and how much? Ans. I shall gain $355,64.

19. To determine the weight of a hollow spherical iron shell, 5 inches in diameter, the thickness of the metal being one inch? Ans. 11.79lb.

20. It is proposed to determine the proportional quantities of matter in the earth and moon; the density of the former being to that of the latter, as 10 to 7, and their diameters as 7930 to 2160. Ans. as 71 to 1 nearly.

21. What difference is there, in point of weight, between a block of marble containing 1 cubic foot and a half, and another of brass of the same dimensions, whose specific gravity is 8000? Ans. 496lb. 14oz.

22. What position in the line between the earth and moon, is their common centre of gravity; supposing the earth's diameter to be 7920 miles, and the moon's 2160; also the density of the former to that of the latter, as 99 to 68, or as 10 to 7 nearly, and their mean distance 30 of the earth's diameters? Ans. 633.65 miles below the surface of the earth.

23. How deep will a cube of oak sink in common water; each side of the cube being 1 foot? Ans. $11\frac{1}{10}$ inches.

24. How deep will a globe of oak sink in water; the diameter being 1 foot? Ans. 9.9867 inches.

25. If a cube of wood, floating in common water, have three inches of it dry above the water, and $4\frac{8}{103}$ inches dry when in sea-water; it is proposed to determine the magnitude of the cube, and what sort of wood it is made of? Ans. the wood is oak, and each side 40 inches.

26. Hiero, king of Sicily, ordered his jeweller to make h a crown, containing 63 ounces of gold. The workm thought that substituting part silver was only a proper perq site; but Hiero, suspecting that fraud had been practis Archimides was appointed to examine it; who on putting into a vessel of water, found it raised the fluid 8.2245 cu inches; and having discovered that the inch of gold more c tically weighed 10.36 ounces, he found by calculation wh part of the king's gold had been changed. And you are d sired to repeat the process. Ans. 28.8 ounces.

27. Supposing the cubic inch of common glass wei 1.4921 ounces troy, the same of sea-water .59542, and brandy .5368; then a seaman having a gallon of this liqu in a glass bottle, which weighs 3.84lb out of water, and, conceal it from the officers of the customs, throws it ov board. It is proposed to determine, if it will sink, how mu force will just buoy it up? Ans. 14.1496 ounces

28. Suppose, by measurement, it be found that a man war, with its ordinance, rigging, and appointments, sinks deep as to displace 50000 cubic feet of fresh water; what the whole weight of the vessel? Ans. $1395\frac{1}{11}$ tons.

29. It is required to determine what would be the height the atmosphere, if it were every where of the same densi as at the surface of the earth, when the quicksilver in the b rometer stands at 39 inches; and also, what would be t height of a water barometer at the same time?

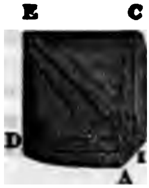
Ans. height of the air $29166\frac{2}{3}$ feet, or 5.5240 miles.
height of water 35 feet.

30. If the inner axis of a hollow globe of copper, exhaust of air, be 100 feet; what thickness must it be of, that it m just float in the air? Ans. .02624 of an inch thick.

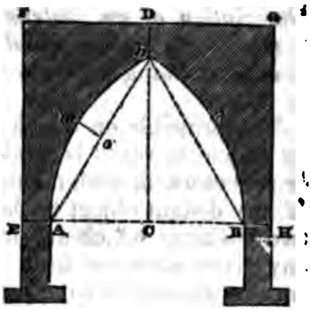
31. If a spherical baloon of copper, of $\frac{1}{100}$ of an inch thic have its cavity of 100 feet diameter, and be filled with infla mable air, of $\frac{1}{10}$ of the gravity of common air, what weig will just balance it, and prevent it from rising up into the a mosphere? Ans. 21785lb.

32. Construct a quantity $\frac{ax^2z}{y}$ and show its value. (S Book IV., Chap. II.)

33. Express in a series of variables, the symmetrical cylindrical ungulas, DBAECAB. where, AB=2ID=x, and IC=z.



33. Required the solidity of the vacuity of a gothic roof, and also the solidity of the materials of the the roof; the span AB = 30 feet, the chords of each arc An and Bn $=40$ feet, the versed sine mo = 6 feet, the thickness of the pear EA or BH = 17 feet, at the spring of the arc, the thickness of the crown of the arch $Dn=4$ feet of the roof 60 feet.



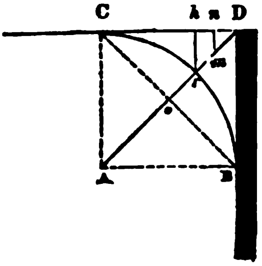
Ans. { 52896.89319 cubic feet, the solidity of the vacuity.
104854.11776 cubic feet, the solidity of the materials.

34. Required the superficies of a dome in the form of a right hexagonal revoloid, each side of the base being 10 feet, and height 10 feet.

Ans. 519.61524 square feet.

35. The circumference of the base of a circular dome is 150 feet, and its height 23.873 feet; required the superficies.

Ans. 3581.1 square feet.



36. If the height BC of a saloon be 3.2 feet, the BoD, of its front 4.5 feet, the distance or, of its middle part, from the arc 9 inches, and the mean circumference at $m=50$ feet; required the solidity of the saloon.

Ans. 138.26489 cubic feet.

37. What is the whole surface of a saloon round a rectangular room, the mean compass at $r=67.3137$ feet, the girt DrB 3.1416 feet, and the ceiling measures 16 feet long, and 12 feet broad?

Ans. 403.4727 square feet.

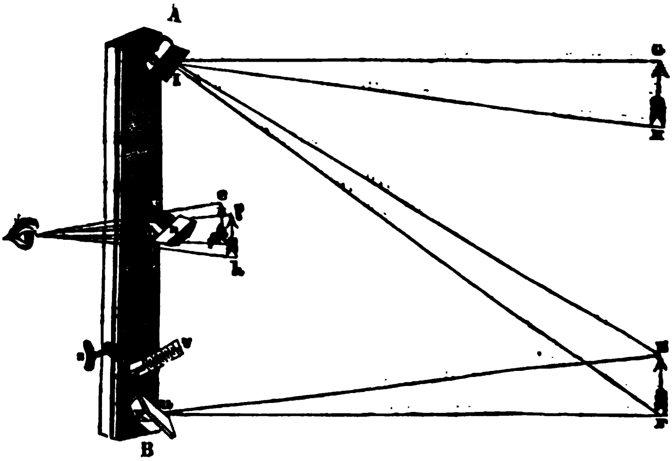
38. Required the concave surface of a circular arch, raised on a rectangular base, whose sides are 27 feet 3 inches by 20 feet 4 inches.

Ans. 632.5415⅓ square feet.

Description of an instrument constructed by the author, for measuring distances and heights, by a single observation, and without changing the position, or measuring any base line.

The principle on which this instrument is constructed, and by which the result is produced, consists in arranging mirrors, or reflectors, in such manner as to convey two distinct images of any distant object to the eye of the observer, as seen from two positions which are indicated by two mirrors, placed at any given distance from each other on the instrument, and causing the object to appear in two positions at the same time, and then measuring the apparent angle under which the two images appear.

For this purpose the following diagram represents one form of the construction of the instrument.



AB, represents the stock or base; at the extremities of which let two mirrors, I, *u*, be placed in the manner of the index glass to a common quadrant, and set perpendicular to the plane of the instrument, and at an angle of 45° with its axis or one of its edges; these we will call the object glasses, one of which, viz: I, is placed on the centre of motion of an index, M*m*, which is moveable about a centre at *e*, and with it the mirror, which for distinction is called also the index glass. There are two other reflectors, *n*, *i*. placed near the middle of the stock of the instrument, one above the other, their edges being in contact, the planes of which cross each other at right angles; one of these reflectors is parallel to one of the object glasses *n*, the other to the index glass, I.

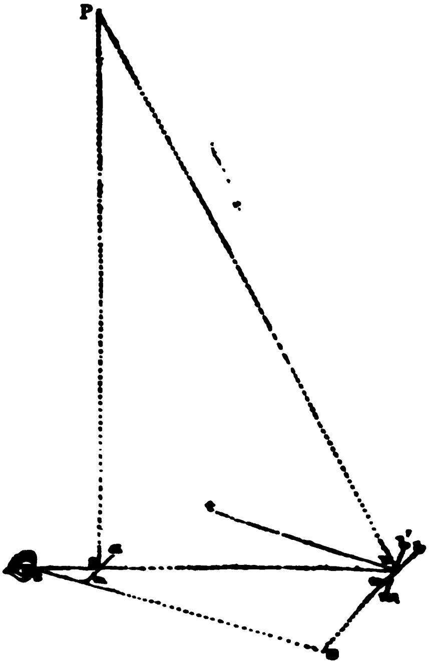
In using this instrument it must be placed or held so that its axis shall be perpendicular to a line from the object EF, to the fixed object glass u, and images of the object will be formed in the two object glasses I, K, and reflected into the two glasses n and i, and hence, to the eye of the observer, forming two distinct images, ef, ph, one of which appears higher than the other; and the angle under which the two objects appear, varies according to the distance of the object. In order to measure the angle contained by the apparent positions as indicated by the two images, the index is moved, and with it the index glass i, till the object shall appear in the same position in both glasses, so that there would appear to be but one image formed in both, or till the images formed in both would appear identical. Now if a *vernier scale*, v, is attached to the index and graduated, it will indicate the apparent angle under which the two images appear, or the angle under which the distance between the two object glasses would appear if placed at the distance of the object. But by the law of reflection, the index would be moved only through half the angular distance of the two images in order to produce an apparent coincidence; and hence the scale should be graduated with double the ordinary divisions, for the angle would be indicated by twice the angular motion of the index. In order that the index may be adjusted with accuracy, a tangent screw s, is provided, by which it can be adjusted with any precision required. When the observation and adjustment is completed, we have a right angled triangle whose base is the distance of the two mirrors, and whose altitude is the distance of the fixed object glass to the object, and since we have one of the acute angles, the other side or distance of the object becomes known.

When the object EF, is at any considerable distance, the angle uFI, or uEI, becomes smaller, and the object will appear to come to the mirror I, from the position GH; so that HI and Fu produced, shall make an angle with each other equal to the apparent distance of the two images. But when the object is at an infinite distance, then the lines Fu, HI, become parallel, and the two images coincide, so that but one image appears to the observer; and hence, in this case, the distance cannot be measured.

In order to save the trouble of calculation in each case, a table may be constructed to accompany the instrument, which shall contain the distance corresponding to any given angle pointed out on the scale, or, the scale itself may be graduated to specific distances, which may be read off instead of the angles.

Another construction of the instrument, using but two glasses is as follows:

Let two mirrors, *a* and *b*, be placed parallel to each other at the opposite extremities of the instrument, making an angle of 45° with its axis as before, and let the mirror *b*, be moveable about a centre by means of an index as described above.



When any distance is to be measured by this instrument, it is placed so that a line from the object P, to the mirror *a*, shall be perpendicular to the axis *rm*, of the instrument, the eye of the observer being at any point *e*, in the axis produced; and the image of the object seen in the mirror *a*, will, by the law of reflection, appear in the direction of this axis; and the image P, in the other mirror, if seen at all, will appear at *o*, not coinciding with *m*. But by turning the mirror *b* about its axis, the image will advance toward *m*, in the line of the axis of the instrument, and will ultimately coincide with it; when the two images will be seen at *e*, in the same line, *em*.

It may be easily shown that the angular motion of the mirror *b*, necessary to bring it into the position *b′ x′* so that the two images shall appear to coincide, will be half the angle at P, hence the distance is determined as before.

In this form of construction, I place the mirror in a wooden case or tube, leaving openings in its side in front of each mirror, and a small hole at the end for the eye of the observer; the index and scale is on the out-side of the case: the mirror *a*, must occupy but half a section of the tube, so that the mirror *b* may be seen over the edge of the mirror *a*; and the object is attained by bringing the images in both mirrors to coincide, or so as to appear as one image.

In order to determine the powers of this instrument, it is only necessary to observe that the distance *m*P, is the cotangent of the angle P to the radius *mr*, and that when *mr* is given, the value of *m*P may be calculated for any assumed value of the angle P, which is half the angle measured by the index and scale. Assuming the distance between the mirrors to be five feet, the angular motion of the mirror *b* from its position parallel to *a*, the zero of the scale, will be for 1000 feet nearly 8′35″, and for 1100 feet 7′50″, a difference of 45″ or three-fourths of a minute for a difference of 100 feet, or ten per cent of the first distance.

If we assume that by the divisions on the scale attached to the index, the motion of the mirror may be correctly found to half minutes, then the distance between the mirrors being taken at five feet, a change of half a minute would correspond at 10 feet to .007 of a foot, at 100 feet to .62 of a foot, at 1000 feet to 63 feet, and at 10000 feet or 1.9 mile the whole angle is but 51″, and considerable variations would entirely escape detection; but by the application of a telescope to the instrument in making the observations, its powers and accuracy may be considerably extended.

After having measured the distance to any object, as a house, or a tree, its altitude may be easily found by moving the index so that the top of the object in one mirror shall coincide with the bottom in the other, when the angle indicated on the scale, less the angle first found, corresponding to the to the distance, is the angle under which the object appears; whence having the angle and distance of the object, its altitude becomes known.

The following investigation of the powers of the instrument, showing its limits of practical accuracy, is taken from a report furnished by a committee of the Franklin Institute of Pennsylvania, to whom the two instruments designated above were submitted by the author in 1833. (Published in Vol. XI., No. 3, Journal of the Franklin Institute.)

Call the variation in the angle P, y, the distance m P a, and mr, b. Suppose the angle P, to become P — y, and that then $a+x$, x denoting the increase of length of a, corresponding to a decrease, y, of the angle P.

By trigonometry,

$$\tan.\ P = \frac{b}{a} \text{ and}$$

$$\tan.\ (P - y) = \frac{b}{a+x}\text{; but}$$

$$\tan.\ (P - y = \frac{\tan.\ P - \tan.\ y}{1 + \tan.\ P \times \tan.\ y}\text{, or by substituting for}$$

tan. P and tan. (P — y) their values found above.

$$\frac{b}{a+x} = \frac{\frac{b}{a} - \tan.\ y}{1 + \frac{b}{a} \div \tan.\ y} \text{ or}$$

$$\frac{b}{a+x} = \frac{b - a \div \tan.\ y}{a + b \div \tan.\ y}\text{, whence}$$

$$x = \frac{(a^2+b^2) \div \tan.\ y}{b - a \times \tan.\ y}\text{, or}$$

$$x = \frac{\frac{a^2+b^2}{b}}{\frac{b}{\tan.\ y} - a}$$

If, as assumed above, $b = 5$, and $y = 1$, the general equation becomes

$$x = \frac{25+a^2}{17241.4 - a}$$

$$\text{When } a = 10, x = \frac{125}{17231.4} = .007.$$

$$\text{For } a = 100, x = \frac{10.025}{17141.4} = .62$$

For $a = 1000$, $x = 61.6$, and for $a = 10{,}000$, $x = 13{,}809$; which is greater than the distance a.

By assuming a limit to the accuracy required, calculation will show how the instrument may be adapted to this limit when possible. For example, let the greatest inaccuracy allowed be one foot in 100, then b and y must be so adjusted that at the greatest distance for which the instrument is to be used $x = \frac{a}{100}$. Calling this value of a, a', we shall have,

$$.01\ a' = \frac{a'^2+b^2}{\dfrac{b}{\tan.\ y} - a'}, \text{ or}$$

$$b^2 - \frac{.01\ a'}{\tan.\ y} \times b = -\ 1.01\ a'^2,$$ an equation which must exist in order that the required accuracy may be attainable. To examine by it the instrument already supposed, let us ascertain whether at 1000 feet, as the greatest distance at which it is to be used, the accuracy will come within the limit of one foot variation, in 100. In this case $a' = 1000$, b, as before, $= 5$, and $\tan.\ y = \tan.\ 1'$. Whence $b^2 = 25$, $\frac{b}{\tan.\ y} = 17241.4$, $01a' = 10$, and $1.01\ a'^2 = 1.010.000$. Substituting these values in the equation above, it requires $25 - 172{,}414 = -\ 1{,}010{,}000$, the equation is not fulfilled, and the instrument does not come up to the requirement. It would be easy to determine values of y and b required for all possible degrees of accuracy, and thus by the possibility of making the half divisions accurate, and by the length which convenience might limit, to ascertain whether the instrument could be constructed to give the required degree of accuracy.

The investigation may be made more general, thus; let $\frac{a}{b}$ express the required limit of accuracy at the greatest distance for which the instrument is to be used, then at that distance $x = \frac{a}{n}$, or calling, as before, the value of a, a',

$$\frac{a}{n} = \frac{a'^2+b^2}{\dfrac{b}{\tan.\ y} - a} \text{ whence}$$

$$b^2 - \frac{a'}{n\ \tan.\ y}\, b = -\ \frac{a'^2}{n} - a'^2 - a'^2\left(1 + \frac{1}{n}\right)$$

$$b = \frac{a'}{2n\ \tan.\ y} \pm \sqrt{\frac{a'^2}{4\ n^2\ \tan.\ ^2y} - a'^2\left(1 + \frac{1}{n}\right)}$$

$$b = \frac{a'\left(1 \pm \sqrt{1 - 4\ n^2\ \tan.\ ^2y\left(1 + \frac{1}{n}\right)}\right)}{2n\ \tan.\ y}.$$

This equation is possible when $4n\ \tan.\ ^2y\ (n+1) < 1$.

NOTES.

BOOK I.

HAVING in the preceding volume treated of the properties of the *Para*
Ellipse, and *Hyperbola*, we show in this, that these curves are the sec
of the cone, and are each formed by a plane passing through a cone, acc
ing to certain conditions. Of these sections the quadrature of the para
is easily attained, by the principles embraced in Prop. IV., in relation t
sectional divisions of a prism. Props. VI. and VII., show the applicati
the principles* to the quadrature of the parabola.

The quadrature of the *Ellipse*, though not so accurately determined as
of the parabola, is nevertheless easily expressed in terms of the cir
quadrature.

The *Hyperbola* is of more difficult determination, in relation to its qua
ture, than the other conic sections, but, is nevertheless susceptible of b
approximately determined to any extent required.

BOOK II.

ON SOLID SECTIONS OR SEGMENTS.

Since the term section, though originally applied only to surfaces, mad
cutting a solid, is extensively used in the arts, as expressing a definite
tion of a solid, I have taken the liberty of making solid sections, as syn
mous with segments of solids. This book, consists mostly of the comp
son of cylindric, and conical ungulas.

BOOK III.

OF REVOLOIDS.

Revoloids are a class of bodies not usually treated of in works on Ge
etry ; but, from the important considerations connected with their organi
tion, and from the relations which they bear, both to rectilineal and curv
neal solids, it is of the highest importance to Geometry, that their propert
should be discussed, and that they should receive a conspicuous pl
among geometrical solids ; more especially, as they are almost the only c
vilinear solids, that are absolutely cubable. For, we have shown, in
progress of the work, that, not only the right or spherical revoloid, and
elliptical revoloid, Props. VI. and VII. are cubable, is absolute terms,
also *Parabolic* and *Hyperbolic Revoloids* ; moreover, we have shown (Pr
III.,) the quadrature of the surface of a right revoloid, without regard to
circle's quadrature, although it is bounded by cylindric surfaces.

Some important principles are derived from the subject of mathemati
transformation, as in Prop. IV., V., XI., and Corollaries.

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* In Prop. VI, the parallel lines kK &c., should be parallel to the axis EF of the para
instead of the position as there expressed, otherwise, the argument is not correct; but since
principle to be established, is not affected by the error in that diagram and argument,
deemed best not to alter the demonstration in this edition. The Scholium to Prop. VII,
be corrected by making AB or CD the axis of the parabola, instead of AC or BD. In all
ferences to these propositions in the succeeding parts of the work, it will be perceived that
conditions of their application is expressed.

BOOK IV.

The quadrature of the revoloidal surface treated of in Book III., furnishes us with data for the quadrature of the circle, through the medium of the revoloidal curve ; this subject is amply discussed in this book ; through the properties of this curve, we are also enabled to deduce some important trigonometrical functions. In Prop. III., it is shown that the revoloidal curve may be by transformation derived from an elliptical curve.

Expressions are obtained in Props. IX. and XII., for the length of the circle's circumference ; it is there shown that these approximations may be extended indefinitely, so that the circumference may be obtained to any degree of exactness required.

By applying the principles of the parabola to those of the revoloidal curve in Prop. XVII., a remarkable approximation is obtained, so that if the sine and cosine of a small arc, and sine of half the given arc is obtained, the arc itself may be expressed in terms of those functions ; and it is shown that these functions may be so taken, that the arc shall be truly expressed to the same number of decimal places, that those functions are truly expressed ; in pursuance of this, will be found, an example in Mensuration, page 194, where the arc is correctly calculated to 20 decimal places, by a simple process, having the sines and cosine given, as data, to 21 decimal places. Other modes of approximation, for the arc of the circumference may be pointed out, depending on the same principles, derived both from the quadrature of the revoloidal surface, and cubature of the reveloid; but, by pursuing the course pointed out, page 125, the arc of the circumference may be found, with certainty, by this process, to any number of decimal places we have patience to pursue it.

Props. XVIII and XIX, prepare us for the construction of a curve, described in Prop. XXI., termed the curve of the circle's quadrature, the properties of which are, that a line drawn from any point in this curve, perpendicular to its conjugate diameter, will be equal to the arc of the inscribed circle cut off by a secant drawn from the centre of the circle to this point, and if another line be drawn from the same point in the curve, to the extremity of the conjugate diameter, the area of the space, intercepted by the two lines without the circle, will be equal to the area of the segment of the circle cut off by the latter line. From these properties, we are enabled to deduce an important theorem, in relation to segments of the circle, viz., that the area of any segment of a circle is equal to the difference between the arc of the segment, and its sine, multiplied by half the radius.

BOOK V.

In this book, all geometrical magnitudes are discussed from their principles of organization, from elementary magnitudes, without referring them to any specific forms or relations. And after introducing, and explaining the principles of the production of geometrical magnitudes, from variable elements, in Chap. I., in order to render this science subject to analytical, and algebraic consideration, a peculiar notation has been introduced as the subject of the second Chapter; and the mode of application, of this notation, to such subjects is there explained.

By this mode of conducting geometrical investigations, and by this notation, we are enabled in a manner, somewhat more obvious than that of the calculus, to arrive at the same results as are obtained by that science.

And although this forms but an introduction to the subject, yet it furnishes evidence of its adaptation to the investigation of the properties of all geometrical magnitudes ; and may, perhaps, be rendered of equal universality with the calculus, with which it is most intimately allied.

Here instead of the infinitely small momentary increments of variable magnitudes, or instead of the differentials of variables, this notation recognizes only the conditions of the variables themselves, in their associated capacity, which, from the principles of the science, may be integrated, as certainly, and with more obvious rationality, than those performed by the calculus from their differentials.

By this notation, we are enabled to get a positive expression for the circle's quarature, in known functions of the diameter ; which, since all Geometricians are satisfied of the incommensurability of the circumference in direct terms of the diameter, should be received as the quadrature itself. For, from this expression, means may be devised of developing, decimally, the quadrature to any desirable extent.

Chapter III: is an introduction to the differential and integral calculus, designed to show the first principles of that science, and to show its connection with that of Chapter II., both of which are evidently in their essential particulars, based on the principles contained in Chapter I.

Chapter IV. is devoted to the application of the principles previously discussed, to determine the position of the virtual centre, or centre of gravity of geometrical magnitudes. This has universally been denominated by all authors, hitherto as the *centre* of *gravity*. I doubt not, I shall have the approbation of most Mathematicians in discarding that term, and supplying in its place that of the *virtual centre*. The term *centre* of *gravity*, though perfectly proper in works on Mechanics and Natural Philosophy, is highly incongruous in a work on pure mathematics, where the physical properties of bodies is not a subject of investigation, and perhaps not intelligibly understood.

MENSURATION.

Such subjects in the mensuration of surfaces and solids, as could not consistently be introduced into the elementary part of the work, is introduced here ; most of the subjects embraced in this, have been discussed in the geometrical part of the work, and the principles demonstrated ; where this is not the case, a reference has been made to the author, where such demonstration may be found ; on this subject free use has been made of Hatton's Mensuration ; limited quotations have also been made from other authors.

The article on Guaging is mostly taken from Day's Mathematics, though originally derived from Hutton.